

**Advanced Partial Differential Equations**  
**Professor Doctor Kaushik Bal**  
**Department of Mathematics and Statistics**  
**Indian Institute of Technology Kanpur**  
**Lecture 26**  
**Wave Equation for  $n = 3$**

(Refer Slide Time: 0:18)

Wave Equation :-  
 Let  $n \geq 2$  or  $m \geq 2$  and  $u \in C^m(\mathbb{R}^n \times [0, \infty))$  solve the I.V.P  
 $u_{tt} - \Delta u = 0$  in  $\mathbb{R}^n \times (0, \infty)$  (1)  
 $u = g; u_t = h$  on  $\mathbb{R}^n \times \{t=0\}$

Aim :- Find an explicit formula for  $u$  in terms of  $g$  and  $h$ .

Define: For  $x \in \mathbb{R}^n, t > 0, r > 0$ . Define  

$$U(x; r, t) := \int_{\partial B(x; r)} u(y, t) ds(y)$$

The average of  $u(\cdot, t)$  over the sphere  $\partial B(x; r)$ .

If  $G(r) := \int_{\partial B(0; r)} g(y) ds(y)$  or  $H(x; r) := \int_{\partial B(x; r)} h(y) ds(y)$ .

Using 'i' we define a new PDE E-P.D. case  
 1-D wave eqn (n-odd)  
 n-even.  
 t  
 (x)

Welcome students and today's class we are going to talk about the wave equation in a higher dimension. So, wave equation, so essentially we want to find out a formula for a wave equation. So, let me write down the problem, we are supposing, so let  $n$  is greater than 2 and  $m$  is greater than 2.

(Refer Slide Time: 11:14)

The image shows a digital whiteboard with handwritten mathematical notes. At the top, there is a blue header bar with navigation icons. The main content is written in green ink on a light-colored background. The notes define the Euler-Poisson-Darboux equation, specify the domain and boundary conditions for  $u$ , and then introduce a function  $V$  that satisfies a similar equation. A note explains that the term  $V_{rr} - \frac{n-1}{r}V_r$  is the radial part of the Laplacian. The equation for  $V$  is labeled as (iii).

$T_h$ : (Euler-Poisson-Darboux equation) Fix  $x \in \mathbb{R}^n$  and let 'u' satisfy

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$
$$u = g; u_t = h \text{ on } \mathbb{R}^n \times \{t=0\}.$$

Then,  $U$  defined by (i) belongs to  $C^m(\bar{\mathbb{R}}_+ \times [0, \infty))$  and

$$\begin{aligned} V_{tt} - V_{rr} - \left(\frac{n-1}{r}\right) V_r &= 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ V = G, V_t &= h \text{ on } \mathbb{R}_+ \times \{t=0\} \end{aligned} \quad \left| \text{--- (iii)} \right.$$

Note:  $V_{rr} - \left(\frac{n-1}{r}\right) V_r$  represents the radial <sup>form</sup> part of Laplacian

(iii) is called the E-P-D eqn.

Wave Equation :-

Let  $n \geq 2$  or  $m \geq 2$  and  $u \in C^m(\mathbb{R}^n \times [0, \infty))$  solve the I.V.P

$$\begin{aligned} u_{tt} - \Delta u &= 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u &= g; \quad u_t = h \text{ on } \mathbb{R}^n \times \{t=0\} \end{aligned} \quad \text{--- (1)}$$

Aim :- Find an explicit formula for  $u$  in terms of  $g$  and  $h$ .

Define, For  $x \in \mathbb{R}^n, t > 0, r > 0$ . Define

$$U(x; r, t) := \int_{\partial B(x; r)} u(y, t) dS(y) \quad \text{--- (2)}$$



The average of  $u(\cdot, t)$  over the sphere  $\partial B(x; r)$ .

$$U(x; r, t) := \int_{\partial B(x; r)} g(y) dS(y) \quad \text{or} \quad H(x; r) := \int_{\partial B(x; r)} h(y) dS(y)$$

Using (1) we define a new I.V.P E-P.D eqn  
 $\downarrow$   
 I.O wave eqn ( $n \geq 2$ )  
 $\downarrow$   
 $n = \text{even}$ .

