Advanced Partial Differential Equations
Professor Doctor Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Lecture: 29
Wave Equation in Even Dimensions and Speed of Propogatiom

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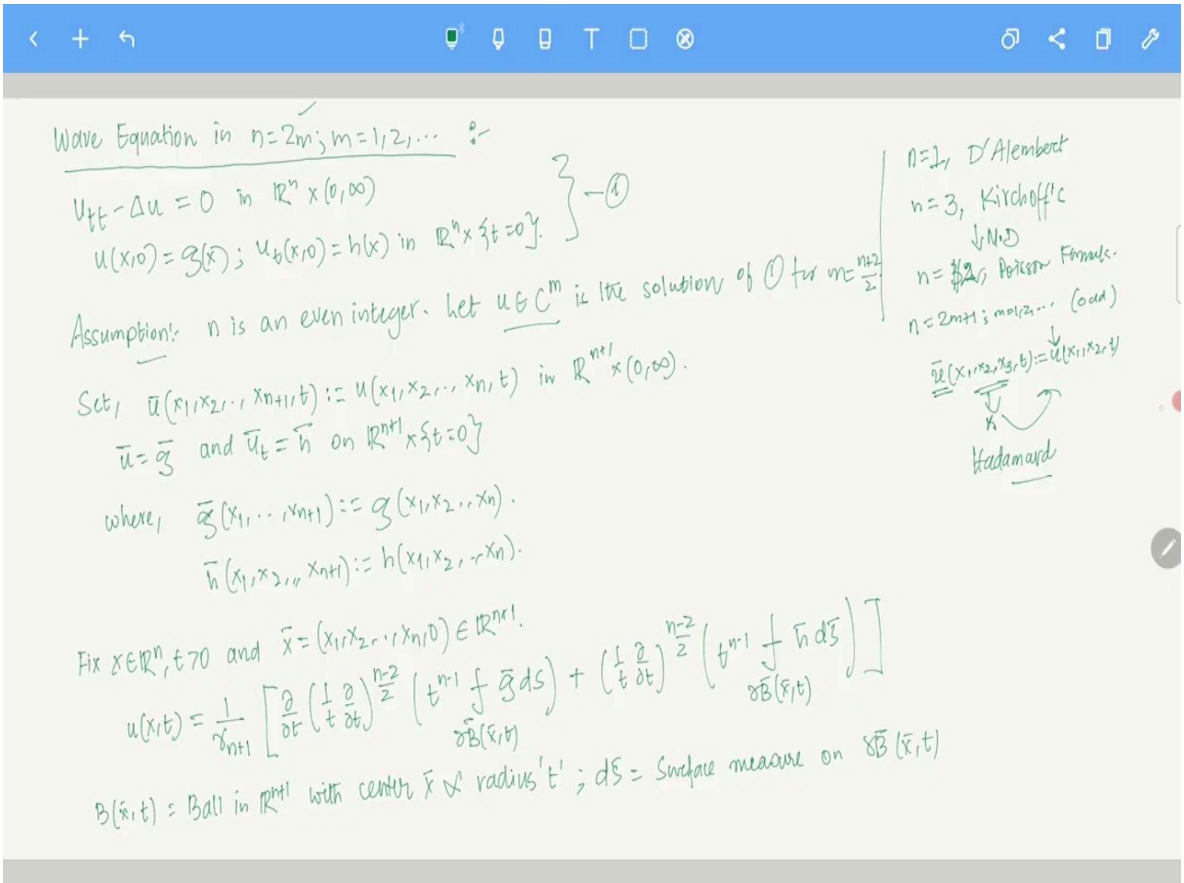

Welcome students in this class, we are going to talk about the wave equation but in even dimension so higher even dimension.

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$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{1}{(n+1)\alpha(n+1)t^n}\int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}$$

[$\alpha(n+1)$ = Volume of the unit ball in $\mathbb{R}^{n+1}$]

$$= \frac{2}{(n+1)\alpha(n+1)t^n}\int_{B(x,t)} g(y)\left(1+|\nabla\gamma(y)|^2\right)^{1/2} dy$$

where $\text{gr}(x),\gamma(y) := \left(t^2-|y-x|^2\right)^{1/2}$

$\partial\bar{B}(x,t)\cap\{y_{n+1}\geq 0\}$

$$\Rightarrow \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{2}{(n+1)\alpha(n+1)t^n}\int_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

$$\left[1+|\nabla\gamma(y)|^2\right]^{1/2} = t\left(t^2-|y-x|^2\right)^{-1/2}$$

$$= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)}\fint_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

$$\therefore u(x,t) = \frac{1}{\gamma_{n+1}}\cdot\frac{2\alpha(n)}{(n+1)\alpha(n+1)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\fint_{B(x,t)}\frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\,dy\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\fint_{B(x,t)}\frac{h(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\,dy\right)\right]$$

where, $\gamma_{n+1} = 1\cdot 3\cdot 5\cdots(n-1)$ and $\alpha(n) = \dfrac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)}$

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For even 'n',

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\underbrace{\left(t^n ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)}_{(*)} + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n ⨍_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right]$$

and $\gamma_n = 2\cdot 4 \cdots (n-2)\cdot n$

Th:- Assume n is even integer ≥ 2 and $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$ for $m = \frac{n+2}{2}$.

