

Advanced Partial Differential Equations
Professor. Dr. Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Lecture No. 30
Conservation Law: Weak Solution

(Refer Slide Time: 00:13)

Conservation Law :

$$u_t + [f(u)]_x = 0 \quad ; x \in \mathbb{R}, t > 0. \quad \text{--- (1)}$$

$$u(x,0) = \phi(x)$$

Assume, f is smooth from \mathbb{R} to \mathbb{R} .
 Let, $f(u) = \frac{u^2}{2}$. then (1) becomes

$$\left\{ \begin{array}{l} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{array} \right\} \text{--- (1)} \quad \text{Quasilinear 1st order PDE}$$

(1) is called the Burgers' Equation. and we showed via the method of characteristics that (1) admits a solution (not global).
 $\sim \phi(x) = x$, the char of (1) intersect. and the points where they intersect, one can no longer define a solution.

Welcome students. In this class we are going to talk about conservation law. Now, what is conservation law and why is it so important? Essentially, we will be talking about something which is in mathematically speaking it is a first order equation.

Now, this is of course a first ordered equation because of highest degree term is like a only 1. And see here if we can assume for now that f is smooth.

(Refer Slide Time: 06:51)

By a Classical solution of $\textcircled{1}$,

$u \in C^1(\mathbb{R}^n \times [0, \infty))$ st u satisfies $\textcircled{1}$ everywhere in $\mathbb{R}^n \times [0, \infty)$
 ($u, u_x, u_t \in C$)


Aim: To define the notion of weak solution.

Definition: " u " is said to be a weak solution of $\textcircled{1}$ if

$$\int_0^\infty \int_{-\infty}^\infty [u v_t + f(u) v_x] dx dt + \int_{-\infty}^\infty \phi(x) v(x, 0) dx = 0 \quad \text{--- (iii)}$$

holds for all $v \in C_c^\infty(\mathbb{R}^n \times [0, \infty))$

Here we assume, $u \in L^\infty(\mathbb{R}^n \times [0, \infty))$. (Bounded and summable).



(Refer Slide Time: 16:01)

Th: If u is a classical solution of \mathcal{D} , then u is a weak solution of \mathcal{D} .

Proof: $u \in C^1$ and

$$u_t(x,t) + [f(u(x,t))]_x = 0, \quad \forall x \in \mathbb{R}, t > 0 \quad (1)$$

$$u(x,0) = \phi(x)$$

|||

$$\begin{cases} u_t(x,t) + f'(u(x,t)) u_x(x,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Let $v \in C_c^\infty(\mathbb{R} \times [0, \infty))$ ($v \equiv 0$ outside some compact set in $\mathbb{R} \times [0, \infty)$)

$$\int_{-\infty}^{\infty} \int_0^{\infty} \{ u_t + [f(u)]_x \} v \, dx \, dt = 0 \quad \text{I.B.P} \Rightarrow - \int_{-\infty}^{\infty} \int_0^{\infty} u v_t \, dt \, dx + \int_{-\infty}^{\infty} u v \Big|_0^{\infty} \, dx$$

$$- \int_0^{\infty} \int_{-\infty}^{\infty} f(u) v_x \, dx \, dt + \int_0^{\infty} f(u) v \Big|_{-\infty}^{\infty} \, dt = 0$$

$\left(\begin{array}{l} \int_{-\infty}^{\infty} u(x,t) v(x,t) \Big|_{t=0}^{t=\infty} \\ - u v \Big|_{t=0} \\ u(x,0) \end{array} \right)$

(Refer Slide Time: 25:20)

The image shows a digital whiteboard with a blue header bar containing navigation icons. The main content is handwritten in green ink. It starts with a mathematical equation:
$$\therefore \int_0^{\infty} \int_{-\infty}^{\infty} [u u_t + f(u) v_x] dx dt + \int_{-\infty}^{\infty} \phi(x) v(x, 0) dx = 0.$$
 Below this, it says: "and the above is true for any $v \in C_c^{\infty}(\mathbb{R}^n \times [0, \infty))$." The final line states: " $\therefore u$ is a weak solution of \mathcal{D} ." To the right of the text is a hand-drawn diagram of a domain \mathcal{D} . It consists of an outer irregular boundary labeled \mathcal{D} and an inner irregular boundary labeled C . The region between the two boundaries is labeled W .

So, this is a much bigger class of function. Why bigger class? Because, we have seen that let us say any classical solution let us say this is the classical solution this is weak solution any classical solution is a weak solution. So, that is why it is a much bigger. So, let us end it here.