Advanced Partial Differential Equations
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Lecture: 28
Wave Equation in n=2 K+1, K=1, 2,…

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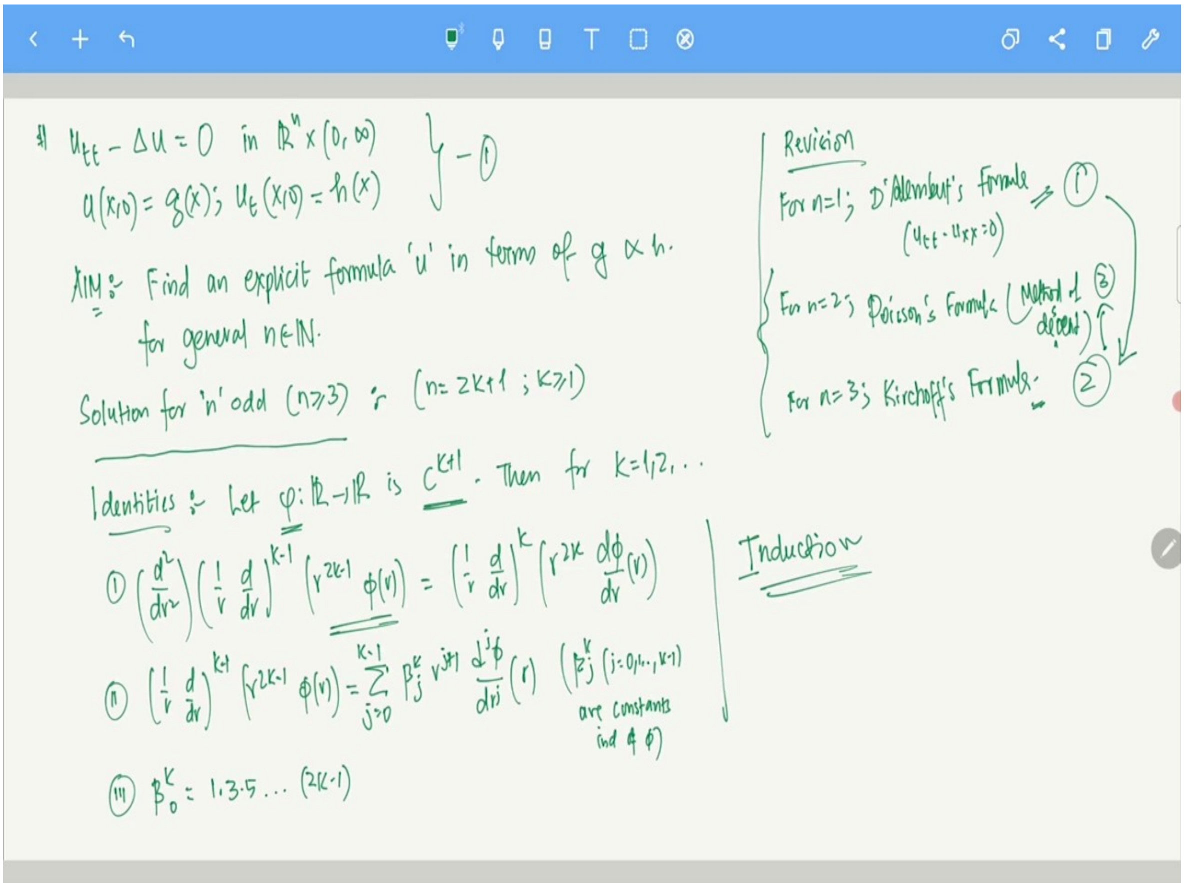

Welcome students, in this week's video, we are going to talk about the norm, the Kirchoff's formula, but for higher dimensions. So, as we have seen that once you can solve the homogeneous problem, you can use the explicit formula to construct a solution for the homogeneous problem via the Duhamel's principle.

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Let $u(x,t)$ be the solution of (1) and $u \in C^{k+1}(\mathbb{R}^n \times (0,\infty))$

Recall, $x \in \mathbb{R}^n, t>0$ & $r>0$ define

$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\, dS(y)$$

and $U \in C^{k+1}$.