If 'u' is defined by $(*)$. Then

(i) $u \in C^2(\mathbb{R}^n \times [0,\infty))$

(ii) $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$

(iii) $\lim\limits_{(x,t)\to(x^0,0)} u(x,t) = g(x^0)$; $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\, t>0}} u_t(x,t) = h(x^0)$

Proof:- Check Yourself

$$⨍_{\partial\bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{1}{(n+1)\alpha(n+1)t^n}\int_{\partial\bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}$$

[$\alpha(n+1)$ = Volume of the unit ball in $\mathbb{R}^{n+1}$]

$$= \frac{2}{(n+1)\alpha(n+1)t^n}\int_{B(x,t)} g(y)\left(1+|D\gamma(y)|^2\right)^{1/2} dy$$

where: graph of γ, $\gamma(y) := (t^2-|y-x|^2)^{1/2}$

$= \partial\bar{B}(\bar{x},t) \cap \{y_{n+1} \geq 0\}$

$$\left[1+|D\gamma(y)|^2\right]^{1/2} = t\,(t^2-|y-x|^2)^{-1/2}$$

$$\Rightarrow ⨍_{\partial\bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{2}{(n+1)\alpha(n+1)t^n}\int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

$$= \frac{2t\,\alpha(n)}{(n+1)\alpha(n+1)} ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

$$\therefore u(x,t) = \frac{1}{\gamma_{n+1}}\cdot\frac{2\alpha(n)}{(n+1)\alpha(n+1)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n ⨍_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n ⨍_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right]$$

where, $\gamma_{n+1} = 1\cdot 3\cdot 5 \cdots (n-1)$ and $\alpha(n) = \dfrac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)}$

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Wave Equation in $n = 2m;\ m = 1, 2, \cdots$:-

$$\left.\begin{array}{l} u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x);\ u_t(x,0) = h(x) \text{ in } \mathbb{R}^n \times \{t=0\}. \end{array}\right\} \longleftarrow (1)$$

Assumption:- n is an even integer. Let $u \in C^m$ is the solution of (1) for $m = \frac{n+2}{2}$.

Set, $\bar{u}(x_1, x_2, \ldots, x_{n+1}, t) := u(x_1, x_2, \ldots, x_n, t)$ in $\mathbb{R}^{n+1} \times (0,\infty)$.

$\bar{u} = \bar{g}$ and $\bar{u}_t = \bar{h}$ on $\mathbb{R}^{n+1} \times \{t=0\}$

where, $\bar{g}(x_1, \ldots, x_{n+1}) := g(x_1, x_2, \ldots, x_n)$.

$\bar{h}(x_1, x_2, \ldots, x_{n+1}) := h(x_1, x_2, \ldots, x_n)$.

Fix $x \in \mathbb{R}^n$, $t > 0$ and $\bar{x} = (x_1, x_2, \ldots, x_n, 0) \in \mathbb{R}^{n+1}$.

$$u(x,t) = \frac{1}{\gamma_{n+1}}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1} \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, dS\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1} \fint_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}\right)\right]$$

$B(\bar{x},t)$ = Ball in $\mathbb{R}^{n+1}$ with center $\bar{x}$ & radius 't'; $d\bar{S}$ = Surface measure on $\partial \bar{B}(\bar{x},t)$

$n=1$, D'Alembert

$n=3$, Kirchoff's

$\downarrow$ N.D

$n = 2$, Poisson Formula.

$n = 2m+1;\ m = 1, 2, \ldots$ (odd)

$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$

x_3

Hadamard

Huygen's Principle :- ('1671)

* If n is odd ≥ 3, the data g & h at $x \in \mathbb{R}^n$ affects the solution 'u' only on the boundary $\{(y,t) \mid t > 0 \text{ \& } |x-y| = t\}$ of the cone $\{(y,t) \mid t > 0 \text{ and } |x-y| < t\} := C$

* If n is even, the data g & h affects 'u' within all of 'C'.