(Refer Slide Time: 19:06)

The image shows a digital whiteboard with handwritten mathematical derivations and notes. The main derivation is for the Poisson integral formula. It starts with the definition of the Poisson kernel  $U(x; r, t) := \int_{\partial B(x, r)} u(y, t) dS(y)$ . This is then transformed into an integral over the boundary of a ball  $\partial B(0, 1)$  by substituting  $y = x + rz$ , resulting in  $U(x; r, t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0, 1)} u(x + rz, t) dS(z)$ . The next step is to differentiate under the integral sign to find  $\Delta U(x; r, t) = \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} \Delta u(x + rz, t) \cdot z \cdot dS(z)$ . Finally, it is shown that  $\Delta U(x; r, t) = \frac{1}{r^n} \int_{\partial B(x, r)} \Delta u(y, t) dy$ .

On the right side of the whiteboard, there are handwritten notes: "1. H.W.E" followed by a downward arrow to "n=1 (D'Alembert)", another downward arrow to "Spherical Means (E.D.D. case)", a third downward arrow to "n-odd", a fourth downward arrow to "1-D wave", and a final downward arrow to "Spherical means n-even".

(Refer Slide Time: 25:05)

$\lim_{r \rightarrow 0} U_r(x; r, t) = 0.$

Again,  $U_{rr}(x; r, t) = ?$

$$U_r = \frac{1}{n(n-1)r^{n-1}} \int_{B(r)} \Delta u(y, t) dy //$$
$$\Rightarrow r^{n-1} U_r = \frac{1}{n(n-1)} \int_{B(r)} \Delta u(y, t) dy - A = B$$

$A \leftarrow r^{n-1} U_r$

$$\therefore A_r = (n-1)r^{n-2} U_r + r^{n-1} U_{rr}$$
$$= r^{n-1} U_{rr} + \left(1 - \frac{1}{n}\right) \frac{1}{n(n-1)r} \int_{B(r)} \Delta u(y, t) dy$$

$B := \frac{1}{n(n-1)} \int_{B(r)} \Delta u(y, t) dy$   
Co-area formula (Polar coordinate formula)

Proof: For  $r > 0$ ,

$$U(x; r, t) := \int_{\partial B(x, r)} u(y, t) dS(y)$$

$$= \frac{1}{n \omega(r)^{n-1}} \int_{\partial B(0, r)} u(x + rz, t) dS(z)$$

$$= \frac{1}{n \omega(r)} \int_{\partial B(0, r)} u(x + rz, t) dS(z)$$

$$\therefore U_r(x; r, t) = \frac{1}{n \omega(r)} \int_{\partial B(0, r)} \nabla u(x + rz, t) \cdot z dS(z)$$

$$= \int_{\partial B(x, r)} \frac{\partial u}{\partial \nu}(y, t) dS(y) \stackrel{\text{G.D.T.}}{=} \frac{1}{\omega(r)} \int_{B(x, r)} \Delta u(y, t) dy.$$

1. H.W.E

$n=1$  (1D Problem)

Spherical Means

(E.D.D. case)

$\downarrow$  n-odd

1D case

$\downarrow$  Special techniques

n-even

(Refer Slide Time: 31:28)

"How to convert n-d integrable to integral over spheres".

o  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and summable. Then

$$\int_{\mathbb{R}^n} f \, dx = \int_0^{\infty} \left( \int_{\partial B(x_0, r)} f \, ds \right) dr$$

for each pt  $x_0 \in \mathbb{R}^n$ .