We write, $\tilde{U}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} U(x;r,t)\right)$

$$\tilde{G}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} G(x;r)\right) \quad (r>0 \text{ \& } t\geq 0)$$

$$\tilde{H}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} H(x;r)\right)$$

#1: $\tilde{U}_{tt} - \tilde{U}_{rr} = 0$ in $\mathbb{R}_+ \times (0,\infty)$

$\tilde{U} = \tilde{G}$; $\tilde{U}_t = \tilde{H}$ on $\mathbb{R}_+ \times \{t=0\}$

$\tilde{U} = 0$ on $\{r=0\} \times (0,\infty)$

Proof: $\tilde{U}_{rr} = \left(\frac{\partial^2}{\partial r^2}\right)\underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} U\right)} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k} U_r\right) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\left(r^{2k} U_r\right)$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1} U_{rr} + 2k r^{2k-2} U_r\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\underbrace{\left(U_{rr} + \frac{n-1}{r} U_r\right)}\right] \quad (n = 2k+1)$$

$\because U$ satisfies the E-P-D equation, one has

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} U_{tt}\right) = \tilde{U}_{tt}.$$

$\therefore \tilde{U}$ solves the 1-D wave equation.

$$\left.\begin{aligned} &u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ &u(x,0) = g(x);\; u_t(x,0) = h(x) \end{aligned}\right\} - (1)$$

AIM: Find an explicit formula 'u' in terms of g & h for general $n \in \mathbb{N}$.

Revision

For $n=1$; D'Alembert's Formula $(u_{tt} - u_{xx} = 0)$ ⇒ (1)

For $n=2$; Poisson's Formula (Method of descent) (3)

For $n=3$; Kirchoff's Formula. (2)

Solution for 'n' odd ($n \geq 3$): ($n = 2k+1$; $k \geq 1$)

Identities: Let $\varphi: \mathbb{R} \to \mathbb{R}$ is C^{k+1}. Then for $k = 1, 2, \ldots$

(i) $\left(\dfrac{d^2}{dr^2}\right)\left(\dfrac{1}{r}\dfrac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\dfrac{1}{r}\dfrac{d}{dr}\right)^{k}\left(r^{2k}\dfrac{d\phi}{dr}(r)\right)$

(ii) $\left(\dfrac{1}{r}\dfrac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \displaystyle\sum_{j=0}^{k-1} \beta_j^k\, r^{j+1} \dfrac{d^j\phi}{dr^j}(r)$ (β_j^k ($j = 0, 1, \ldots, k-1$) are constants ind of ϕ)

