Advanced Partial Differential Equations
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Lecture 27
Wave Equation: Poison and Duhamel Formulae

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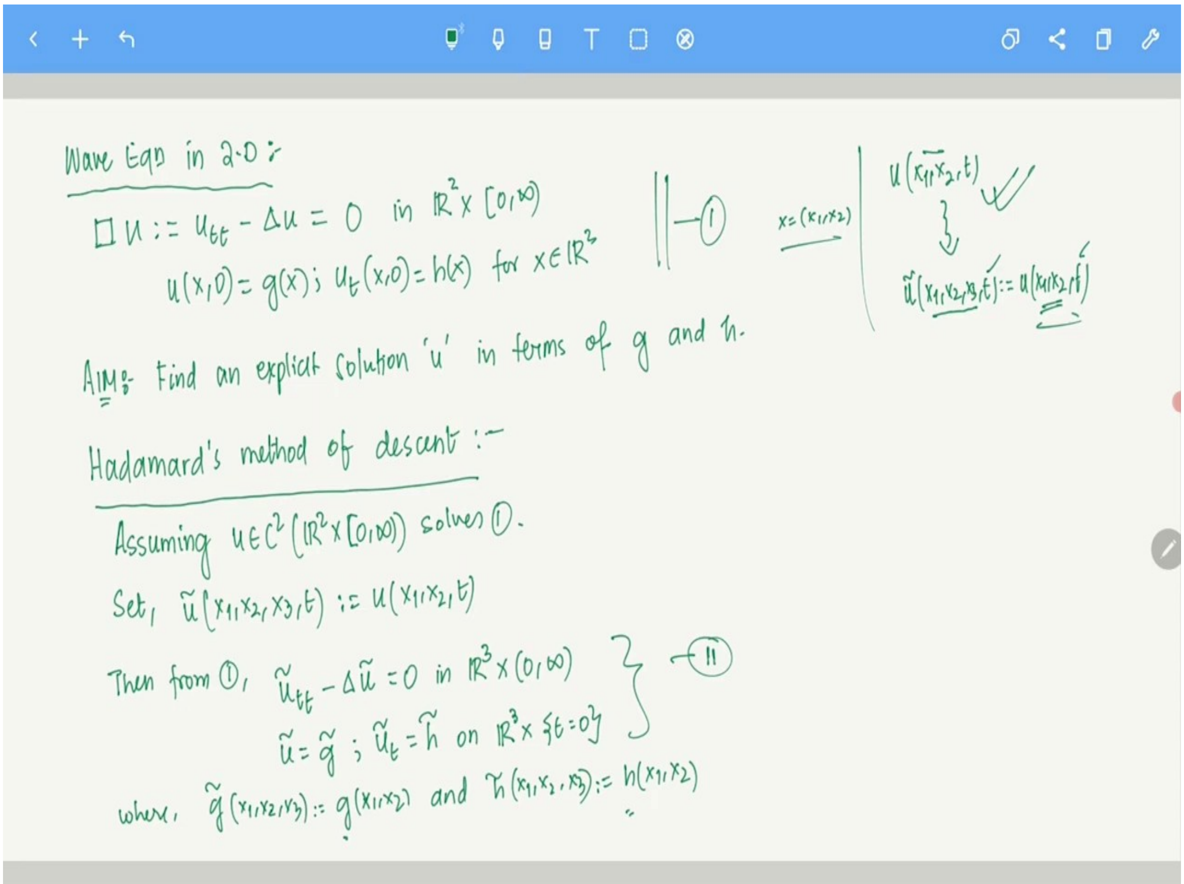


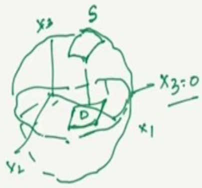
Welcome students, this video lecture we are going to talk about the wave equation in 2-dimension. So, as I told you in the last lecture, we have seen how to solve the wave equation in 2-dimension. It is essentially the, so this is called the d'Alembertian, you guys already know that. We did this thing.

