Advanced Partial Differential Equations
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Lecture 32
Conservation Law: Example and Entropy Condition

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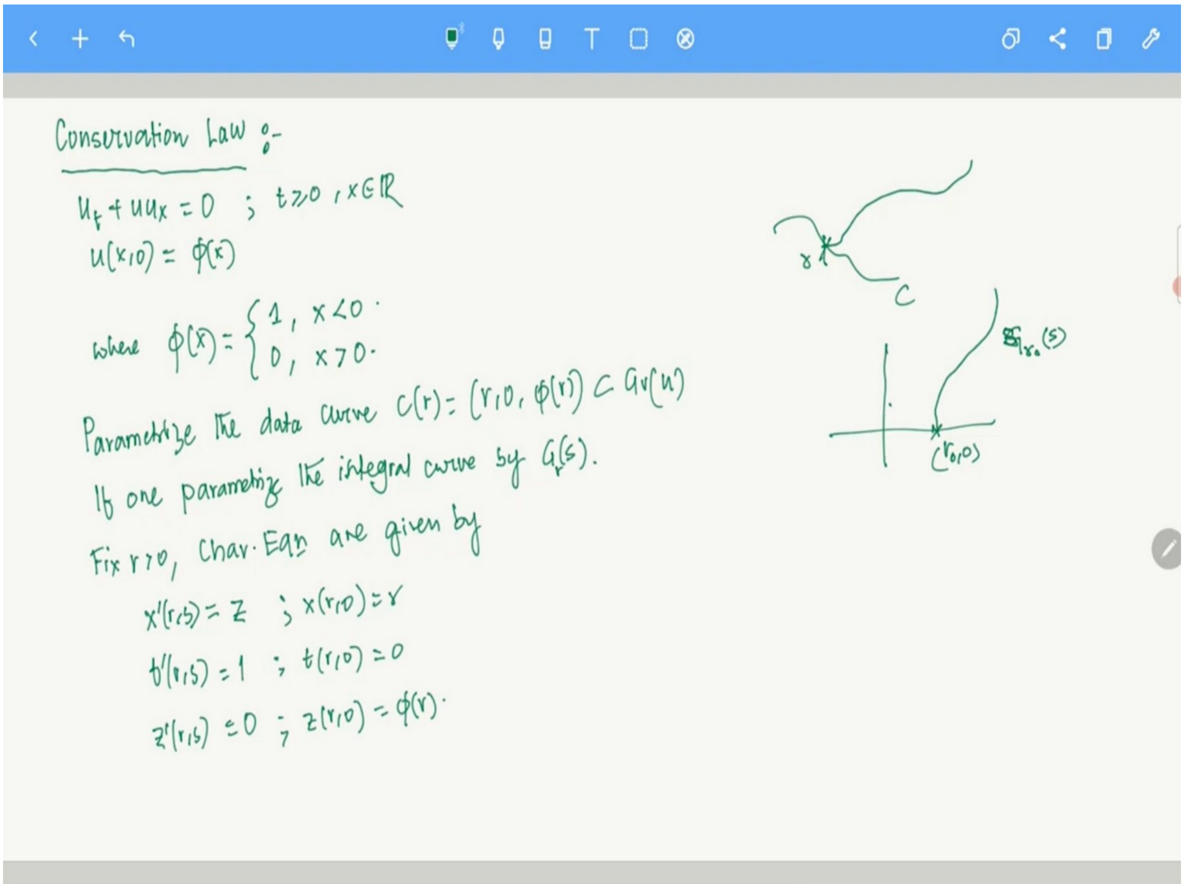

So, welcome students. This video, we are going to look at a problem called Burger's equation. This will actually give you an idea about the Rankine-Hugoniot condition and weak solutions.

