Advanced Partial Differential Equations
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Indian Institute of Technology, Kanpur
Lecture – 31
Rankine-Hugonoit Jump Condition

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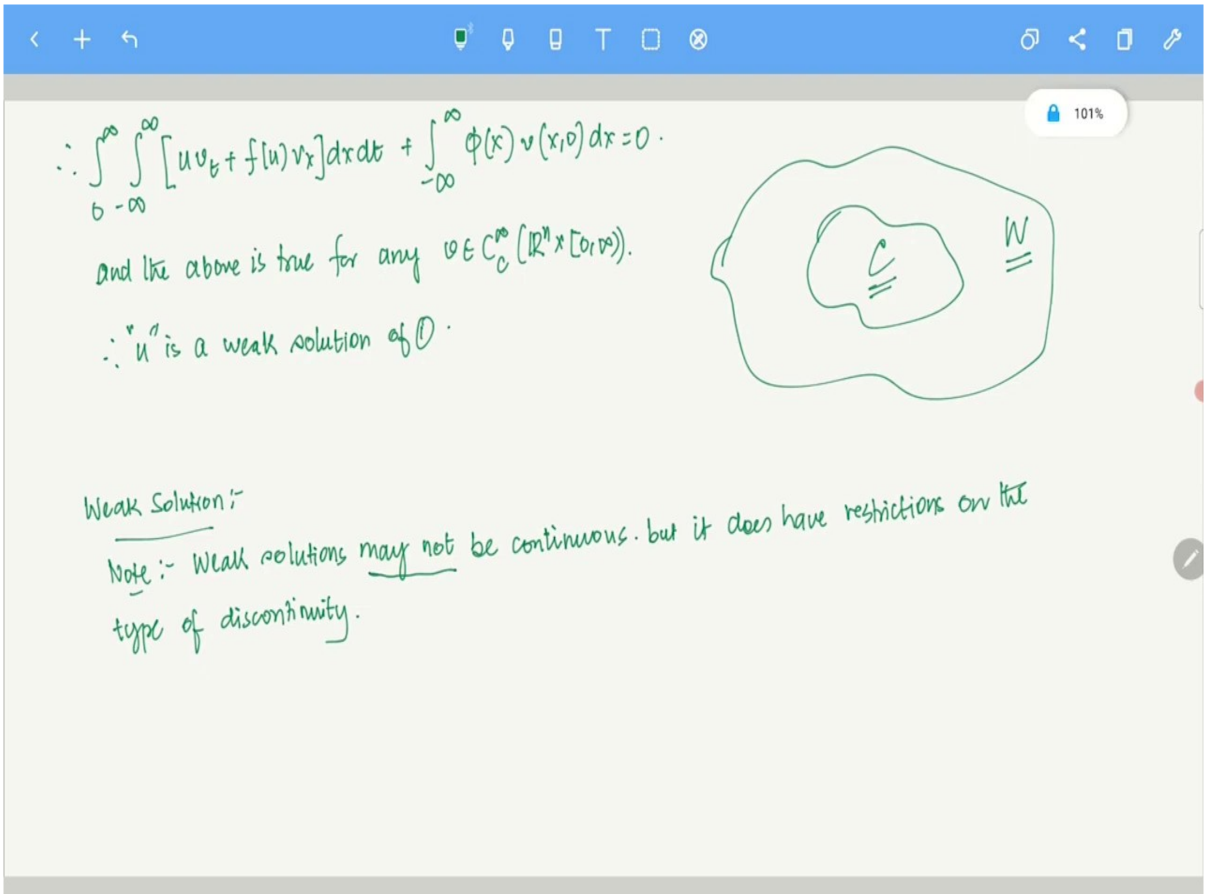

So, welcome students and in this video we are going to continue our study of weak solutions. And the basically the as we have seen that weak solutions may not be continuous. So, let me start with a weak solution.