From Kirchoff's formula : $\tilde{x} = (x_1, x_2, 0)$ & $x = (x_1, x_2)$

$$u(x,t) = \tilde{u}(\tilde{x}, t)$$

$$= \frac{\partial}{\partial t}\left(t \fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} \right) + t \fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{h}\, d\tilde{S}$$

where $\tilde{B}(\tilde{x},t)$ is the ball in $\mathbb{R}^3$ with center $\tilde{x}$ and $t>0$; $d\tilde{S}$ denotes the 2-D surface measure on $\partial\tilde{B}(\tilde{x},t)$.



$$\fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} = \frac{1}{4\pi t^2} \int_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S}(y) \quad \text{—— (14)}$$

[Let the surface is given by $x_3 = p(x_1, x_2)$

Then the surface integral is given by

$$\iint_S f(x_1,x_2,x_3)\, dS = \iint_D f(x_1, x_2, p(x,y)) \sqrt{(p_x)^2 + (p_y)^2 + 1}\; dA \;]$$

$\because \tilde{g}$ does not depend on y_3.

Wave Eqn in 2-D :-

$$\Box u := u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^2 \times [0,\infty)$$
$$u(x,0) = g(x);\; u_t(x,0) = h(x) \quad \text{for } x \in \mathbb{R}^2 \qquad \text{—— (1)}$$

$\xrightarrow{x=(x_1,x_2)}$ $u(x_1, x_2, t)$ $\leadsto$ $\tilde{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$

AIM:- Find an explicit solution 'u' in terms of g and h.

Hadamard's method of descent :-

Assuming $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves (1).

Set, $\tilde{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$

Then from (1),
$$\left.\begin{aligned} &\tilde{u}_{tt} - \Delta \tilde{u} = 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty) \\ &\tilde{u} = \tilde{g}\;;\; \tilde{u}_t = \tilde{h} \quad \text{on } \mathbb{R}^3 \times \{t=0\} \end{aligned}\right\} \text{—— (11)}$$

where, $\tilde{g}(x_1,x_2,x_3) := g(x_1,x_2)$ and $\tilde{h}(x_1,x_2,x_3) := h(x_1,x_2)$

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From (iii) we have,

$$\fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} = \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)\left(1+|\nabla \gamma(y)|^2\right)^{1/2} dy$$

where $\gamma(y) = \left(t^2 - |y-x|^2\right)^{1/2}$ for $y \in B(x,t)$.

$$\because \left(1+|\nabla \gamma(y)|^2\right)^{1/2} = t\left(t^2-|y-x|^2\right)^{-1/2}. \text{ (Check)}$$

$$\therefore \fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} = \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

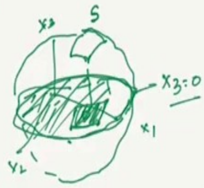
$$= \frac{t}{2} \fint_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

From Kirchoff's formula : $\tilde{x} = (x_1, x_2, 0)$ & $x = (x_1, x_2)$

$$u(x,t) = \tilde{u}(\tilde{x},t)$$

$$= \frac{\partial}{\partial t}\left(t \fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} \right) + t \fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{h}\, d\tilde{S}$$

where $\tilde{B}(\tilde{x},t)$ is the ball in $\mathbb{R}^3$ with center $\tilde{x}$ and $t>0$; $d\tilde{S}$ denotes the 2-D surface measure on $\partial \tilde{B}(\tilde{x},t)$.



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$$\iint_S f(x_1,x_2,x_3)\, dS = \iint_D f(x_1,x_2,p(x_1,x_2))\sqrt{(p_{x_1})^2+(p_{x_2})^2+1}\; dA \text{]}$$

$\because \tilde{g}$ does not depend on y_3.