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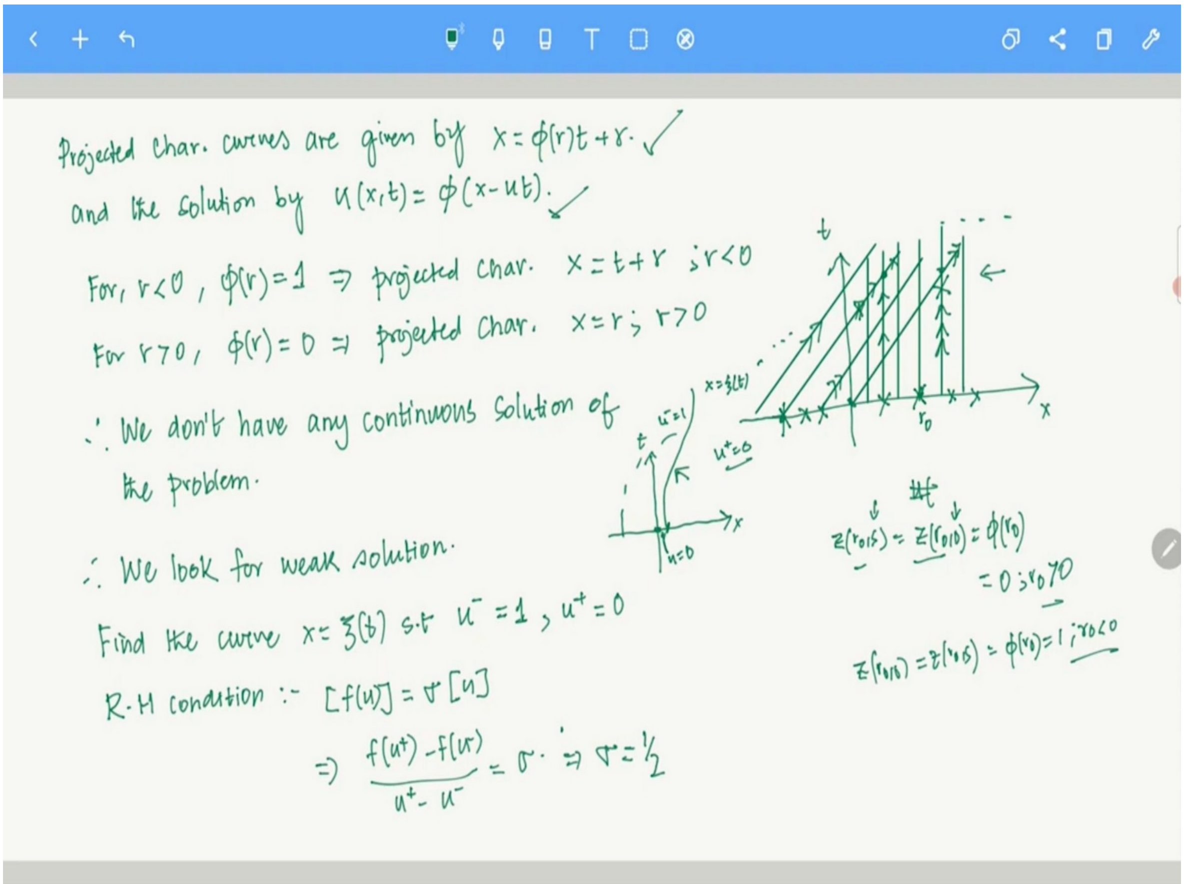