(iii) $\beta_0^k = 1 \cdot 3 \cdot 5 \ldots (2k-1)$

Induction

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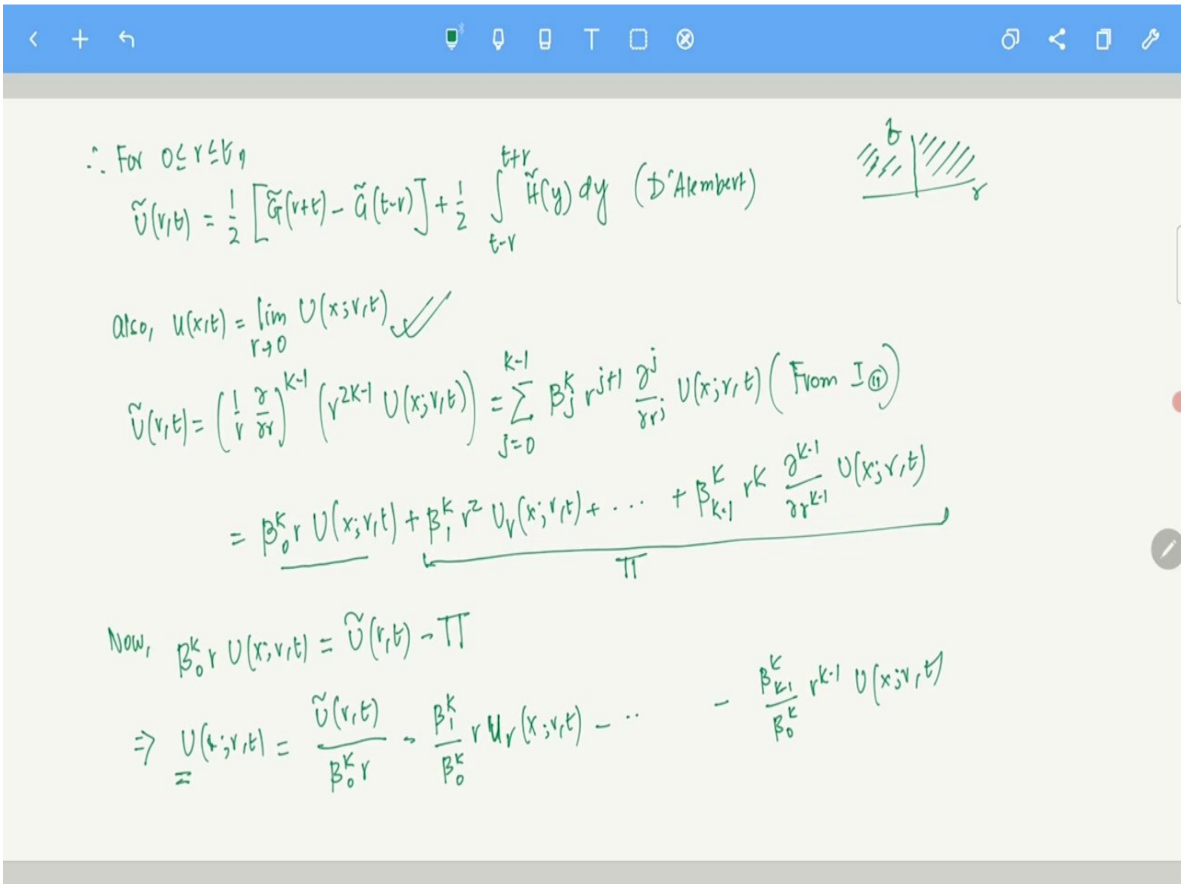