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$$\therefore\ u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) + \frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

Now,

$$t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy = t\fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz$$

and so,

$$\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) = \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz + t\fint_{B(0,1)} \frac{\nabla g(x+tz)\cdot z}{(1-|z|^2)^{1/2}}\,dz$$

$$= t\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy + t\fint_{B(x,t)} \frac{\nabla g(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy$$

From (iii) we have,

$$\fint_{\partial\tilde{B}(\bar{x},t)} \tilde{g}\,d\tilde{S} = \frac{2}{4\pi t^2}\int_{B(x,t)} g(y)\left(1+|\nabla\gamma(y)|^2\right)^{1/2}dy$$

where $\gamma(y) = (t^2-|y-x|^2)^{1/2}$ for $y \in B(x,t)$.

$$\therefore\ \left(1+|\nabla\gamma(y)|^2\right)^{1/2} = t\left(t^2-|y-x|^2\right)^{-1/2}. \quad \text{(Check)}$$

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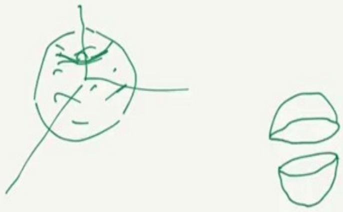
From Kirchoff's formula : $\tilde{x} = (x_1, x_2, 0)$ & $x = (x_1, x_2)$



$$u(x,t) = \tilde{u}(\tilde{x}, t)$$

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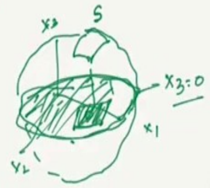
where $\tilde{B}(\tilde{x},t)$ is the ball in $\mathbb{R}^3$ with center $\tilde{x}$ and $t>0$; $d\tilde{S}$ denotes the 2-D surface measure on $\partial\tilde{B}(\tilde{x},t)$.



$$\fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} = \frac{1}{4\pi t^2} \int_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S}(y) \quad \text{—— (14)}$$

[Let the surface is given by $x_3 = p(x_1, x_2)$
then the surface integral is given by

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$\because \tilde{g}$ does not depend on y_3.

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$$\therefore u(x,t) = \frac{1}{2} \fint_{B(x,t)} \frac{t g(y) + t^2 h(y) + t \nabla g(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{1/2}} \, dy \quad ; x \in \mathbb{R}^2 \ \& \ t > 0 \quad \text{—} \ (*)$$

(*) is called the Poisson's formula for the solution of (1) in 2-D.

[$n=3$ - Kirchoff's formula

$n=2$ - Poisson formula].

The inhomogeneous Cauchy problem :

$(n \in \mathbb{N})$

$$\Box u = F \quad \text{—} \ (IV)$$

$$u = g \ \& \ u_t = h \ \text{ in } \mathbb{R}^n \times \{t=0\}$$

$$\therefore u(x,t) = \frac{1}{2} \frac{\partial}{\partial t} \left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy \right) + \frac{t^2}{2} \fint_{B(x,t)} \frac{h(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy$$

Now,
$$t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy = t \fint_{B(0,1)} \frac{g(x+tz)}{(1 - |z|^2)^{1/2}} \, dz$$

and so,
$$\frac{\partial}{\partial t} \left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy \right) = \fint_{B(0,1)} \frac{g(x+tz)}{(1 - |z|^2)^{1/2}} \, dz + t \fint_{B(0,1)} \frac{\nabla g(x+tz) \cdot z}{(1 - |z|^2)^{1/2}} \, dz$$

$$= t \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy + t \fint_{B(x,t)} \frac{\nabla g(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{1/2}} \, dy$$

Wave Eqn in 2-D :-

$$\Box u := u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^2 \times [0,\infty)$$
$$u(x,0) = g(x);\; u_t(x,0) = h(x) \quad \text{for } x \in \mathbb{R}^2 \quad \Bigg\} \longrightarrow (1)$$

$\xrightarrow{x=(x_1,x_2)}$

$u(x_1,x_2,t)$

$\tilde{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$

AIM:- Find an explicit solution 'u' in terms of g and h.

Hadamard's method of descent :-

Assuming $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves (1).