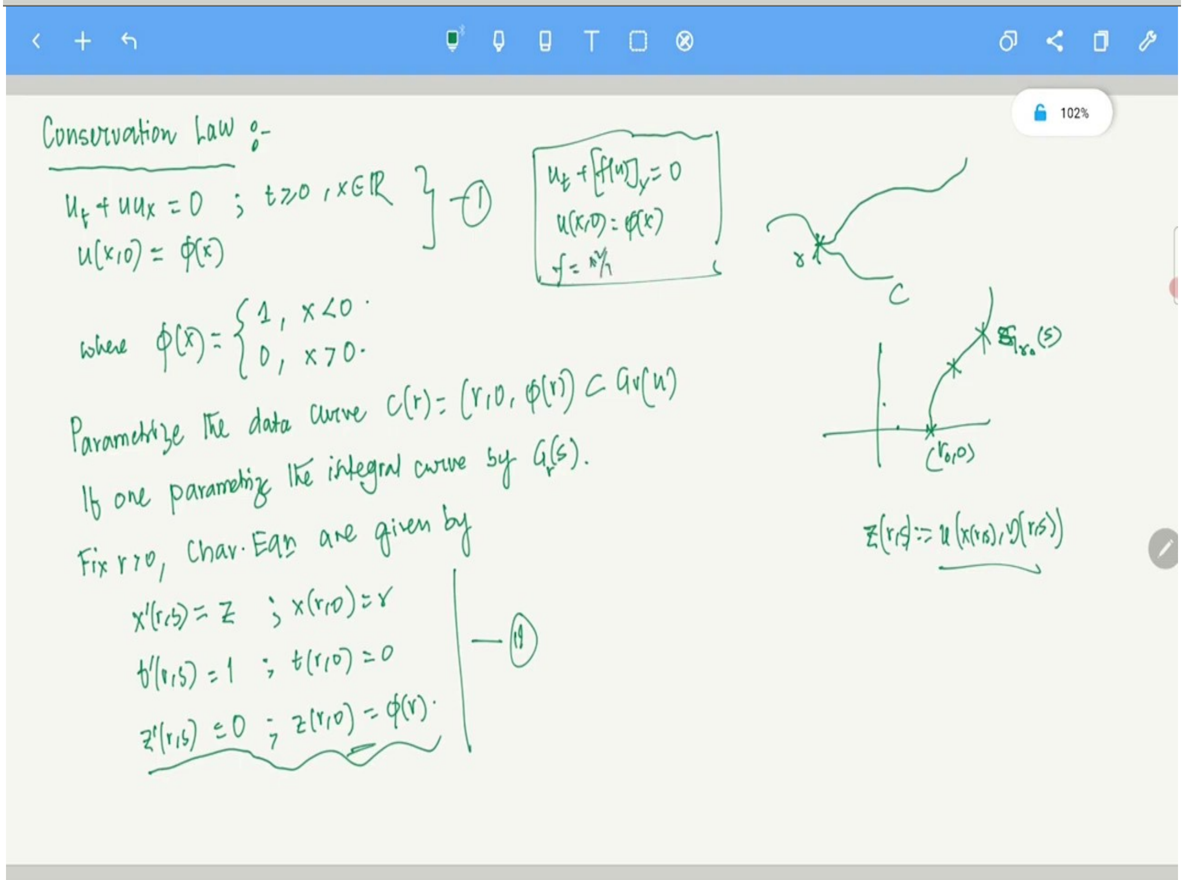

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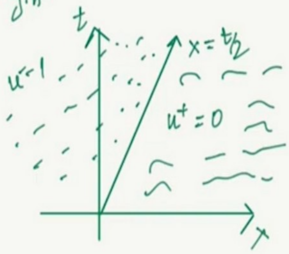
Also we want our curve to pass through the origin.

Now, $x = \xi(t)$ is the required curve.

$$\sigma = \xi'(t) = \tfrac{1}{2} \Rightarrow \xi(t) = t/2 \Rightarrow x = t/2.$$



Our weak solution is given by

$$u(x,t) = \begin{cases} 1, & x < t/2 \\ 0, & x > t/2 \end{cases}$$

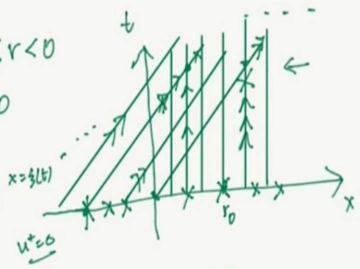
- It is classical solution on each side of the curve
- The solution is bounded.
- It satisfies RH condition.

Projected char. curves are given by $x = \phi(r)t + r$.

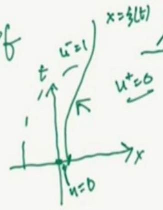
and the solution by $u(x,t) = \phi(x - ut)$.

For, $r<0$, $\phi(r) = 1 \Rightarrow$ projected char. $x = t + r$; $r<0$

For $r>0$, $\phi(r) = 0 \Rightarrow$ projected char. $x = r$; $r>0$



∴ We don't have any continuous solution of the problem.



∴ We look for weak solution.

#t

$z(r_0,s) = z(r_0,0) = \phi(r_0)$
$= 0 ; r_0 > 0$

$z(r_0,s) = z(r_0,0) = \phi(r_0) = 1 ; r_0 < 0$

Find the curve $x = \xi(t)$ s.t $u^- = 1$, $u^+ = 0$

R-H condition :- $[f(u)] = \sigma [u]$

$$\Rightarrow \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \sigma \Rightarrow \sigma = \tfrac{1}{2}$$