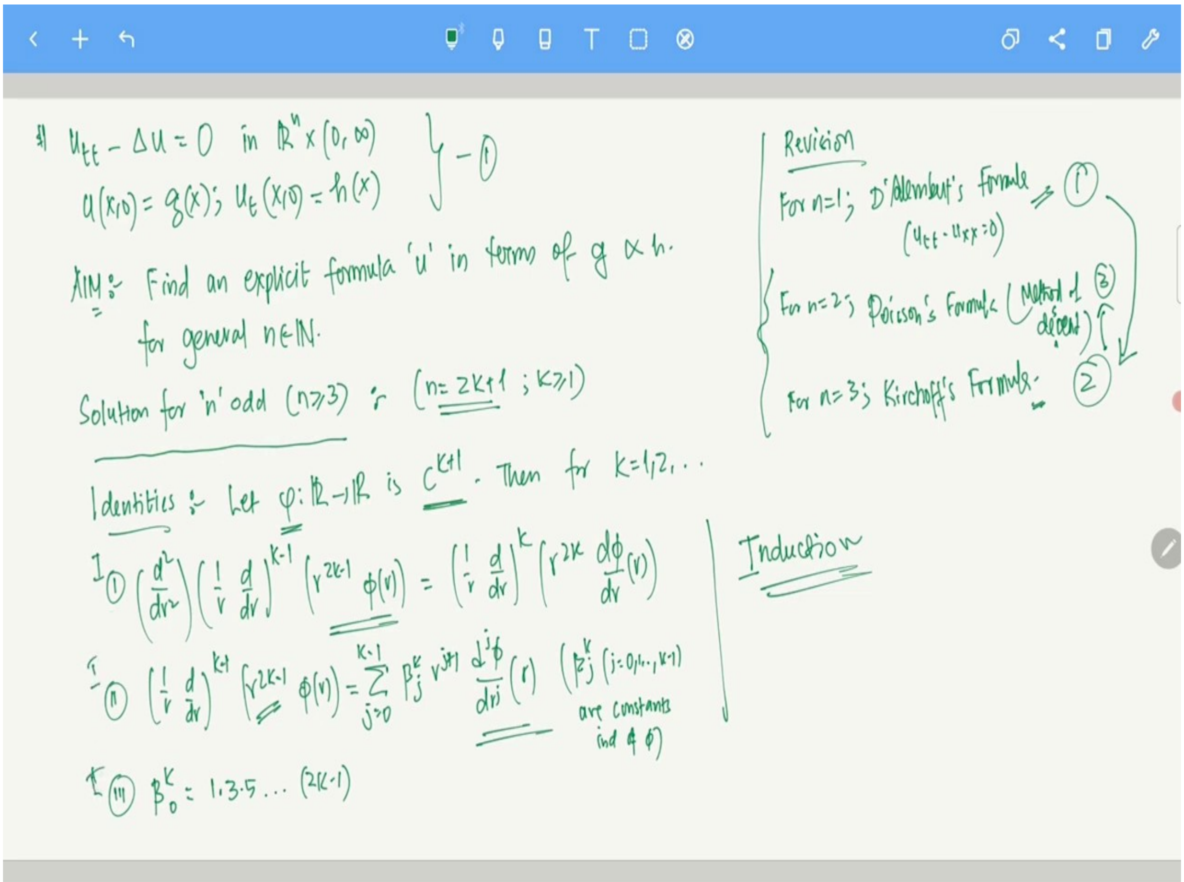

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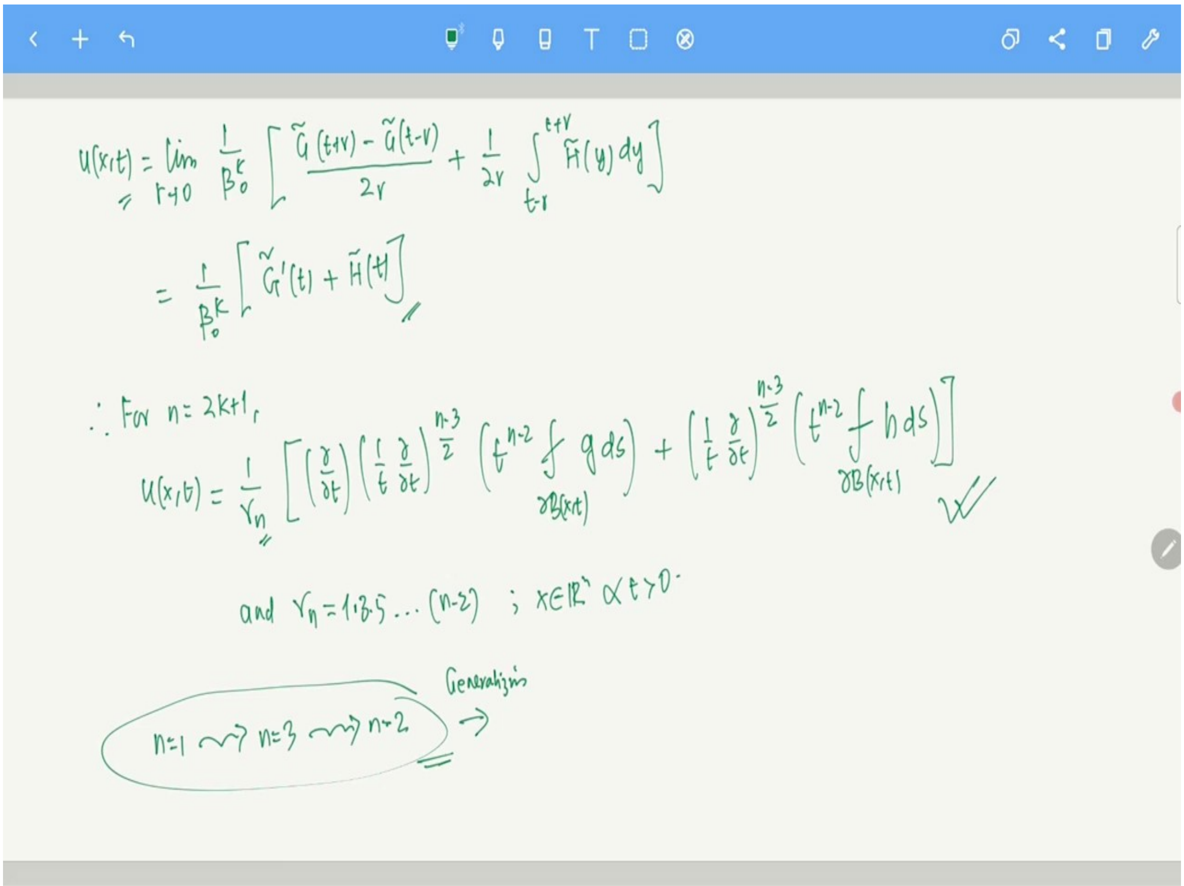
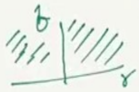