Set, $\tilde{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$

Then from (1),
$$\left.\begin{aligned} &\tilde{u}_{tt} - \Delta\tilde{u} = 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty) \\ &\tilde{u} = \tilde{g}\;;\; \tilde{u}_t = \tilde{h} \quad \text{on } \mathbb{R}^3 \times \{t=0\} \end{aligned}\right\} \longrightarrow (11)$$

where, $\tilde{g}(x_1,x_2,x_3) := g(x_1,x_2)$ and $\tilde{h}(x_1,x_2,x_3) := h(x_1,x_2)$

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$\Box u = F$ is a linear equation.

$\Box u = 0$
$u = g\,;\, u_t = h$

(I, II) → (u)

$u(x,t) = u_1(x,t) + u_2(x,t)$

where, $\Box u_1(x,t) = 0$
$u_1(x,0) = g(x)\,;\,(u_1)_t(x,0) = h(x)$ — (V) and

$\Box u_2(x,t) = F$
$u_2(x,0) = 0 \;\&\; (u_2)_t(x,0) = 0$ — (VI)

Homogeneous Wave Eqn + Inhomogeneous wave eqn

Let, $W_F(x,t;s)$ the solution of the homogeneous problem.

$$\Box w = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty)$$
$$w(x,s) = 0\,;\; w_t(x,s) = F(x,s)$$

Duhamel's principle says that, $u(x,t) := \int_0^t W_F(x,t;s)\,ds \quad (x \in \mathbb{R}^n, t > 0)$

is the solution of

$$\therefore u(x,t) = \frac{1}{2} ⨍_{B(x,t)} \frac{t g(y) + t^2 h(y) + t \nabla g(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\,dy \;;\; x \in \mathbb{R}^2 \;\&\; t > 0 \quad \text{— } (*)$$

(*) is called the Poisson's formula for the solution of (I) in 2-D.

[n=3 - Kirchoff's formula.
n=2 - Poisson formula.]

The inhomogeneous Cauchy problem: $(n \in \mathbb{N})$

$\Box u = F$ — (IV)

$u = g \;\&\; u_t = h$ in $\mathbb{R}^n \times \{t=0\}$

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$\Box u = F$ in $\mathbb{R}^n \times (0,\infty)$
$u = 0;\ u_t = 0$ on $\mathbb{R}^n \times \{t=0\}$

Theorem: (Nonhomogeneous wave equation). Assume $n \geq 2$ and $F \in C^{[n/2]+1}(\mathbb{R}^n \times [0,\infty))$ and define "u" as (**). Then

(a) $u \in C^2(\mathbb{R}^n \times [0,\infty))$

(b) $\Box u = F$ in $\mathbb{R}^n \times (0,\infty)$

(c) $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$; $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u_t(x,t) = 0$ for each pt $x_0 \in \mathbb{R}^n$.

$\Box u = F$ is a linear equation.

$\Box u = 0$
$u = g;\ u_t = h$

(I, II) → (u)

$u(x,t) = u_1(x,t) + u_2(x,t)$

where, $\Box u_1(x,t) = 0$, $u_1(x,0) = g(x)$; $(u_1)_t(x,0) = h(x)$ (I) and $\Box u_2(x,t) = F$, $u_2(x,0) = 0$ & $(u_2)_t(x,0) = 0$ (II)

Homogeneous Wave Eqn + Inhomogeneous wave eqn

Let, $W_F(x,t;s)$ the solution of the homogeneous problem.

$$\Box w = 0 \text{ in } \mathbb{R}^n \times [0,\infty)$$
$$w(x,s) = 0;\ w_t(x,s) = F(x,s)$$

Duhamel's principle says that, $u(x,t) := \int_0^t W_F(x,t;s)\,ds \quad (x \in \mathbb{R}^n, t > 0)$ is the solution of

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Proof: (a) If n is odd, $\left[\frac{n}{2}\right]+1 = \frac{n+1}{2}$. [Check using Kirchoff formula for $n=3$]

If $F \in C^{\frac{n+1}{2}}(\mathbb{R}^n \times [0,\infty))$ then

$W_F(x,t;s) \in C^2(\mathbb{R}^n \times [s,\infty))$ for each $s \geq 0$.

and hence, $u \in C^2(\mathbb{R}^n \times [0,\infty))$.