You can solve the inhomogeneous problem. How do you solve it? You just use Duhamel’s principle using this. Now, we move on to something called so let me do it in a new page.

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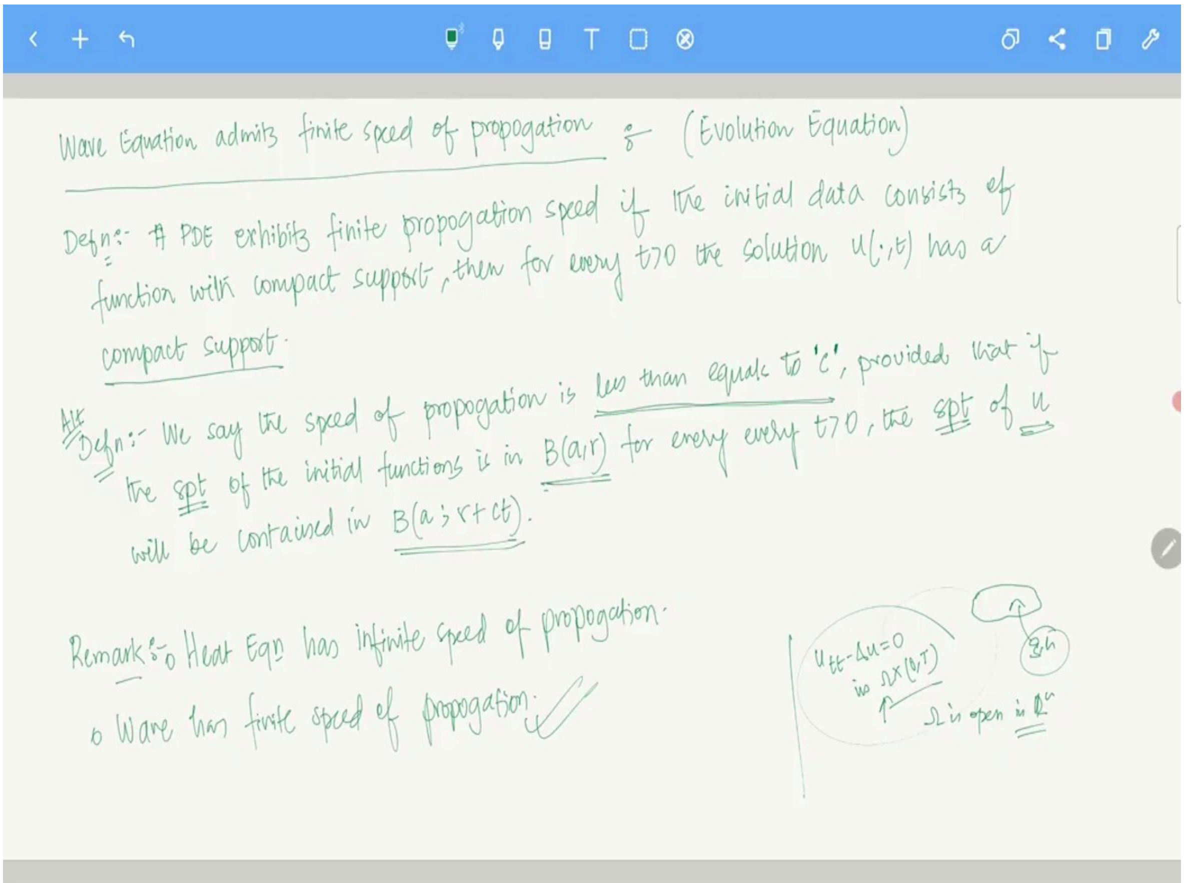

We move on to something which we did in the heat equation also, here we are going to do something called speed of propagation. So, we are going to show that wave equation, in our setting, wave equation admits finite speed of propagation. Let me explain to you what all of this means, what finite speed of propagation means? First of all, in the mathematical sense of course.