$\therefore$ For $0 \le r \le t$,

$$\tilde{U}(r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy \quad (\text{D'Alembert})$$



Also, $u(x,t) = \lim_{r\to 0} U(x;r,t)$

$$\tilde{U}(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(x;r,t) \quad (\text{From I(ii)})$$

$$= \beta_0^k r\, U(x;r,t) + \underbrace{\beta_1^k r^2 U_r(x;r,t) + \dots + \beta_{k-1}^k r^k \frac{\partial^{k-1}}{\partial r^{k-1}} U(x;r,t)}_{\Pi}$$

Now, $\beta_0^k r\, U(x;r,t) = \tilde{U}(r,t) - \Pi$

$$\Rightarrow \underset{\substack{\downarrow r\to 0\\ u(x,t)}}{U(x;r,t)} = \frac{\tilde{U}(r,t)}{\beta_0^k r} - \underbrace{\frac{\beta_1^k}{\beta_0^k} r\, U_r(x;r,t) - \dots - \frac{\beta_{k-1}^k}{\beta_0^k} r^{k-1} U(x;r,t)}_{\downarrow r\to 0 \; ?}$$

Let $u(x,t)$ be the solution of (1) and $u \in C^{k+1}(\mathbb{R}^n \times (0,\infty))$

Recall, $x \in \mathbb{R}^n$, $t>0$ & $r>0$ define

$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\,dS(y)$$

and $U \in C^{k+1}$.

We write, $\tilde{U}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)$

$$\tilde{G}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}G(x;r)\right) \quad (r>0 \;\&\; t>0)$$