If n is even, $\left[\frac{n}{2}\right]+1 = \frac{n+2}{2}$. and since $F \in C^{\frac{n+2}{2}} \Rightarrow u \in C^2(\mathbb{R}^n \times [0,\infty))$ (Think of this using Poisson formula).

(b) $u_t(x,t) = W_F(x,t;t) + \int_0^t (W_F)_t(x,t;s)\,ds = \int_0^t (W_F)_t(x,t;s)\,ds$. (Differentiation under the sign of integration)

and, $u_{tt}(x,t) = (W_F)_t(x,t;t) + \int_0^t (W_F)_{tt}(x,t;s)\,ds = F(x,t) + \int_0^t (W_F)_{tt}(x,t;s)\,ds$

also, $\Delta u(x,t) = \int_0^t \Delta W_F(x,t;s)\,ds = \int_0^t (W_F)_{tt}(x,t;s)\,ds$ ∎

Wave Eqn in 2-D:

$$\Box u := u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times [0,\infty)$$
$$u(x,0) = g(x);\ u_t(x,0) = h(x) \text{ for } x \in \mathbb{R}^2 \qquad (1)$$

$u(x_1,x_2,t)$

$\tilde{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$

AIM: Find an explicit solution 'u' in terms of g and h.

Hadamard's method of descent:-

Assuming $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves (1).

Set, $\tilde{u}(x_1,x_2,x_3,t)$:

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For the case $n=1$;

$$u_{tt} - u_{xx} = f \text{ in } \mathbb{R}\times(0,\infty)$$
$$u = g;\ u_t = h \text{ on } \mathbb{R}\times\{t=0\} \quad — P$$

$$\begin{cases} \Box u = f \\ u = 0;\ u_t = 0 \end{cases} \quad — P_1$$

$$\begin{cases} \Box u = 0 \\ u = g;\ u_t = h. \end{cases} \quad — P_2.$$

We know how to solve (P_2) using d'Alembert formula

$\Box u = F$ is a linear equation.

$$\Box u = 0$$
$$u = g;\ u_t = h$$

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

where, $\Box u_1(x,t) = 0$, $u_1(x,0) = g(x)$; $(u_1)_t(x,0) = h(x)$ — (I) and $\Box u_2(x,t) = F$, $u_2(x,0) = 0$ & $(u_2)_t(x,0) = 0$ — (II)

Homogeneous Wave Eqn + Inhomogeneous wave eqn

(I, II) → (u)

Let, $W_F(x,t;s)$ the solution of the homogeneous problem. $\Box w = w_{tt} - \Delta w = 0$

$$\Box w = 0 \text{ in } \mathbb{R}^n \times [0,\infty)$$
$$w(x,s) = 0;\ w_t(x,s) = F(x,s)$$

Duhamel's principle says that, $u(x,t) := \int_0^t W_F(x,t;s)\,ds \quad (x\in\mathbb{R}^n, t>0)$ — (**) is the solution of