$$o \int_{B(x_0, r)} f \, dx = \int_0^r \int_{\partial B(x_0, s)} f \, ds \, ds \quad \text{for } r > 0.$$

$$\therefore \frac{d}{dr} \left( \int_{B(x_0, r)} f \, dx \right) = \int_{\partial B(x_0, r)} f \, ds$$

(Refer Slide Time: 35:03)

$$B := \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t) dy$$

$$B_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} \Delta u(y,t) dS(y)$$

$$\therefore U_{rr}(x;r,t) = \int_{\partial B(x,r)} \Delta u dS + \left(\frac{1}{n}-1\right) \int_{B(x,r)} \Delta u dy \quad ; r > 0. \quad \text{--- (iv)}$$

$$\text{and } \lim_{r \rightarrow 0^+} U_{rr}(x;r,t) = \frac{1}{n} \Delta u(x,t)$$

Now, using (iv) one can calculate  $\underline{U_{rr}}$  ...

$$\therefore U \in C^m(\bar{\mathbb{R}}_+^n \times [0, \infty))$$



"How to convert n-d integrable to integral over spheres":

o  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and summable. Then

$$\int_{\mathbb{R}^n} f \, dx = \int_0^\infty \left( \int_{\partial B(x_0, r)} f \, dS \right) dr$$

for each pt  $x_0 \in \mathbb{R}^n$ .

$$o \int_{B(x_0, r)} f \, dx = \int_0^r \int_{\partial B(x_0, s)} f \, dS(s) \, ds \quad \text{for } r > 0.$$

$$\therefore \frac{d}{dr} \left( \int_{B(x_0, r)} f \, dx \right) = \int_{\partial B(x_0, r)} f \, dS$$

$$\lim_{r \rightarrow 0} U_r(x; r, b) = 0.$$

Again,  $U_{rr}(x; r, b) = ?$

$$U_r = \frac{1}{n(n-1)r^{n-1}} \int_{B(r)} \Delta u(y; r) dy //$$

$$\Rightarrow r^{n-1} U_r = \frac{1}{n(n-1)} \int_{B(r)} \Delta u(y; r) dy - A = B$$

$$A = r^{n-1} U_r$$

$$\therefore A_r = (n-1)r^{n-2} U_r + r^{n-1} U_{rr}$$

$$= r^{n-1} U_{rr} + \left(1 - \frac{1}{n}\right) \frac{1}{n(n-1)r} \int_{B(r)} \Delta u(y; r) dy$$

$$B := \frac{1}{n(n-1)} \int_{B(r)} \Delta u(y; r) dy$$

(Refer Slide Time: 40:47)

$$U(x, y, z, t) = \int_{\partial B(x, r)} u(y, t) dS(y)$$
$$\therefore U_{tt}(x, y, z, t) = \int_{\partial B(x, r)} u_{tt}(y, t) dS(y) \quad \text{--- (1)}$$
$$U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = \int_{\partial B(x, r)} u_{tt}(y, t) dS(y) - \left(\frac{1}{n-1}\right) \int_{\partial B(x, r)} \Delta u(y, t) dy - \frac{n-1}{r} \int_{\partial B(x, r)} \Delta u(y, t) dy - \int_{\partial B(x, r)} \Delta u(y, t) dS(y)$$
$$= \int_{\partial B(x, r)} [u_{tt}(y, t) dS(y) - \Delta u(y, t)] dS(y)$$
$$= 0$$

also,  $U = G_t$  and  $U_t = H'$

$$B := \frac{1}{nd(n)} \int_{B(x,r)} \Delta u(y,t) dy$$

$$B_r = \frac{1}{nd(n)} \int_{\partial B(x,r)} \Delta u(y,t) dS(y)$$

$$\therefore U_r(x,y,t) = \int_{\partial B(x,r)} \Delta u dS + \left(\frac{1}{n}-1\right) \int_{B(x,r)} \Delta u dy \quad ; r > 0. \quad \text{--- (iv)}$$

$$\text{and } \lim_{r \rightarrow 0^+} U_r(x,y,t) = \frac{1}{n} \Delta u(x,t)$$

Now, using (iv) one can calculate  $\underline{U_{rr}}$  ...