$$\tilde{H}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}H(x;r)\right)$$

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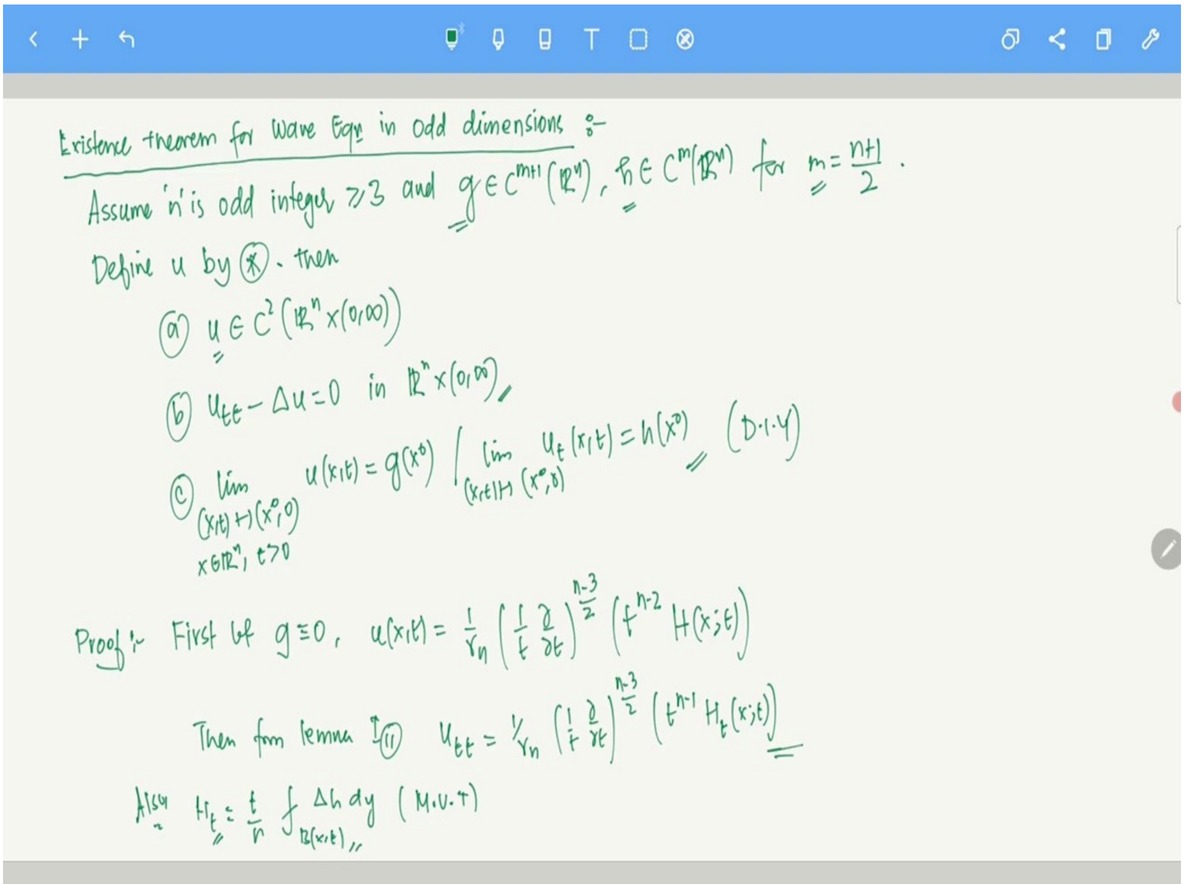

So, for that the existence theorem for wave equation in odd dimensions:so basically we want to show that the formula which you obtained.

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Existence theorem for wave Eqn in odd dimensions :-

Assume 'n' is odd integer ≥ 3 and $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$ for $m = \frac{n+1}{2}$.

Define u by (*). Then

(a) $u \in C^2(\mathbb{R}^n \times (0,\infty))$

(b) $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$,

(c) $\lim\limits_{\substack{(x,t) \to (x^0,0) \\ x \in \mathbb{R}^n,\, t>0}} u(x,t) = g(x^0)$ / $\lim\limits_{(x,t) \to (x^0,0)} u_t(x,t) = h(x^0)$ (D.1.4)

Proof :- First let $g \equiv 0$, $u(x,t) = \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} H(x;t)\right)$

Then from lemma 1(ii) $u_{tt} = \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-1} H_t(x;t)\right)$

Also $H_t = \frac{t}{n} ⨍_{B(x,t)} \Delta h \, dy$ (M.V.T)

$$u(x,t) = \lim_{r \to 0} \frac{1}{\beta_0^k}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy\right]$$

$$= \frac{1}{\beta_0^k}\left[\tilde{G}'(t) + \tilde{H}(t)\right]$$

$\therefore$ For $n = 2k+1$,

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} ⨍_{\partial B(x,t)} g\,dS\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} ⨍_{\partial B(x,t)} h\,dS\right)\right]$$

(n-odd ≥ 3)

and $\gamma_n = 1 \cdot 3 \cdot 5 \ldots (n-2)$; $x \in \mathbb{R}^n$ & $t > 0$. —— (*)

$n=1 \rightsquigarrow n=3 \rightsquigarrow n-2$ $\xrightarrow{\text{Generalization}}$

This is our Kirchoff's formula for n greater than equal 3. And this formula solves our wave equation. So let us end it here.