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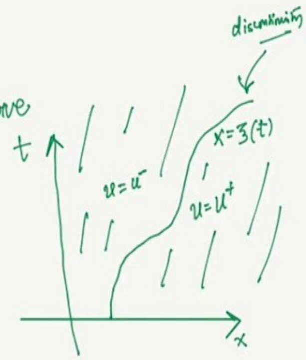
$$\left.\begin{aligned} &u_t + [f(u)]_x = 0 \quad ; x \in \mathbb{R}, t > 0 \\ &u(x,0) = \phi(x) \end{aligned}\right\} \quad -(1)$$

Let 'u' is a weak solution of (1) such that 'u' is discontinuous across some curve $x = \xi(t)$, but 'u' is smooth on either side of the curve



Define, $u^-(x,t)$ = limit of 'u' approaching (x,t) from the left

$u^+(x,t)$ = limit of 'u' approaching (x,t) from the right.

Claim: $x = \xi(t)$ cannot be arbitrary.

Th: If 'u' is a weak solution of (1) such that 'u' is discontinuous across the curve $x = \xi(t)$ but 'u' is smooth on either side of $x = \xi(t)$, then u must satisfy the condition.

$$\frac{f(u^-) - f(u^+)}{u^- - u^+} = \xi'(t)$$

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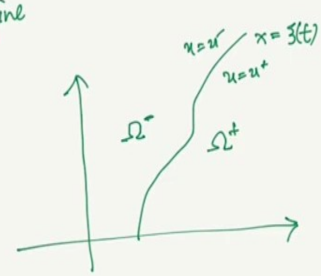
across the curve of discontinuity.

Proof:- If u is a weak soln of (1) then

$$\int_0^\infty \int_{-\infty}^\infty [u v_t + f(u) v_x]\, dx\, dt + \int_{-\infty}^\infty \phi(x) v(x,0)\, dx = 0. \quad \forall\, v \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$$

Let, v be a smooth function with $v(x,0) = 0$. and define



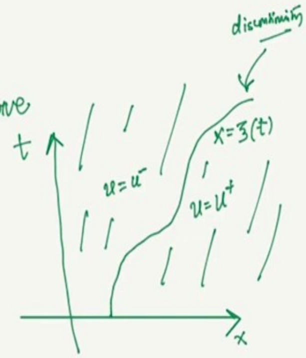
$$\Omega^- \equiv \{(x,t) : 0 < t < \infty ; -\infty < x < \infty\}$$

$$\Omega^+ \equiv \{(x,t) : 0 < t < \infty ; \xi(t) < x < +\infty\}$$

$$\therefore \int_0^\infty \int_{-\infty}^\infty [u v_t + f(u) v_x]\, dx\, dt = \int_{\Omega^-} [u v_t + f(u) v_x]\, dx\, dt + \int_{\Omega^+} [u v_t + f(u) v_x]\, dx\, dt = 0.$$

$$\left.\begin{array}{l} u_t + [f(u)]_x = 0 \quad ; x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{array}\right\} \quad -(1)$$

Let 'u' is a weak solution of (1) such that 'u' is discontinuous across some curve $x = \xi(t)$, but 'u' is smooth on either side of the curve



Define, $u^-(x,t)$ = limit of 'u' approaching (x,t) from the left

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$$\frac{f(u^-) - f(u^+)}{u^- - u^+} = \xi'(t)$$

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$$\iint_{\Omega^-} \left[u v_t + f(u) v_x\right] dx\,dt = -\iint_{\Omega^-} \left\{u_t + [f(u)]_x\right\} v \, dx\,dt + \int_{x=\xi(t)} \left[u^- v \nu_2 + f(u^-) v \nu_1\right] ds \quad \text{— (II)}$$

where $\nu = (\nu_1, \nu_2)$ are outward unit normal to Ω^-.

