

**Advanced Partial Differential Equations**  
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**Lecture: 29**

**Wave Equation in Even Dimensions and Speed of Propagation**

(Refer Slide Time: 00:12)

Wave Equation in  $n=2m; m=1,2,\dots$  :-

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

$$u(x,0) = g(x); u_t(x,0) = h(x) \text{ in } \mathbb{R}^n \times \{t=0\} \quad \text{--- (1)}$$

Assumption:  $n$  is an even integer. let  $u \in C^m$  is the solution of (1) for  $n = \frac{n+2}{2}$

Set,  $\bar{u}(x_1, x_2, \dots, x_{n+1}, t) := u(x_1, x_2, \dots, x_n, t)$  in  $\mathbb{R}^{n+1} \times (0, \infty)$ .

$$\bar{u} = \bar{g} \text{ and } \bar{u}_t = \bar{h} \text{ on } \mathbb{R}^{n+1} \times \{t=0\}$$

where,  $\bar{g}(x_1, \dots, x_{n+1}) := g(x_1, x_2, \dots, x_n)$ .

$$\bar{h}(x_1, x_2, \dots, x_{n+1}) := h(x_1, x_2, \dots, x_n)$$

Fix  $x \in \mathbb{R}^n, t > 0$  and  $\bar{x} = (x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1}$ .

$$u(x,t) = \frac{1}{\omega_{n+1}} \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^{n+1} \int_{\partial \bar{B}(\bar{x}, t)} \bar{g} dS \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^{n+1} \int_{\partial \bar{B}(\bar{x}, t)} \bar{h} dS \right) \right]$$

$B(\bar{x}, t) =$  Ball in  $\mathbb{R}^{n+1}$  with centre  $\bar{x}$  & radius 't';  $dS =$  Surface measure on  $\partial \bar{B}(\bar{x}, t)$

*Handwritten notes on the right side of the slide:*

- $n=1$ , D'Alembert
- $n=3$ , Kirchoff's
- $n=2$ , Poisson Formula
- $n=2m+1; m=1,2,\dots$  (odd)
- $\bar{u}(x_1, x_2, \dots, x_{n+1}, t) = u(x_1, x_2, \dots, x_n, t)$
- Hadamard

Welcome students in this class, we are going to talk about the wave equation but in even dimension so higher even dimension.

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$$\int_{\partial B(x,t)} \bar{g} d\bar{S} = \frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial B(x,t)} \bar{g} d\bar{S} \quad [\alpha(n+1) = \text{volume of the unit ball in } \mathbb{R}^{n+1}]$$

$$= \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} g(y) (1 + |\nabla \varphi(y)|^2)^{1/2} dy$$

where:  $g(y) := (t^2 - |y-x|^2)^{1/2}$   
 $\partial B(x,t) \cap \{y_{n+1} > 0\}$   
 $[1 + |\nabla \varphi(y)|^2]^{1/2} = t (t^2 - |y-x|^2)^{-1/2}$

$$\Rightarrow \int_{\partial B(x,t)} \bar{g} d\bar{S} = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} dy$$

$$= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} dy$$

$$\therefore u(x,t) = \frac{1}{\gamma_{n+1}} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} dy \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \int_{B(x,t)} \frac{h(y)}{(t^2 - |y-x|^2)^{1/2}} dy \right) \right]$$

where  $\gamma_{n+1} = 1 \cdot 3 \cdot 5 \cdots (n-1)$  and  $\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}$ .

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For even 'n',

$$u(x,t) = \frac{1}{\gamma_n} \left[ \left( \frac{\partial}{\partial t} \right) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{n}{2}}} dy \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \int_{B(x,t)} \frac{h(y)}{(t^2 - |y-x|^2)^{\frac{n}{2}}} dy \right) \right]$$

and  $\gamma_n = 2 \cdot 4 \cdot \dots \cdot (n-2) \cdot n$

Th%: Assume n is even integer  $\geq 2$  and  $g \in C^{m+1}(\mathbb{R}^n)$ ,  $h \in C^m(\mathbb{R}^n)$  for  $m = \frac{n+2}{2}$ .