$$\therefore U \in C^m(\bar{\mathbb{R}}_+ \times [0, \infty))$$

(Refer Slide Time: 45:58)

Solution for  $n=3$ :- Suppose  $u \in C^2(\mathbb{R}^3 \times [0, \infty))$  solves ①. Let  $U, G$  and  $H$  are defined as

① and ② resp. set

$$\tilde{U} = rU, \quad \tilde{G} = rG \quad \text{and} \quad \tilde{H} = rH$$

$$\begin{aligned} \text{Ans: } \tilde{U}_{tt} - \tilde{U}_{rr} &= 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \quad \checkmark \\ \tilde{U} &= \tilde{G} \quad \text{and} \quad \tilde{U}_t = \tilde{H} \text{ on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{U} &= 0 \text{ on } \{r=0\} \times (0, \infty) \end{aligned}$$

$$\text{Note, } \tilde{U}_{tt} = rU_{tt} = r \left[ U_{rr} + \frac{2}{r}U_r \right] \quad (\because U \text{ satisfy E.P.D eqn})$$

$$\begin{aligned} &= rU_{rr} + 2U_r \\ &= (U + rU_r)_r = \tilde{U}_{rr} \end{aligned}$$

$$\text{Also } \tilde{U}_{rr}(0) = 0 \quad (\text{D.L.Y})$$

Wave Equation :-

Let  $n \geq 2$  or  $m \geq 2$  and  $u \in C^m(\mathbb{R}^n \times [0, \infty))$  solve the I.V.P

$$\begin{aligned} u_{tt} - \Delta u &= 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u &= g; \quad u_t = h \text{ on } \mathbb{R}^n \times \{t=0\} \end{aligned} \quad \text{--- (1) (Spherical Mean)}$$

Aim :- Find an explicit formula for  $u$  in terms of  $g$  and  $h$ .

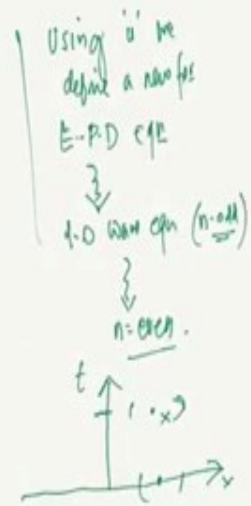
Define, For  $x \in \mathbb{R}^n, t > 0, r > 0$ . Define

$$U(x; r, t) := \int_{\partial B(x; r)} u(y, t) dS(y) \quad \text{--- (1)}$$



The average of  $u(\cdot, t)$  over the sphere  $\partial B(x; r)$ .

$$U(x; r, t) := \int_{\partial B(x; r)} g(y) dS(y) \quad \text{or} \quad H(x; r) := \int_{\partial B(x; r)} h(y) dS(y)$$



Th: (Euler-Poisson-Darboux equation) Fix  $x \in \mathbb{R}^n$  and let 'u' satisfy

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

$$u = g; u_t = h \text{ on } \mathbb{R}^n \times \{t=0\}.$$

Then,  $U$  defined by (ii) belongs to  $C^m(\bar{\mathbb{R}}_+ \times [0, \infty))$  and

$$U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \quad \left| \text{--- (iii)}\right.$$

$$V = G, V_t = h \text{ on } \mathbb{R}^+ \times \{t=0\}$$

Note:  $U_{rr} - \left(\frac{n-1}{r}\right) U_r$  represents the radial <sup>form</sup> part of Laplacian

(i) (iii) is called the E-P-D eqn.

