Advanced Partial Differential Equations
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Lecture No. 30
Conservation Law: Weak Solution

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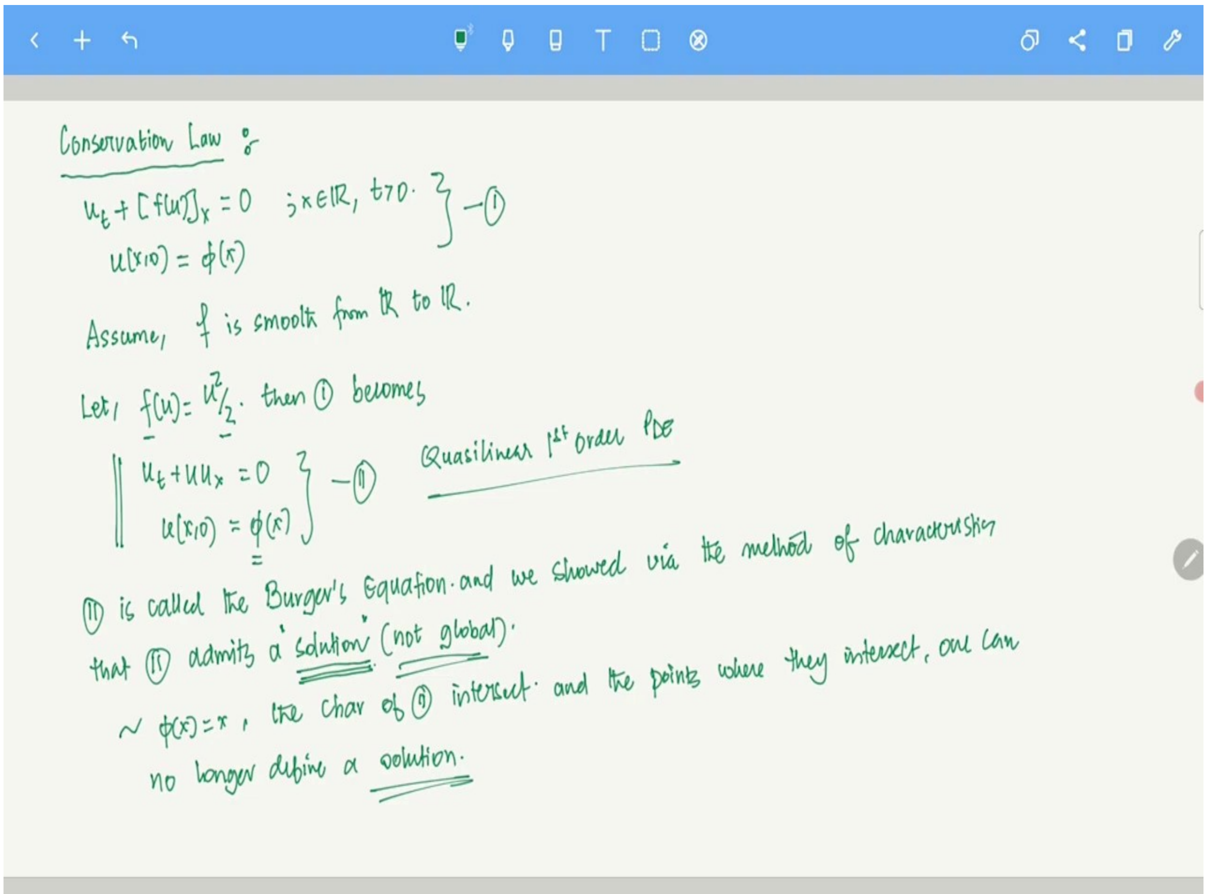

Welcome students. In this class we are going to talk about conservation law. Now, what is conservation law and why is it so important? Essentially, we will be talking about something which is in mathematically speaking it is a first order equation.

Now, this is of course a first ordered equation because of highest degree term is like a only 1. And see here if we can assume for now that f is smooth.

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By a (Classical) solution of (I),

$u \in C^1(\mathbb{R}\times[0,\infty))$ st u satisfies (I) everywhere in $\mathbb{R}\times[0,\infty)$

$(u, u_x, u_t \in C)$

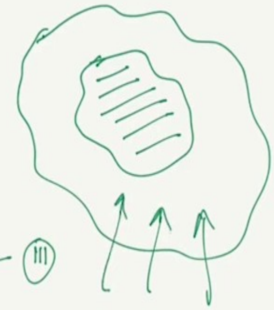
AIM: To define the notion of weak solution.

Definition: "u" is said to be a weak solution of (I) if

$$\int_0^\infty\int_{-\infty}^{\infty} [u v_t + f(u) v_x]\,dx\,dt + \int_{-\infty}^{\infty} \phi(x)\, v(x,0)\,dx = 0 \quad \text{— (III)}$$

holds for all $v \in C_c^\infty(\mathbb{R}\times[0,\infty))$

Here we assume, $u \in L^\infty(\mathbb{R}^n\times[0,\infty))$. (Bounded and summable).



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Th: If u is a classical solution of (1), then u is a weak solution of (1).