If 'u' is defined by (\*). Then

- (i)  $u \in C^2(\mathbb{R}^n \times [0, \infty))$
- (ii)  $u_{tt} - \Delta u = 0$  in  $\mathbb{R}^n \times (0, \infty)$
- (iii)  $\lim_{(x,t) \rightarrow (x^0, 0)} u(x,t) = g(x^0)$ ;  $\lim_{\substack{(x,t) \rightarrow (x^0, 0) \\ x \in \mathbb{R}^n, t > 0}} u_t(x,t) = h(x^0)$

Proof: Check Yourself

$$\int_{\partial B(x,t)} g dS = \frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial B(x,t)} g dS$$

[ $\alpha(n+1)$  = volume of the unit ball in  $\mathbb{R}^{n+1}$ ]

$$= \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} g(y) (1 + |v \cdot x(y)|^2)^{\frac{n}{2}} dy$$

where:  $g(y) = (t^2 - |y-x|^2)^{\frac{n}{2}}$

$$\Rightarrow \int_{\partial B(x,t)} g dS = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{n}{2}}} dy$$

$\int_{\partial B(x,t) \cap \{y_{n+1} \geq 0\}} [1 + |v \cdot x(y)|^2]^{\frac{n}{2}} = t (t^2 - |y-x|^2)^{\frac{n}{2}}$

$$= \frac{2\alpha(n)}{(n+1)\alpha(n+1)} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{n}{2}}} dy$$

$$\therefore u(x,t) = \frac{1}{\gamma_n} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{n}{2}}} dy \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \int_{B(x,t)} \frac{h(y)}{(t^2 - |y-x|^2)^{\frac{n}{2}}} dy \right) \right]$$

where  $\gamma_{n+1} = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-1)$  and  $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})}$

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Wave Equation in  $n=2m; m=1,2,\dots$  :-

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

$$u(x,0) = g(x); u_t(x,0) = h(x) \text{ in } \mathbb{R}^n \times \{t=0\} \quad \text{--- (1)}$$

Assumption:  $n$  is an even integer. Let  $u \in C^m$  is the solution of (1) for  $m = \frac{n+2}{2}$

Set,  $\bar{u}(x_1, x_2, \dots, x_{n+1}, t) := u(x_1, x_2, \dots, x_n, t)$  in  $\mathbb{R}^{n+1} \times (0, \infty)$ .

$$\bar{u} = \bar{g} \text{ and } \bar{u}_t = \bar{h} \text{ on } \mathbb{R}^{n+1} \times \{t=0\}$$

where,  $\bar{g}(x_1, \dots, x_{n+1}) := g(x_1, x_2, \dots, x_n)$ .

$$\bar{h}(x_1, x_2, \dots, x_{n+1}) := h(x_1, x_2, \dots, x_n)$$

Fix  $x \in \mathbb{R}^n, t > 0$  and  $\bar{x} = (x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1}$ .

$$u(x,t) = \frac{1}{\omega_{n+1}} \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^{n+1} \int_{\partial B(\bar{x}, t)} \bar{g} ds \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^{n+1} \int_{\partial B(\bar{x}, t)} \bar{h} ds \right) \right]$$

$B(\bar{x}, t) =$  Ball in  $\mathbb{R}^{n+1}$  with centre  $\bar{x}$  & radius 't';  $ds =$  Surface measure on  $\partial B(\bar{x}, t)$

*Notes on the right side of the slide:*

- $n=1$ , D'Alembert
- $n=3$ , Kirchoff's
- $n=2$ , Poisson Formula
- $n=2m+1; m=1,2,\dots$  (odd)
- $\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$
- Hadamard

Huygen's Principle :- ('1671)

- \* If  $n$  is odd  $\geq 3$ , the data  $g$  &  $h$  at  $x \in \mathbb{R}^n$  affects the solution 'u' only on the boundary  $\{(y,t) | t > 0 \text{ and } |x-y|=t\}$  of the cone  $\{(y,t) | t > 0 \text{ and } |x-y| \leq t\} := C$
- \* If  $n$  is even, the data  $g$  &  $h$  affects 'u' within all of 'C'.