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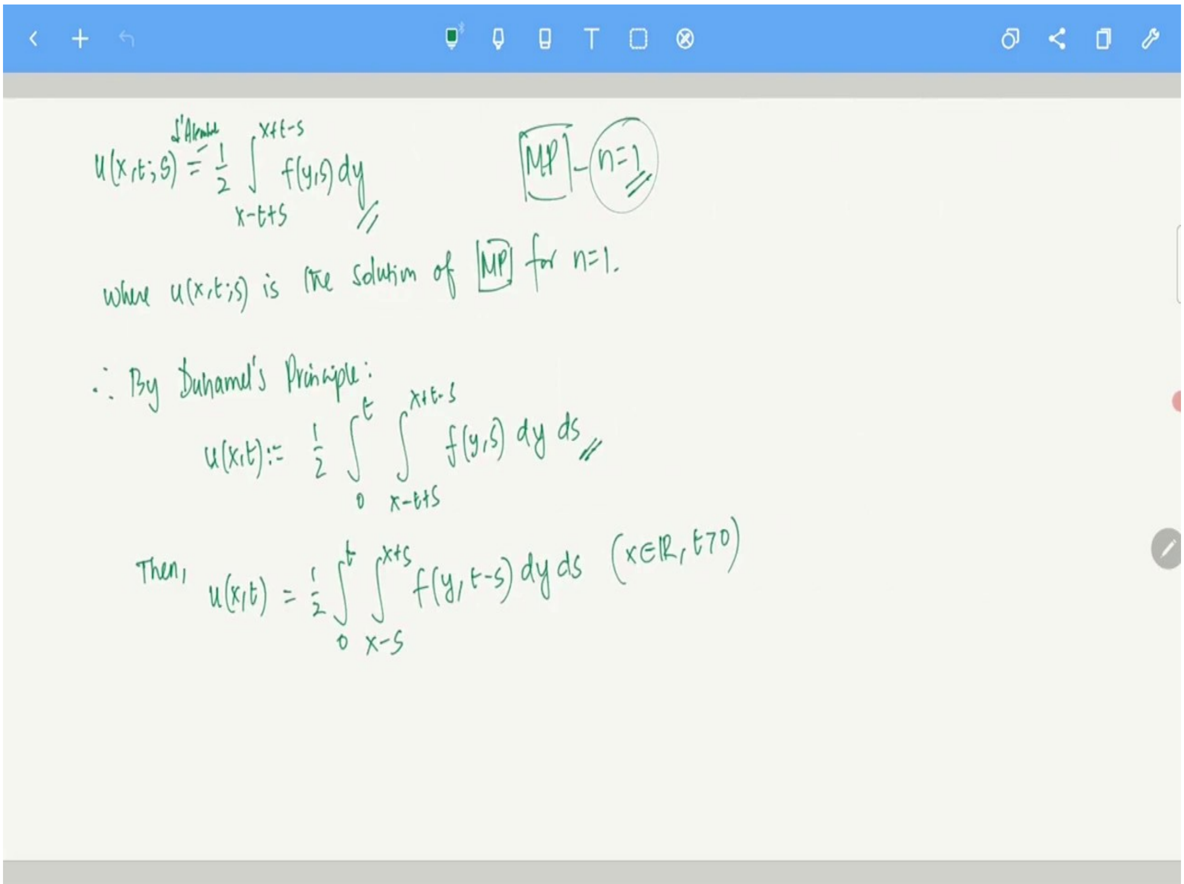

$\square u = F$ is a linear equation.

$\square u = 0$
$u = g \,;\, u_t = h$

I.V.P $\rightarrow$ u

$u(x,t) = u_1(x,t) + u_2(x,t)$

where, $\square u_1(x,t) = 0$, $u_1(x,0) = g(x) \,;\, (u_1)_t(x,0) = h(x)$ — (I) and $\square u_2(x,t) = F$, $u_2(x,0) = 0 \;\&\; (u_2)_t(x,0) = 0$ — (II)

Homogeneous Wave Eqn + Inhomogeneous wave eqn

Let, $W_F(x,t;s)$ the solution of the homogeneous problem. $\square w = w_{tt} - \Delta w = 0$

$$\square w = 0 \text{ in } \mathbb{R}^n \times [0,\infty)$$
$$w(x,s) = 0 \,;\, w_t(x,s) = F(x,s) \quad \text{— MP}$$

Duhamel's principle says that, $u(x,t) := \int_0^t W_F(x,t;s)\, ds \quad (x \in \mathbb{R}^n, t \geq 0)$ — (**)

is the solution of

For the case $n=1$;

$$u_{tt} - u_{xx} = f \text{ in } \mathbb{R} \times (0,\infty)$$
$$u = g \,;\, u_t = h \text{ on } \mathbb{R} \times \{t=0\} \quad \text{— P}$$

$$\begin{cases} \square u = f \\ u = 0 \,;\, u_t = 0 \end{cases} \quad \text{— } P_1$$

$$\begin{cases} \square u = 0 \\ u = g \,;\, u_t = h. \end{cases} \quad \text{— } P_2$$

We know how to solve P_2 using d'Alembert formula.