

Advanced Partial Differential Equations
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Lecture 32
Conservation Law: Example and Entropy Condition

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Conservation Law :-
 $u_t + uu_x = 0 \quad ; \quad t \geq 0, x \in \mathbb{R}$
 $u(x, 0) = \phi(x)$

where $\phi(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$

Parametrize the data curve $c(r) = (r, 0, \phi(r)) \subset G_0(u)$
 If one parametrizes the integral curve by $G_r(s)$.

Fix $r > 0$, Char. Eqns are given by

$$\begin{aligned}
 x'(r, s) &= z & ; & \quad x(r, 0) = r \\
 t'(r, s) &= 1 & ; & \quad t(r, 0) = 0 \\
 z'(r, s) &= 0 & ; & \quad z(r, 0) = \phi(r)
 \end{aligned}$$

The diagram shows a curve c in the (x, t) plane starting from a point $(r, 0)$ on the $t=0$ axis. A vertical line $x=r$ is drawn, and the characteristic curve $G_r(s)$ is shown as a curve starting from $(r, 0)$ and moving upwards and to the right.

So, welcome students. This video, we are going to look at a problem called Burger's equation. This will actually give you an idea about the Rankine-Hugoniot condition and weak solutions.

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Projected char. curves are given by $x = \phi(r)t + r$. ✓
 and the solution by $u(x,t) = \phi(x-ut)$. ✓

For $r < 0$, $\phi(r) = 1 \Rightarrow$ projected char. $x = t + r$; $r < 0$
 For $r > 0$, $\phi(r) = 0 \Rightarrow$ projected char. $x = r$; $r > 0$

\therefore We don't have any continuous solution of the problem.

\therefore We look for weak solution.

Find the curve $x = \xi(t)$ s.t. $u^- = 1, u^+ = 0$

R-H condition :- $[f(u)] = \sigma [u]$
 $\Rightarrow \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \sigma \Rightarrow \sigma = \frac{1}{2}$

$\xi(r_0) = \xi(r_0) = \phi(r_0) = 0$; $r_0 > 0$
 $\xi(r_0) = \xi(r_0) = \phi(r_0) = 1$; $r_0 < 0$

Conservation Law :-

$u_t + uu_x = 0$; $t \geq 0, x \in \mathbb{R}$
 $u(x,0) = \phi(x)$

where $\phi(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$

Parametrize the data curve $c(r) = (r, 0, \phi(r)) \subset G(u)$
 If one parametrizes the integral curve by $q(s)$.

Fix $r > 0$, Char. Eqns are given by

$x'(r,s) = z$; $x(r,0) = r$
 $t'(r,s) = 1$; $t(r,0) = 0$
 $z'(r,s) = 0$; $z(r,0) = \phi(r)$

$u_t + [f(u)]_x = 0$
 $u(x,0) = \phi(x)$
 $f = u^2/2$

$\xi(r,s) := u(x(r,s), t(r,s))$

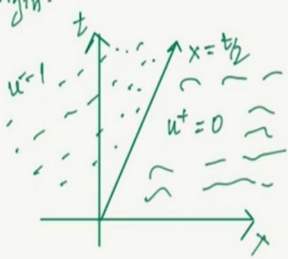
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Also we want our curve to pass through the origin.

Now $x = \xi(t)$ is the required curve.

$\sigma = \xi'(t) = 1/2 \Rightarrow \xi(t) = t/2 \Rightarrow x = t/2.$

Our weak solution is given by

$$u(x,t) = \begin{cases} 1, & x < t/2 \\ 0, & x > t/2 \end{cases}$$


- It is classical solution on each side of the curve
- The solution is bounded.
- It satisfies RH condition.

Projected char. curves are given by $x = \phi(r)t + r$. ✓

and the solution by $u(x,t) = \phi(x-ut)$. ✓

For $r < 0$, $\phi(r) = 1 \Rightarrow$ projected char. $x = t + r$; $r < 0$

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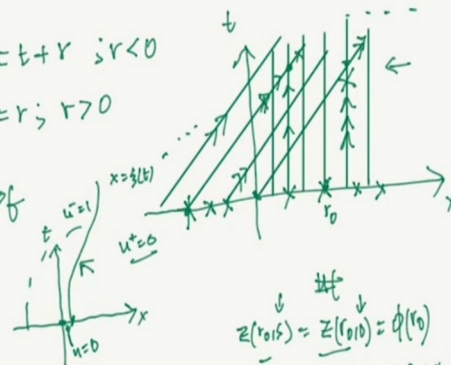
\therefore We don't have any continuous solution of the problem.

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Find the curve $x = \xi(t)$ s.t. $u^- = 1, u^+ = 0$

R-H condition: $[f(u)] = \sigma [u]$

$$\Rightarrow \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \sigma \Rightarrow \sigma = \frac{1}{2}$$



$$\begin{aligned} z(\tau_0/\delta) &= z(\tau_0/0) = \phi(\tau_0) \\ &= 0; \tau_0 > 0 \\ z(\tau_0/\delta) &= z(\tau_0/\delta) = \phi(\tau_0) = 1; \tau_0 < 0 \end{aligned}$$

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Ex 2:- $u_t + uu_x = 0$
 $u(x,0) = \phi(x)$ } - (ii)

IC:- $\phi(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

Projected Char Curves are given by $x = \phi(v)t + r$

\therefore If $r < 0$; $x = r$.