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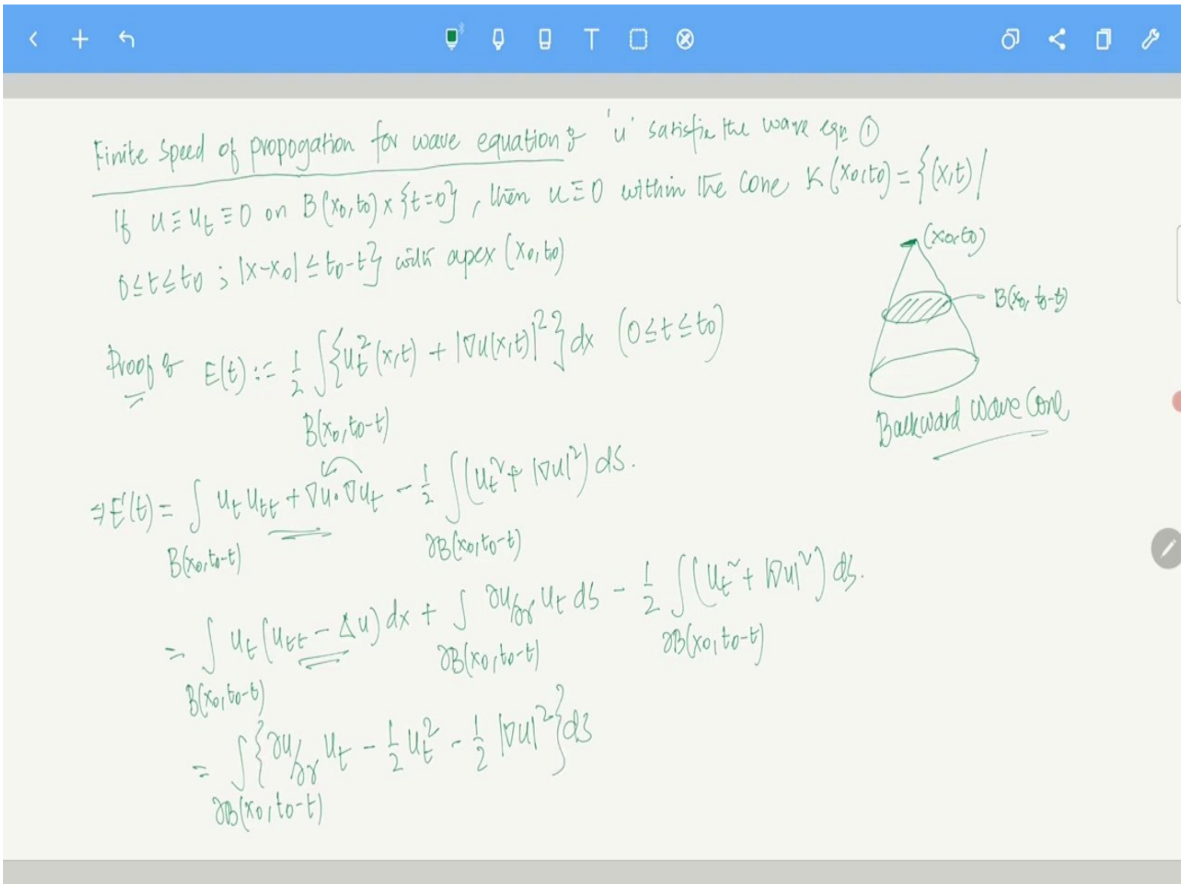

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Wave Equation in $n = 2m;\ m = 1, 2, \dots$:-

$$\left.\begin{aligned} &u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ &u(x,0) = g(x);\ u_t(x,0) = h(x) \text{ in } \mathbb{R}^n \times \{t=0\}. \end{aligned}\right\} \leftarrow ①$$

Assumption:- n is an even integer. Let $u \in C^m$ is the solution of ① for $m = \frac{n+2}{2}$.

Set, $\bar{u}(x_1, x_2, \dots, x_{n+1}, t) := u(x_1, x_2, \dots, x_n, t)$ in $\mathbb{R}^{n+1} \times (0,\infty)$.

$\bar{u} = \bar{g}$ and $\bar{u}_t = \bar{h}$ on $\mathbb{R}^{n+1} \times \{t=0\}$

where, $\bar{g}(x_1, \dots, x_{n+1}) := g(x_1, x_2, \dots, x_n)$.

$\bar{h}(x_1, x_2, \dots, x_{n+1}) := h(x_1, x_2, \dots, x_n)$.

Fix $x \in \mathbb{R}^n$, $t > 0$ and $\bar{x} = (x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1}$.

$$u(x,t) = \frac{1}{\gamma_{n+1}} \left[\frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n-1} \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, dS \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n-1} \fint_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\xi \right) \right]$$

$B(\bar{x}, t)$ = Ball in $\mathbb{R}^{n+1}$ with center $\bar{x}$ & radius 't' ; $d\xi$ = Surface measure on $\partial \bar{B}(\bar{x}, t)$

$n = 1$, D'Alembert

$n = 3$, Kirchoff'c

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$n = 2$, Poisson Formula.