Proof: $u \in C^1$ and

$$\left.\begin{aligned} &u_t(x,t) + [f(u(x,t))]_x = 0, \quad t > 0,\ x \in \mathbb{R} \\ &u(x,0) = \phi(x) \end{aligned}\right\} - (iv)$$

$$\vert\vert\vert$$

$$\left[\begin{aligned} &u_t(x,t) + f'(u(x,t))\, u_x(x,t) = 0 \\ &u(x,0) = \phi(x) \end{aligned}\right.$$

Let, $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$ ($v \equiv 0$ outside some compact set in $\mathbb{R} \times [0,\infty)$)

$$\int_0^\infty \int_{-\infty}^\infty \left\{u_t + [f(u)]_x\right\} v \, dx\, dt = 0 \overset{I.B.P}{\Longrightarrow} - \int_{-\infty}^\infty \int_0^\infty u v_t \, dt\, dx + \int_{-\infty}^\infty u v \Big|_0^\infty dx$$

$$- \int_0^\infty \int_{-\infty}^\infty f(u) v_x \, dx\, dt + \int_0^\infty \cancel{f(u) v \Big|_{-\infty}^\infty} dt = 0$$

$$\left(f(u(x,t))\, \underset{\underset{0}{\parallel}}{v(x,t)} \Big|_{x=-\infty}^{x=+\infty} \right)$$

$$\left|\begin{aligned} &\overset{\overset{0}{\downarrow}}{u(x,t) v(x,t)} \Big|_{t=0}^{t=\infty} \\ &- u v \Big|_{t=0} \\ &\ \ u(x,0) \end{aligned}\right.$$

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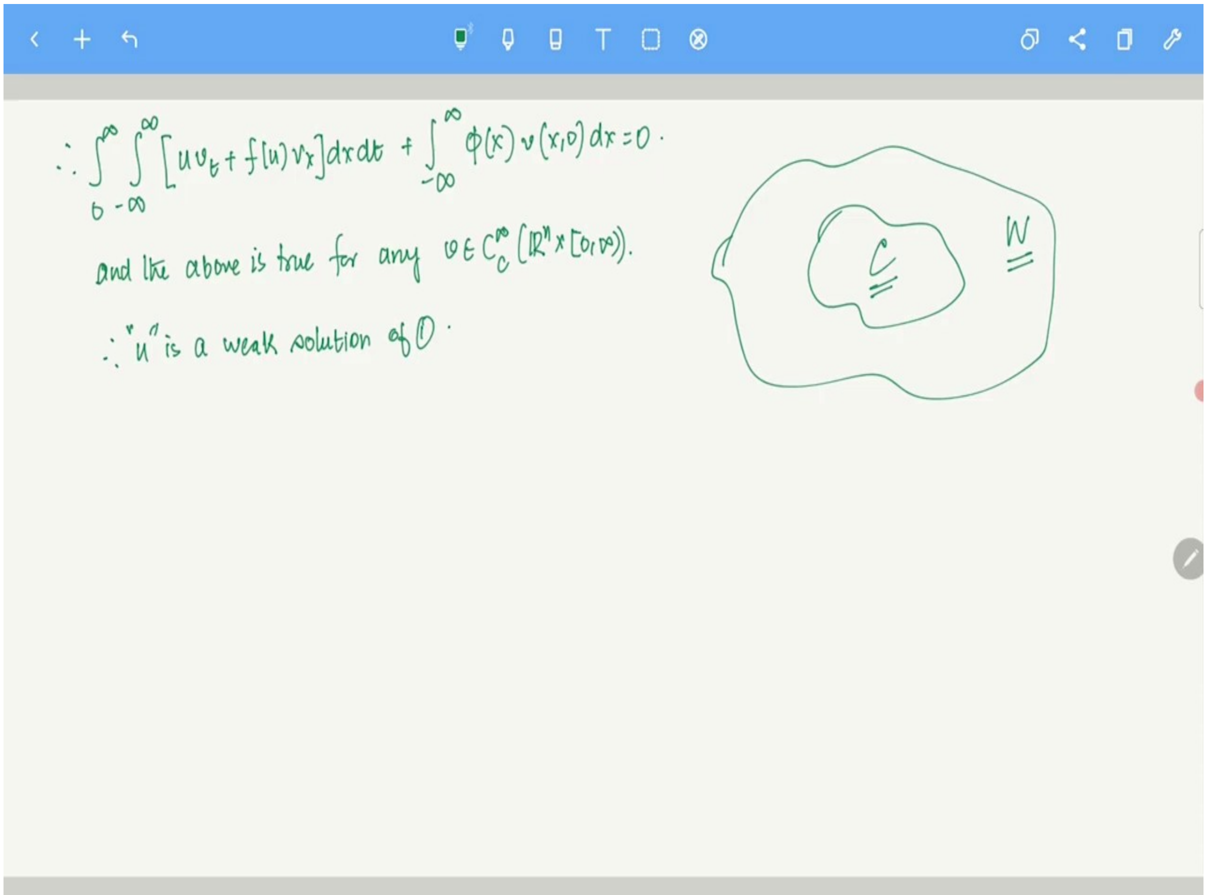

So, this is a much bigger class of function. Why bigger class? Because, we have seen that let us say any classical solution let us say this is the classical solution this is weak solution any classical solution is a weak solution. So, that is why it is a much bigger. So, let us end it here.