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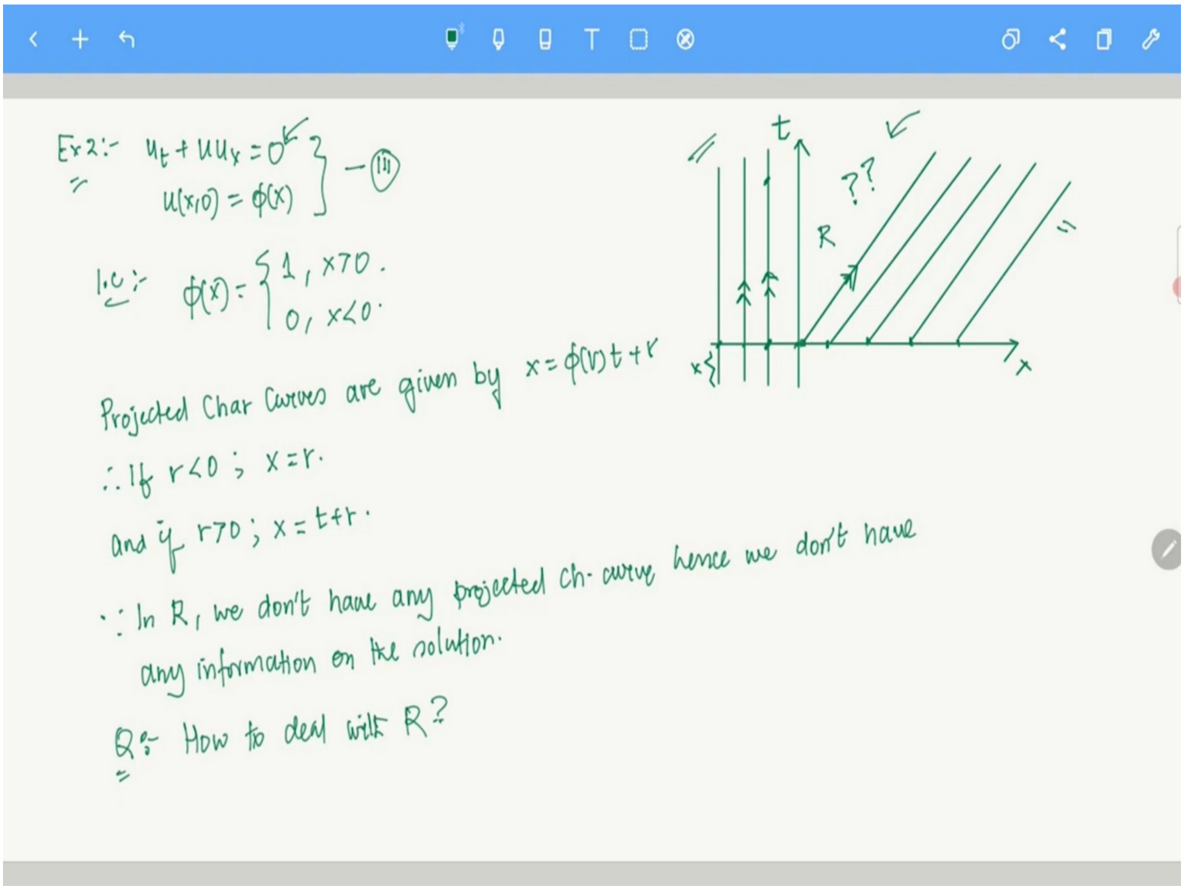

Now, we are exploring more on this Burger's equation. So, essentially, let us take another example and see that, how to deal with this sort of problems.