And if $r > 0$; $x = t + r$.

\therefore In R , we don't have any projected ch. curve hence we don't have any information on the solution.

Q:- How to deal with R ?

The graph shows a coordinate system with x and t axes. A vertical line at $x=0$ is labeled $x=0$. A region R is indicated by a shaded area in the upper right quadrant, bounded by the t -axis and a curve. The region R is marked with "??". The t -axis is labeled t and the x -axis is labeled x . A point $x=0$ is marked on the x -axis. The region R is bounded by the t -axis and a curve that starts at the origin and goes up and to the right. The region R is shaded and labeled R . The t -axis is labeled t and the x -axis is labeled x . A point $x=0$ is marked on the x -axis. The region R is bounded by the t -axis and a curve that starts at the origin and goes up and to the right. The region R is shaded and labeled R .

Now, we are exploring more on this Burger's equation. So, essentially, let us take another example and see that, how to deal with this sort of problems.

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Let $x = \xi(t)$ be a curve st either side of curve the
 soln is classical
 $u^- = 0 ; u^+ = 1.$
 $\sigma = \xi'(t) = \frac{u^+ + u^-}{2} = \frac{1}{2}$
 $\Rightarrow x = t/2$

One possible solution is given by

$$u_1(x,t) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2. \end{cases}$$

the other possibility

$$u_2(x,t) = \begin{cases} 0, & x \leq 0 \\ x/t, & 0 \leq x \leq t \\ 1, & x \geq t \end{cases}$$

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Remark: Solution like this "which fills the void region" $\{0 < x < t\}$ is called a rarefaction wave.

o We want a physically realistic solution.

Entropy Condition: $f \in C^1(\mathbb{R}, \mathbb{R})$

$$u_t + [f(u)]_x = 0 \Rightarrow u_t + f'(u) u_x = 0$$

For $x > 0$:

$$\begin{cases} x'(t) = f'(z) \\ t'(t) = 1 \\ z'(t) = 0 \end{cases} \Rightarrow \frac{dx}{dt} = \frac{dx/ds}{dt/ds} = \frac{f'(z)}{1} = f'(u)$$

\therefore the speed of the solution 'u' is given by $f'(u)$.

Let $x = \xi(t)$ be a curve st either side of curve the soln is classical

$$u^- = 0 \quad ; \quad u^+ = 1.$$

$$t = \xi'(t) = \frac{u^+ + u^-}{2} = \frac{1}{2}$$

$$\Rightarrow x = t/2$$

One possible solution is given by

$$u_1(x,t) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2. \end{cases}$$

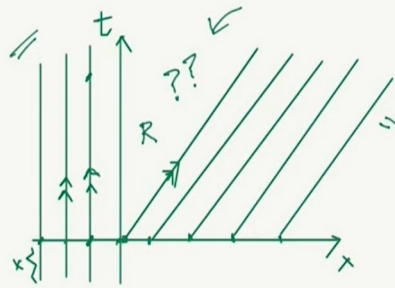
the other possibility

$$u_2(x,t) = \begin{cases} 0, & x \leq 0 \\ x/t, & 0 \leq x \leq t \\ 1, & x \geq t \end{cases}$$

Ex 2:
$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases} \quad \text{--- (11)}$$

"u" represent the height of the wave

IC:
$$\phi(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$



Projected Char Curves are given by $x = \phi(t)t + r$

\therefore If $r < 0$; $x = r$.

and if $r > 0$; $x = t+r$.

\therefore In R, we don't have any projected ch. curve, hence we don't have any information on the solution.

Q: How to deal with R?

Conservation Law :-

$$\begin{cases} u_t + [f(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases} \quad \text{--- (1)}$$

$$\begin{cases} u_t + [f(u)]_x = 0 \\ u(x,0) = \phi(x) \\ f = \frac{1}{2}u^2 \end{cases}$$

where
$$\phi(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

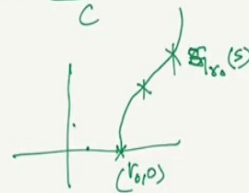
Parametrize the data curve $c(r) = (r, 0, \phi(r)) \in \text{Gr}(u)$

If one parametrize the integral curve by $q(s)$.

Fix $r > 0$, Char. Eqn are given by

$$\begin{cases} x'(r,s) = z & ; & x(r,0) = r \\ t'(r,s) = 1 & ; & t(r,0) = 0 \\ z'(r,s) = 0 & ; & z(r,0) = \phi(r) \end{cases} \quad \text{--- (1)}$$

$$\frac{dx}{dt} = u$$



$$z(r,s) := u(x(r,s), t(r,s))$$

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∴ In Burger's Eqn $\frac{dx}{dt} = u$. Hence the bigger the wave the faster it is.