$n = 2m+1;\ m = 1, 2, \dots$ (odd)

$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$

x_3

Hadamard

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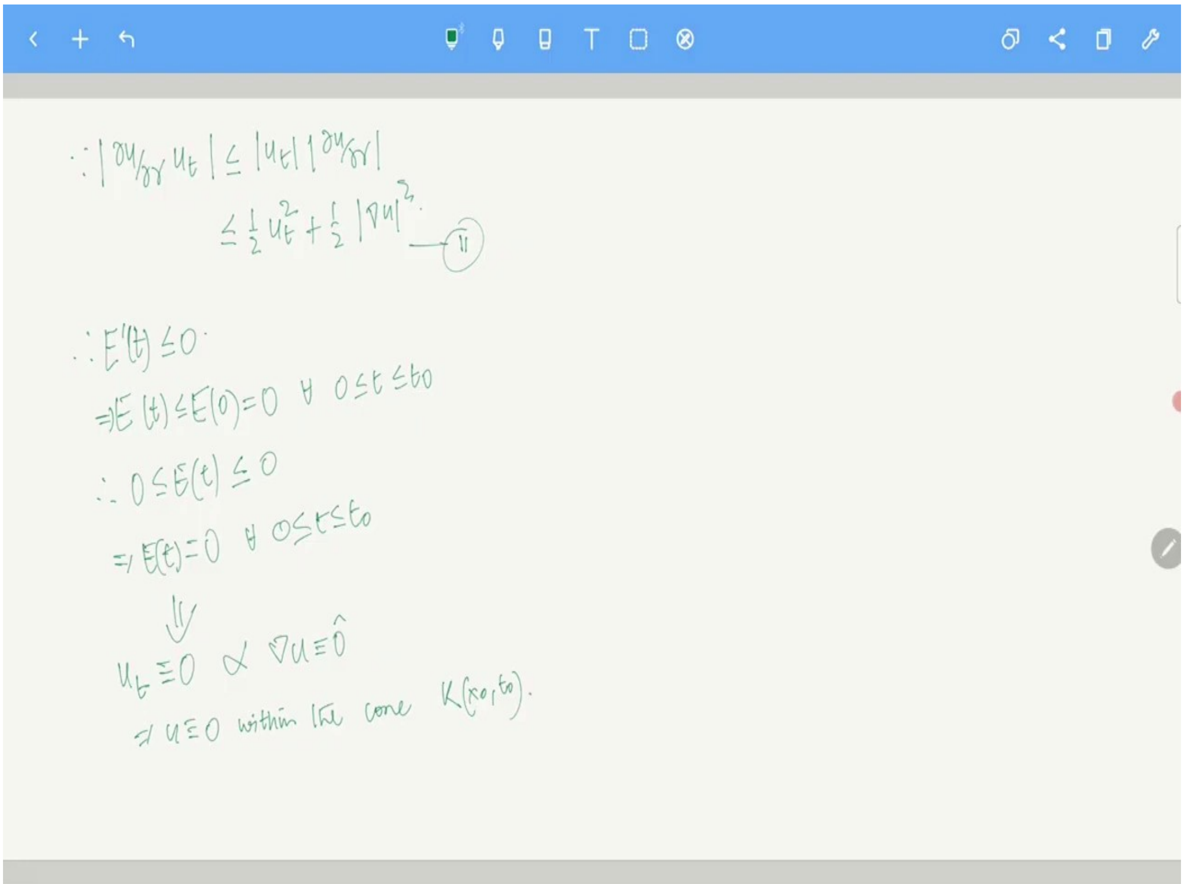
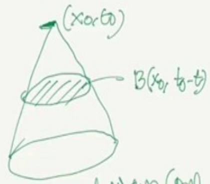


Finite speed of propogation for wave equation: 'u' satisfie the wave eqn (1)

If $u \equiv u_t \equiv 0$ on $B(x_0,t_0)\times\{t=0\}$, then $u \equiv 0$ within the cone $K(x_0,t_0)=\{(x,t) \mid 0\le t\le t_0 ;\ |x-x_0|\le t_0-t\}$ with apex (x_0,t_0)



Backward Wave Cone

Proof: $E(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} \{u_t^2(x,t) + |\nabla u(x,t)|^2\}\,dx \quad (0\le t\le t_0)$

$$\Rightarrow E'(t) = \int_{B(x_0,t_0-t)} u_t u_{tt} + \nabla u\cdot\nabla u_t - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(u_t^2 + |\nabla u|^2\right) dS.$$

$$= \int_{B(x_0,t_0-t)} u_t(u_{tt}-\Delta u)\,dx + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial r} u_t\, dS - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(u_t^2 + |\nabla u|^2\right) dS.$$

$$= \int_{\partial B(x_0,t_0-t)} \left\{\frac{\partial u}{\partial r} u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2\right\} dS$$