Ily,

$$\iint_{\Omega^+} \left[u v_t + f(u) v_x\right] dx\,dt = -\iint_{\Omega^+} \left[u_t + [f(u)]_x\right] v \, dx\,dt - \int_{x=\xi(t)} \left[u^+ v \nu_2 + f(u^+) v \nu_1\right] ds \quad \text{— (III)}$$

$\because$ 'u' is smooth on Ω^- and Ω^+, hence u is a classical solution in Ω^- & Ω^+.

From (II),

$$\iint_{\Omega^-} \left[u v_t + f(u) v_x\right] dx\,dt = \int_{x=\xi(t)} \left[u^- v \nu_2 + f(u^-) v \nu_1\right] ds \quad \text{— (IV)}$$

and,

$$\iint_{\Omega^+} \left[u v_t + f(u) v_x\right] dx\,dt = -\int_{x=\xi(t)} \left[u^+ v \nu_2 + f(u^+) v \nu_2\right] ds \quad \text{— (V)}$$

across the curve of discontinuity.

Proof: If u is a weak soln of (1) then

$$\int_0^\infty \int_{-\infty}^\infty \left[u v_t + f(u) v_x \right] dx\,dt + \int_{-\infty}^\infty \phi(x)\, v(x,0)\, dx = 0. \quad \forall\, v \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$$

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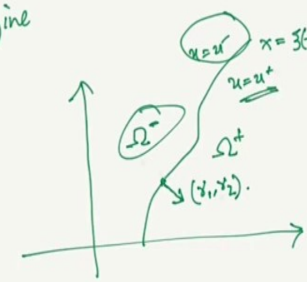

$$\Omega^- \equiv \{(x,t) : 0<t<\infty ; -\infty < x < \infty\}$$

$$\Omega^+ \equiv \{(x,t) : 0<t<\infty ; \ \xi(t) < x < +\infty\}$$

$$\therefore \int_0^\infty \int_{-\infty}^\infty \left[u v_t + f(u) v_x \right] dx\,dt = \iint_{\Omega^-} \left[u v_t + f(u) v_x \right] dx\,dt + \iint_{\Omega^+} \left[u v_t + f(u) v_x \right] dx\,dt = 0.$$

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$\because$ This is true for any smooth function 'v', we have

$$u^- \gamma_2 + f(u^-)\gamma_1 = u^+ \gamma_2 + f(u^+)\gamma_1 \checkmark$$

$$\left[\int_{x=\xi(t)} [u^- v \gamma_2 + f(u^-) v \gamma_1] ds - \int_{x=\xi(t)} [u^+ v \gamma_2 + f(u^+) v \gamma_1] ds = 0. \right]$$

$$\Rightarrow \frac{f(u^+) - f(u^-)}{u^+ - u^-} = -\frac{\gamma_2}{\gamma_1}.$$

Again, $\xi'(t) = -\dfrac{\gamma_2}{\gamma_1} = \dfrac{f(u^-) - f(u^+)}{u^- - u^+}$

$\therefore$ The soln 'u' has a discontinuity along a curve $x = \xi(t)$, the solution 'u' on the curve $x = \xi(t)$ must satisfy

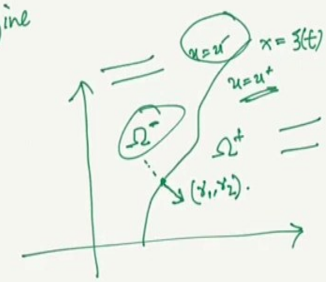
$$\xi'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+}.$$

acron the curve of discontinuity.