You can solve the inhomogeneous problem. How do you solve it? You just use Duhamel's principle using this. Now, we move on to something called so let me do it in a new page.

(Refer Slide Time: 41:53)

Wave Equation admits finite speed of propagation (Evolution Equation)

Defn: A PDE exhibits finite propagation speed if the initial data consists of function with compact support, then for every  $t > 0$  the solution  $u(\cdot, t)$  has a compact support.

Defn: We say the speed of propagation is less than equal to 'c', provided that if the spt of the initial functions is in  $B(a, r)$  for every every  $t > 0$ , the spt of  $u$  will be contained in  $B(a, r+ct)$ .

Remark: Heat Eqn has infinite speed of propagation.  
 o Wave has finite speed of propagation.

$u_{tt} - \Delta u = 0$   
 in  $\Omega \times (0, T)$   
 $\Omega$  is open in  $\mathbb{R}^n$


We move on to something which we did in the heat equation also, here we are going to do something called speed of propagation. So, we are going to show that wave equation, in our setting, wave equation admits finite speed of propagation. Let me explain to you what all of this means, what finite speed of propagation means? First of all, in the mathematical sense of course.

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Finite speed of propagation for wave equation & 'u' satisfies the wave eqn ①

If  $u \equiv u_t \equiv 0$  on  $B(x_0, t_0) \times \{t=0\}$ , then  $u \equiv 0$  within the cone  $K(x_0, t_0) = \{(x, t) / 0 \leq t \leq t_0; |x - x_0| \leq t_0 - t\}$  with apex  $(x_0, t_0)$

Proof of  $E(t) := \frac{1}{2} \int_{B(x_0, t_0-t)} \{u_t^2(x, t) + |\nabla u(x, t)|^2\} dx \quad (0 \leq t \leq t_0)$



$\Rightarrow E'(t) = \int_{B(x_0, t_0-t)} u_t u_{tt} + \nabla u \cdot \nabla u_t - \frac{1}{2} \int_{\partial B(x_0, t_0-t)} (u_t^2 + |\nabla u|^2) dS.$

$= \int_{B(x_0, t_0-t)} u_t (u_{tt} - \Delta u) dx + \int_{\partial B(x_0, t_0-t)} \frac{\partial u}{\partial \nu} u_t dS - \frac{1}{2} \int_{\partial B(x_0, t_0-t)} (u_t^2 + |\nabla u|^2) dS.$

$= \int_{\partial B(x_0, t_0-t)} \left\{ \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \right\} dS$

Wave Equation in  $n=2m$ ;  $m=1,2,\dots$  :-

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

$$u(x,0) = g(x); u_t(x,0) = h(x) \text{ in } \mathbb{R}^n \times \{t=0\}$$

} - (1)

Assumption:  $n$  is an even integer. Let  $u \in C^m$  is the solution of (1) for  $m = \frac{n+2}{2}$

Set,  $\bar{u}(x_1, x_2, \dots, x_{n+1}, t) := u(x_1, x_2, \dots, x_n, t)$  in  $\mathbb{R}^{n+1} \times (0, \infty)$ .

$$\bar{u} = \bar{g} \text{ and } \bar{u}_t = \bar{h} \text{ on } \mathbb{R}^{n+1} \times \{t=0\}$$

$$\text{where, } \bar{g}(x_1, \dots, x_{n+1}) := g(x_1, x_2, \dots, x_n)$$

$$\bar{h}(x_1, x_2, \dots, x_{n+1}) := h(x_1, x_2, \dots, x_n)$$

Fix  $\bar{x} \in \mathbb{R}^n$ ,  $t > 0$  and  $\bar{x} = (x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1}$ .