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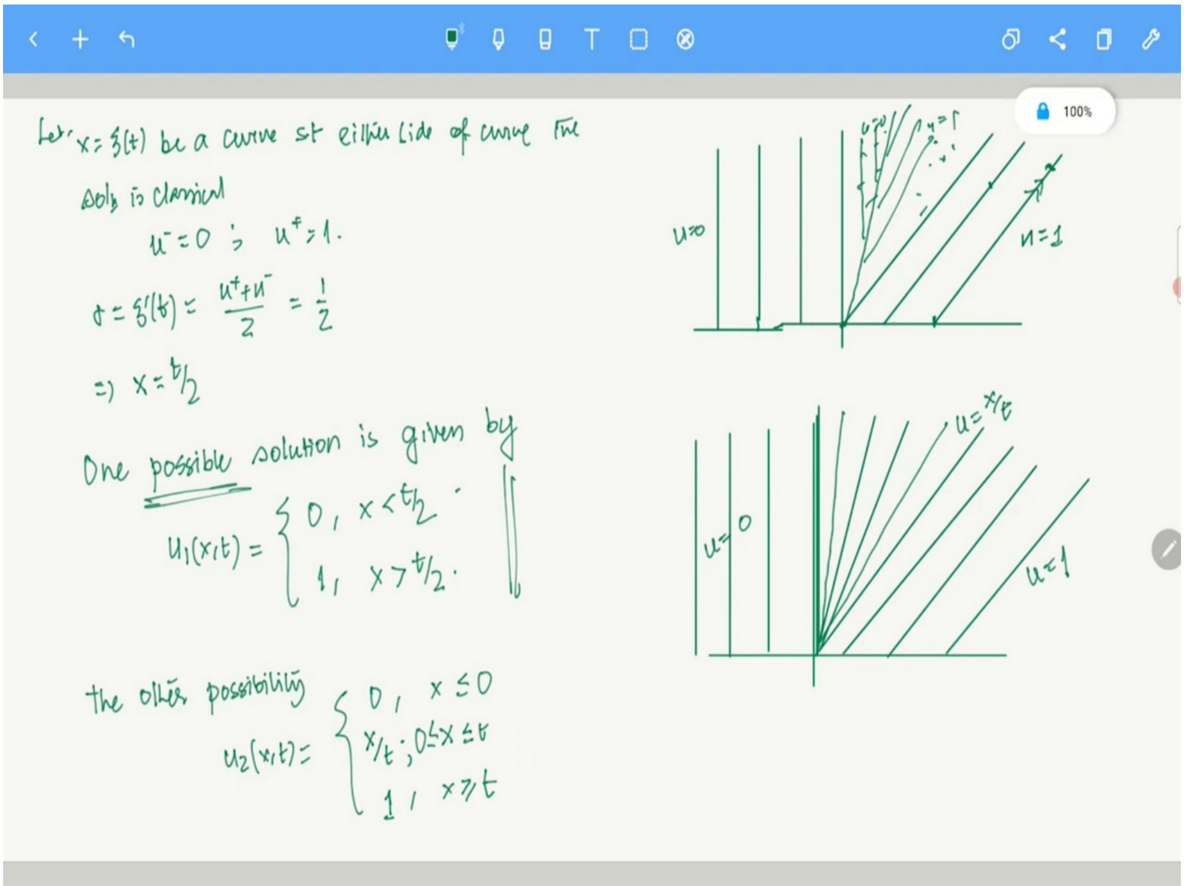

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Remark: o Solution like this "which fills the void region" $\{0<x<t\}$ is called a rarefaction wave.

o We want a physically realistic solution.

Entropy Condition: $f \in C^1(\mathbb{R},\mathbb{R})$

$$u_t + [f(u)]_x = 0 \Rightarrow u_t + f'(u)\, u_x = 0$$

Fix $r \geq 0$:

$$\left.\begin{array}{l} x'(r,s) = f'(z) \\ t'(r,s) = 1 \\ z'(r,s) = 0 \end{array}\right| \Rightarrow \frac{dx}{dt} = \frac{dx/ds}{dt/ds} = \frac{f'(z)}{1} = f'(u)$$

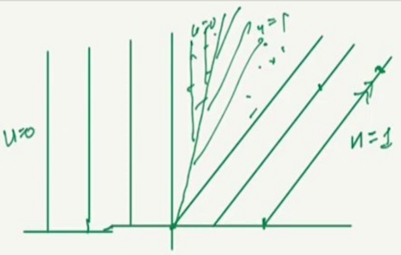
$\therefore$ The speed of the solution 'u' is given by $f'(u)$.

Let $x = \xi(t)$ be a curve s.t. either side of curve the sol^n is classical

$$u^- = 0 \;;\; u^+ = 1.$$

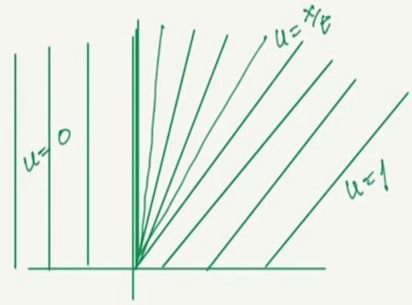
$$\sigma = \xi'(t) = \frac{u^+ + u^-}{2} = \frac{1}{2}$$

$$\Rightarrow x = t/2$$



One possible solution is given by

$$u_1(x,t) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2. \end{cases}$$



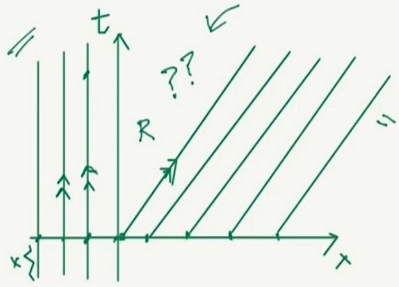
The other possibility

$$u_2(x,t) = \begin{cases} 0, & x \leq 0 \\ x/t; & 0 \leq x \leq t \\ 1, & x \geq t \end{cases}$$

Ex 2:- $\left.\begin{array}{l} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{array}\right\} - (iii)$

" u " represent the height of the wave.