(Refer Slide Time: 53:22)

For,  $0 \leq r \leq t$ ,

$$\tilde{v}(x; r, t) = \frac{1}{2} [\tilde{g}(r+t) - \tilde{g}(t-r)] + \frac{1}{2} \int_{t-r}^{t+r} \tilde{h}(y) dy$$

$$\text{now, } u(x; t) = \lim_{r \rightarrow 0^+} \frac{\tilde{v}(x; r, t)}{r} \quad (u(x; t) = \lim_{r \rightarrow 0^+} v(x; r, t))$$

$$= \lim_{r \rightarrow 0^+} \left[ \frac{\tilde{g}(t+r) - \tilde{g}(t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{h}(y) dy \right]$$

$$= \tilde{g}'(t) + \tilde{h}(t)$$

$$\therefore u(x; t) = \frac{\partial}{\partial t} \left( t \int_{\partial B(x; t)} g ds \right) + t \int_{\partial B(x; t)} h ds$$



Solution for  $n=3$ :- Suppose  $u \in C^2(\mathbb{R}^3 \times [0, \infty))$  solves ①. Let  $U, G$  and  $H$  are def

① and ② resp. set

$$\tilde{U} = rU, \quad \tilde{G} = rG \quad \text{and} \quad \tilde{H} = rH$$

$$\begin{aligned} \text{Ans: } \tilde{U}_{tt} - \tilde{U}_{rr} &= 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \checkmark \\ \tilde{U} &= \tilde{G} \text{ and } \tilde{U}_t = \tilde{H} \text{ on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{U} &= 0 \text{ on } \{r=0\} \times (0, \infty) \end{aligned} \quad \parallel$$

$$\text{Note, } \tilde{U}_{tt} = rU_{tt} = r \left[ U_{rr} + \frac{2}{r}U_r \right] \quad (\because U \text{ satisfy E.P.D eqn})$$

$$\begin{aligned} &= rU_{rr} + 2U_r \\ &= (U + rU_r)_r = \tilde{U}_{rr} \end{aligned}$$

$$\text{Also } \tilde{U}_{rr}(0) = 0 \quad (D+Y)$$

(Refer Slide Time: 58:50)

$$\therefore \int_{\partial B(x,t)} q(y) ds(y) = \int_{\partial B(0,t)} q(x+tz) ds(z)$$

$$\begin{aligned} \therefore \frac{\partial}{\partial t} \left( \int_{\partial B(x,t)} q(y) ds(y) \right) &= \int_{\partial B(0,t)} \nabla q(x+tz) \cdot z ds(z) \\ &= \int_{\partial B(x,t)} \nabla q(y) \cdot \left( \frac{y-x}{t} \right) ds(y) \end{aligned}$$

$$\text{Hence } u(x,t) = \int_{\partial B(x,t)} t h(y) + q(y) + \nabla q(y) \cdot (y-x) ds(y) \quad (x \in \mathbb{R}^3, t > 0)$$

—  $\boxed{A}$

This formula  $\boxed{A}$  is called the Kirchhoff's formula for  $\textcircled{1}$  in  $n=3$ .

For,  $0 \leq r \leq t$ ,

$$\tilde{v}(x; r, t) = \frac{1}{2} [\tilde{g}(t+r) - \tilde{g}(t-r)] + \frac{1}{2} \int_{t-r}^{t+r} \tilde{h}(y) dy$$

$$\text{now, } u(x; t) = \lim_{r \rightarrow 0^+} \frac{\tilde{v}(x; r, t)}{r} \quad (u(x; t) = \lim_{r \rightarrow 0^+} v(x; r, t))$$

$$= \lim_{r \rightarrow 0^+} \left[ \frac{\tilde{g}(t+r) - \tilde{g}(t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{h}(y) dy \right]$$

$$= \tilde{g}'(t) + \tilde{h}(t)$$

$$\therefore u(x; t) = \frac{\partial}{\partial t} \left( t \int_{\mathcal{B}(x; t)} g ds \right) + t \int_{\mathcal{B}(x; t)} h ds$$