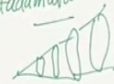
$$u(x,t) = \frac{1}{\omega_{n+1}} \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^{n+1} \int_{\partial \bar{B}(\bar{x}, t)} \bar{g} \, dS \right) + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^{n+1} \int_{\partial \bar{B}(\bar{x}, t)} \bar{h} \, dS \right) \right]$$

$B(\bar{x}, t) = \text{Ball in } \mathbb{R}^{n+1} \text{ with center } \bar{x} \text{ \& radius 't' ; } dS = \text{Surface measure on } \partial B(\bar{x}, t)$

$n=1$ , D'Alembert  
 $n=3$ , Kirchoff's  
 $\downarrow$  N.D  
 $n=2$ , Poisson Formls.  
 $n=2m+1; m=1,2,\dots$  (odd)

$$\bar{u}(x_1, x_2, \dots, x_n, t) = u(x_1, x_2, \dots, x_n, t)$$

Hadamard



(Refer Slide Time: 58:45)

The image shows a digital whiteboard with handwritten mathematical notes. At the top, there is a blue header bar with navigation icons. The main content is written in black ink on a light green background. The derivation starts with an inequality involving the time derivative of the norm of  $u_t$ . It then shows that the energy  $E(t)$  is non-increasing and starts at zero, which implies  $E(t) = 0$  for all  $t$  in the interval  $[0, t_0]$ . This leads to the conclusion that  $u_t = 0$  and  $\nabla u = \hat{0}$ , meaning  $u$  is zero within the cone  $K(x_0, t_0)$ .

$$\begin{aligned} \therefore |\partial_t u_t| &\leq |u_t| |\partial_t \gamma| \\ &\leq \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \quad \text{--- (ii)} \end{aligned}$$
$$\begin{aligned} \therefore E'(t) &\leq 0 \\ \Rightarrow E(t) &\leq E(0) = 0 \quad \forall 0 \leq t \leq t_0 \\ \therefore 0 &\leq E(t) \leq 0 \\ \Rightarrow E(t) &= 0 \quad \forall 0 \leq t \leq t_0 \end{aligned}$$

$\Downarrow$

$$u_t \equiv 0 \quad \wedge \quad \nabla u \equiv \hat{0}$$

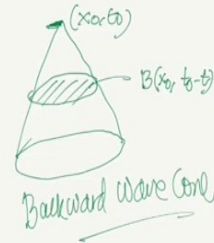
$\Rightarrow u \equiv 0$  within the cone  $K(x_0, t_0)$ .



Finite speed of propagation for wave equation: 'u' satisfies the wave eqn ①

If  $u \equiv u_t \equiv 0$  on  $B(x_0, t_0) \times \{t=0\}$ , then  $u \equiv 0$  within the cone  $K(x_0, t_0) = \{(x, t) \mid 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$  with apex  $(x_0, t_0)$

Proof:  $E(t) := \frac{1}{2} \int_{B(x_0, t_0-t)} \{u_t^2(x, t) + |\nabla u(x, t)|^2\} dx \quad (0 \leq t \leq t_0)$



$$\Rightarrow E'(t) = \int_{B(x_0, t_0-t)} u_t u_{tt} + \nabla u \cdot \nabla u_t - \frac{1}{2} \int_{\partial B(x_0, t_0-t)} (u_t^2 + |\nabla u|^2) dS.$$

$$= \int_{B(x_0, t_0-t)} u_t (u_{tt} - \Delta u) dx + \int_{\partial B(x_0, t_0-t)} \frac{\partial u}{\partial \nu} u_t dS - \frac{1}{2} \int_{\partial B(x_0, t_0-t)} (u_t^2 + |\nabla u|^2) dS.$$

$$= \int_{\partial B(x_0, t_0-t)} \left\{ \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \right\} dS$$