I.C :- $\phi(x) = \begin{cases} 1, & x > 0. \\ 0, & x < 0. \end{cases}$



Projected Char Curves are given by $x = \phi(r)t + r$

$\therefore$ If $r < 0$; $x = r$.

and if $r > 0$; $x = t + r$.

$\because$ In R, we don't have any projected Ch. curve hence we don't have any information on the solution.

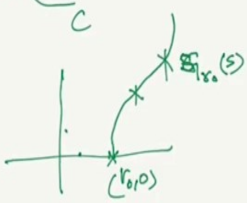
Q :- How to deal with R ?

Conservation Law :-

$\left.\begin{array}{l} u_t + uu_x = 0 \;\; ; \;\; t \geq 0, x \in \mathbb{R} \\ u(x,0) = \phi(x) \end{array}\right\} - (I)$

$\boxed{\begin{array}{l} u_t + [f(u)]_x = 0 \\ u(x,0) = \phi(x) \\ f = u^2/2 \end{array}}$

where $\phi(x) = \begin{cases} 1, & x < 0. \\ 0, & x > 0. \end{cases}$



Parametrize the data curve $C(r) = (r, 0, \phi(r)) \subset Gr(u)$

If one parametrize the integral curve by $G(s)$.

$z(r,s) := u(x(r,s), y(r,s))$

Fix $r > 0$, Char. Eqn are given by

$\left.\begin{array}{ll} x'(r,s) = z & ; \; x(r,0) = r \\ t'(r,s) = 1 & ; \; t(r,0) = 0 \\ z'(r,s) = 0 & ; \; z(r,0) = \phi(r). \end{array}\right| - (ii)$

$\frac{dx}{dt} = u$

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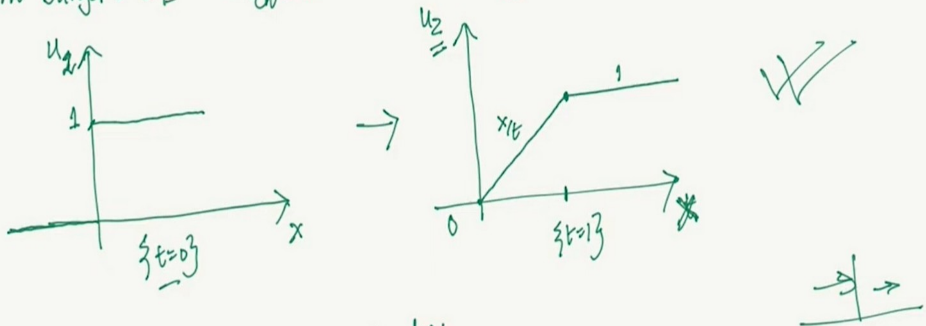
$\because$ In Burger's Eqn $dx/dt = u$. hence the bigger the wave the faster it is.



In Ex 1:- Initial waves was taller on the left.

$\Downarrow$

The wave is moving faster than the wave on the right.

"We want to allow for a curve of discontinuity if the wave moving from left is faster than the wave moving from right."

Remark:- o Solution like this "which fills the void region" $\{0<x<t\}$ is called a rarefaction wave.

o We want a physically realistic solution.

Entropy Condition:- $f \in C^1(\mathbb{R}, \mathbb{R})$

$$u_t + [f(u)]_x = 0 \Rightarrow u_t + f'(u)\, u_x = 0$$

Fix r:

$$x'(r,s) = f'(z)$$

$$t'(r,s) = 1$$

$$z'(r,s) = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{dx/ds}{dt/ds} = \frac{f'(z)}{1} = f'(u)$$

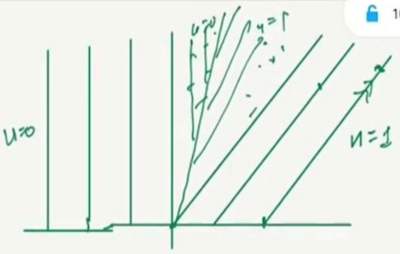
$\therefore$ The speed of the solution 'u' is given by $f'(u)$.