In Ex 1:- Initial waves was taller on the left.
 ↓
 The wave is moving faster than the wave on the right.

"We want to allow for a curve of discontinuity if the wave moving from left is faster than the wave moving from right."

Remark: Solution like this "which fills the void region" $\{0 < x < t\}$ is called a rarefaction wave.

o We want a physically realistic solution.

Entropy Condition:- $f \in C^1(\mathbb{R}, \mathbb{R})$

$$u_t + [f(u)]_x = 0 \Rightarrow u_t + f'(u) u_x = 0$$

For $\gamma \geq 0$:

$$\begin{cases} x'(r/s) = f(\xi) \\ t'(r/s) = 1 \\ z'(r/s) = 0 \end{cases} \Rightarrow \frac{dx}{dt} = \frac{dx/ds}{dt/ds} = \frac{f(\xi)}{1} = f'(u)$$

∴ the speed of the solution 'u' is given by $f'(u)$.

Let $x = \xi(t)$ be a curve st either side of curve line

Soln is classical

$$u^- = 0 \quad ; \quad u^+ = 1.$$

$$t = \xi(t) = \frac{u^+ + u^-}{2} = \frac{1}{2}$$

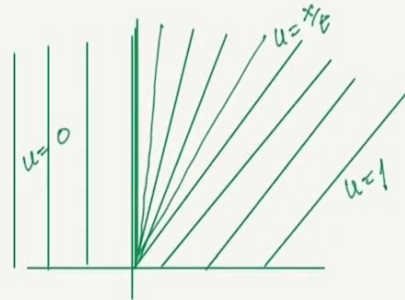
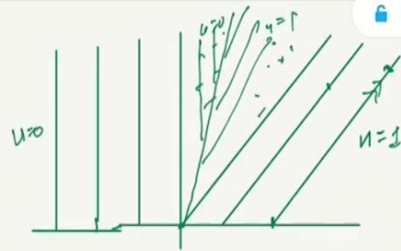
$$\Rightarrow x = t/2$$

One possible solution is given by

$$u_1(x,t) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

The other possibility

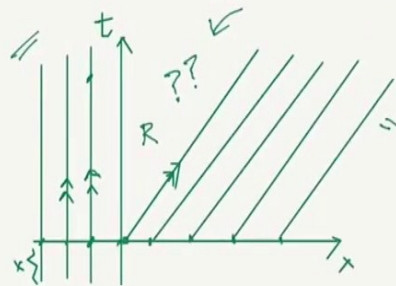
$$u_2(x,t) = \begin{cases} 0, & x \leq 0 \\ x/t, & 0 \leq x \leq t \\ 1, & x \geq t \end{cases}$$



Ex 2: $u_t + uu_x = 0$ (11)
 $u(x,0) = \phi(x)$

$$\phi(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

"u" represent the height of the wave.



Projected Char Curves are given by $x = \phi(0)t + r$

\therefore If $r < 0$; $x = r$.

and if $r > 0$; $x = t + r$.

\therefore In R, we don't have any projected ch-curve hence we don't have any information on the solution.

Q: How to deal with R?

Speed of each 'u' is $\frac{dx}{dt} = u$

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Entropy Condition $x = \xi(t)$ is the curve of discontinuity then

$$f'(u^-) > \sigma = \xi'(t) > f'(u^+)$$

And the curve of discontinuity is called Shock Curve for a solution 'u' if it satisfies the Rankine-Hugoniot condition and the entropy condition.

Ex 38 In ex 2 :

$$f'(u_i^-) = u_i^- = 0$$

$$f'(u_i^+) = u_i^+ = 1.$$

Hence Entropy condition is not satisfied

$x = t/2$ is not a shock curve.

bcz: Entropy condition is violated

Let $x = \xi(t)$ be a curve st either side of curve the

sols is classical

$$u^- = 0 \quad ; \quad u^+ = 1.$$

$$\sigma = \xi'(t) = \frac{u^+ + u^-}{2} = \frac{1}{2}$$

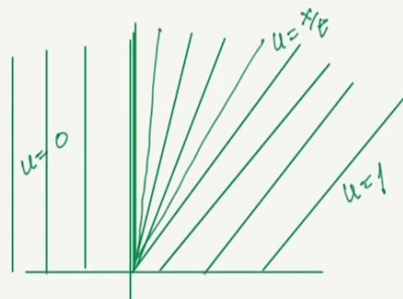
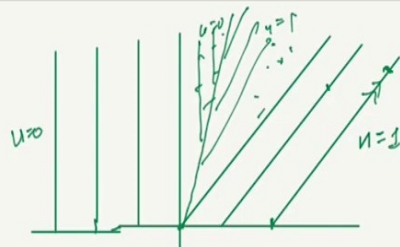
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And moreover, you see, u^2 which you are getting is a continuous function, u^2 is a continuous function, this where is u^2 , this is a continuous function and you can say that this satisfies the entropic condition and hence this is a physically relevant condition and an acceptable condition. So, with this, we are going to end this part.