Proof :- If u is a weak soln of (1) then

$$\int_0^\infty \int_{-\infty}^{\infty} \left[u v_t + f(u) v_x \right] dx\,dt + \int_{-\infty}^{\infty} \phi(x)\, v(x,0)\, dx = 0 \quad \forall\; v \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$$

Let, v be a smooth function with $v(x,0) = 0$ and define



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$$\therefore \int_0^\infty \int_{-\infty}^{\infty} [u v_t + f(u) v_x]\, dx\,dt = \iint_{\Omega^-} [u v_t + f(u) v_x]\, dx\,dt + \iint_{\Omega^+} [u v_t + f(u) v_x]\, dx\,dt = 0. \quad \text{— } (*)$$

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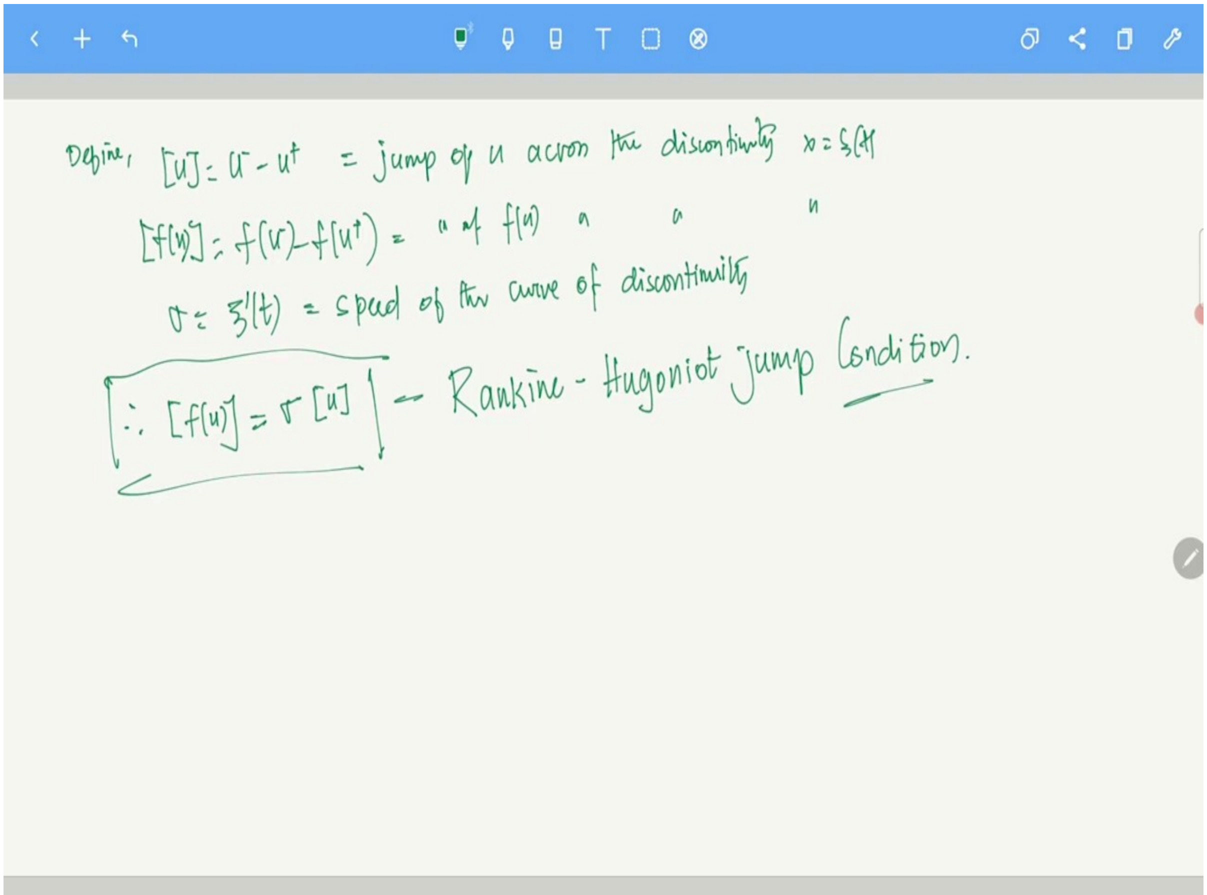

This relation this is called the Rankine-Hugoniot jump condition. Rankine-Hugoniot jump condition. So, with this we are going to end this video in the next video we are going to take up in problem and I am going to show you how we can use this condition to find solutions.