Let $x=\xi(t)$ be a curve st either side of curve the soln is classical

$$u^-=0\ ;\ u^+=1.$$

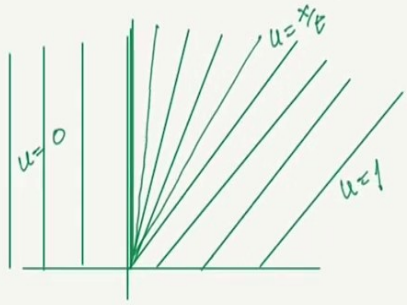
$$\sigma=\xi'(t)=\frac{u^++u^-}{2}=\frac{1}{2}$$

$$\Rightarrow x=t/2$$



One possible solution is given by

$$u_1(x,t)=\begin{cases}0, & x<t/2\\ 1, & x>t/2.\end{cases}$$



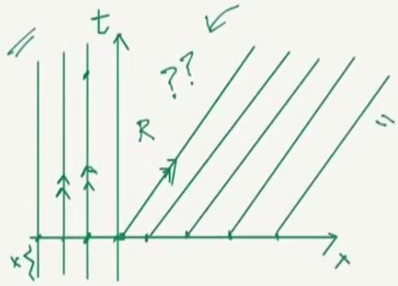
The other possibility

$$u_2(x,t)=\begin{cases}0, & x\le 0\\ x/t; & 0\le x\le t\\ 1, & x\ge t\end{cases}$$

Ex 2:- $\left.\begin{aligned}&u_t+uu_x=0\\&u(x,0)=\phi(x)\end{aligned}\right\}$ — (iii)

"u" represent the height of the wave.

I.C:- $\phi(x)=\begin{cases}1, & x>0.\\ 0, & x<0.\end{cases}$



Projected Char Curves are given by $x=\phi(r)t+r$

∴ If $r<0$; $x=r$.

and if $r>0$; $x=t+r$.

∵ In R, we don't have any projected Ch-curve hence we don't have any information on the solution.

Speed of soln 'u' is $\frac{dx}{dt}=$

Q:- How to deal with R?

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Entropy Condition $x=\xi(t)$ is the curve of discontinuity then

$$f'(u^-) > \sigma = \xi'(t) > f'(u^+)$$

And the curve of discontinuity is called Shock Curve for a solution 'u' if it satisfies the Rankine-Hugonoit condition and the entropy condition.

Ex 3: In ex 2 :

$f'(u_1^-) = u_1^- = 0$

$f'(u_1^+) = u_1^+ = 1.$

Hence Entropy condition is not satisfied.

$x = t/2$ is not a shock curve.

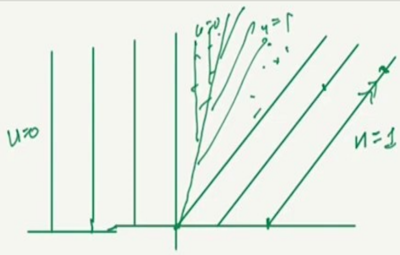
bcoz: Entropy condition is violated

Let $x=\xi(t)$ be a curve st either side of curve the soln is classical

$u^- = 0$; $u^+ = 1.$

$\sigma = \xi'(t) = \dfrac{u^+ + u^-}{2} = \dfrac{1}{2}$

$\Rightarrow x = t/2$



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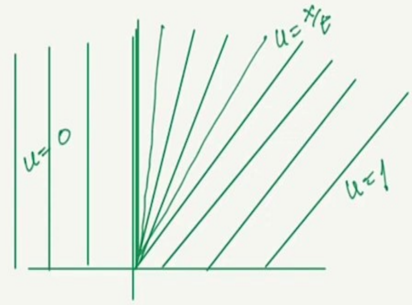

The other possibility

$$u_2(x,t) = \begin{cases} 0, & x \le 0 \\ x/t; & 0 \le x \le t \\ 1, & x \ge t \end{cases}$$

And moreover, you see, u 2 which you are getting is a continuous function, u2 is a continuous function, this where is u2, this is a continuous function and you can say that this satisfies the entropic condition and hence this is a physically relevant condition and an acceptable condition. So, with this, we are going to end this part.