Advanced Partial Differential Equations
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Lecture 26
Wave Equation for n = 3

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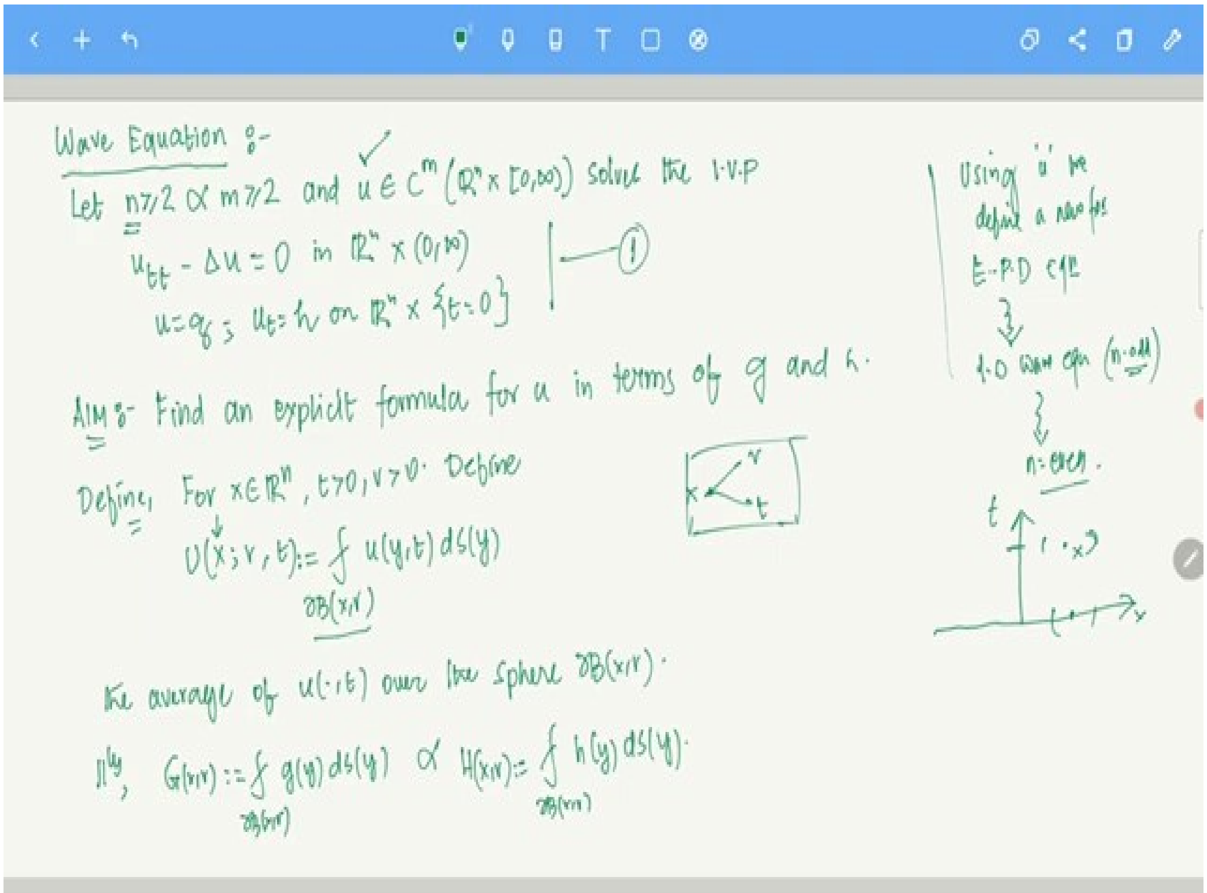

Welcome students and today’s class we are going to talk about the wave equation in a higher dimension. So, wave equation, so essentially we want to find out a formula for a wave equation. So, let me write down the problem, we are supposing, so let n is greater than 2 and m is greater than 2.

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Th: (Euler-Poisson-Darboux equation) Fix $x \in \mathbb{R}^n$ and let 'u' satisfy

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty)$$
$$u = g;\ u_t = h \text{ on } \mathbb{R}^n \times \{t=0\}.$$

Then, U defined by (II) belongs to $C^m(\bar{\mathbb{R}}_+ \times [0,\infty))$ and

$$\left.\begin{aligned} &U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = 0 \text{ in } \mathbb{R}_+ \times (0,\infty) \\ &U = G,\ U_t = H \text{ on } \mathbb{R}^+ \times \{t=0\} \end{aligned}\right| \text{ — (III)}$$

Note: $U_{rr} - \left(\frac{n-1}{r}\right) U_r$ represents the radial form ~~part~~ of Laplacian

(III) is called the E-P-D eqn.

Wave Equation :-

Let $n \geq 2$ & $m \geq 2$ and $u \in C^m(\mathbb{R}^n \times [0,\infty))$ solve the I.V.P

$$\left.\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g \,;\ u_t &= h \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}\right| \text{— (1)}$$

Using (1) we define a new fn

E.P.D eqn

$\Downarrow$

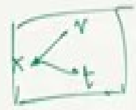
1-D wave eqn (n = odd)

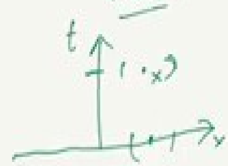
$\Downarrow$

n = even.

AIM :- Find an explicit formula for u in terms of g and h.

Define, For $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. Define





$$U(x; r, t) := ⨍_{\partial B(x,r)} u(y,t)\, dS(y) \quad \text{— (*)}$$

the average of $u(\cdot, t)$ over the sphere $\partial B(x,r)$.

$\|^{ly}$, $G(x;r) := ⨍_{\partial B(x,r)} g(y)\, dS(y)$ & $H(x;r) := ⨍_{\partial B(x,r)} h(y)\, dS(y)$.

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Proof: For $r > 0$,

$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\, dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,1)} u(x+rz,t)\, r^{n-1} dS(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz,t)\, dS(z)$$

$$\therefore U_r(x;r,t) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz,t)\cdot z\, dS(z)$$

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial r}(y,t)\, dS(y) \overset{\text{G.D.T}}{=} \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy.$$

I.H.W.E

$\downarrow$

$n=1$ (D'Alembert)

$\downarrow$

Spherical Means

(E-P-D eqn)

$\downarrow$ n-odd

1-D wave

$\downarrow$ Special techniques

n-even

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$$\lim_{r \to 0} U_r(x;r,t) = 0.$$

Again, $U_{rr}(x;r,t) = ?$

$$U_r = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t)\,dy$$

$$\Rightarrow r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\,dy \quad - \quad A = B$$

$$A = r^{n-1} U_r$$

$$\therefore A_r = (n-1) r^{n-2} U_r + r^{n-1} U_{rr}$$

$$= r^{n-1} U_{rr} + \left(1 - \frac{1}{n}\right) \frac{1}{n\alpha(n) r} \int_{B(x,r)} \Delta u(y,t)\,dy$$

$$B := \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\,dy$$

Co-area formula (Polar . coordinate formula)

Proof: For $r>0$,

$$U(x;r,t) := ⨍_{\partial B(x,r)} u(y,t)\, dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,1)} u(x+rz,t)\, r^{n-1}\, dS(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz,t)\, dS(z)$$

$$\therefore U_r(x;r,t) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz,t)\cdot z\, dS(z)$$

$$= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial r}(y,t)\, dS(y) \overset{G.D.T}{=} \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\, dy.$$

I.H.W.E

$\downarrow$

$n=1$ (D'Alembert)

$\downarrow$

Spherical Means (E-P-D eqn)

$\downarrow$ n-odd

1-D wave

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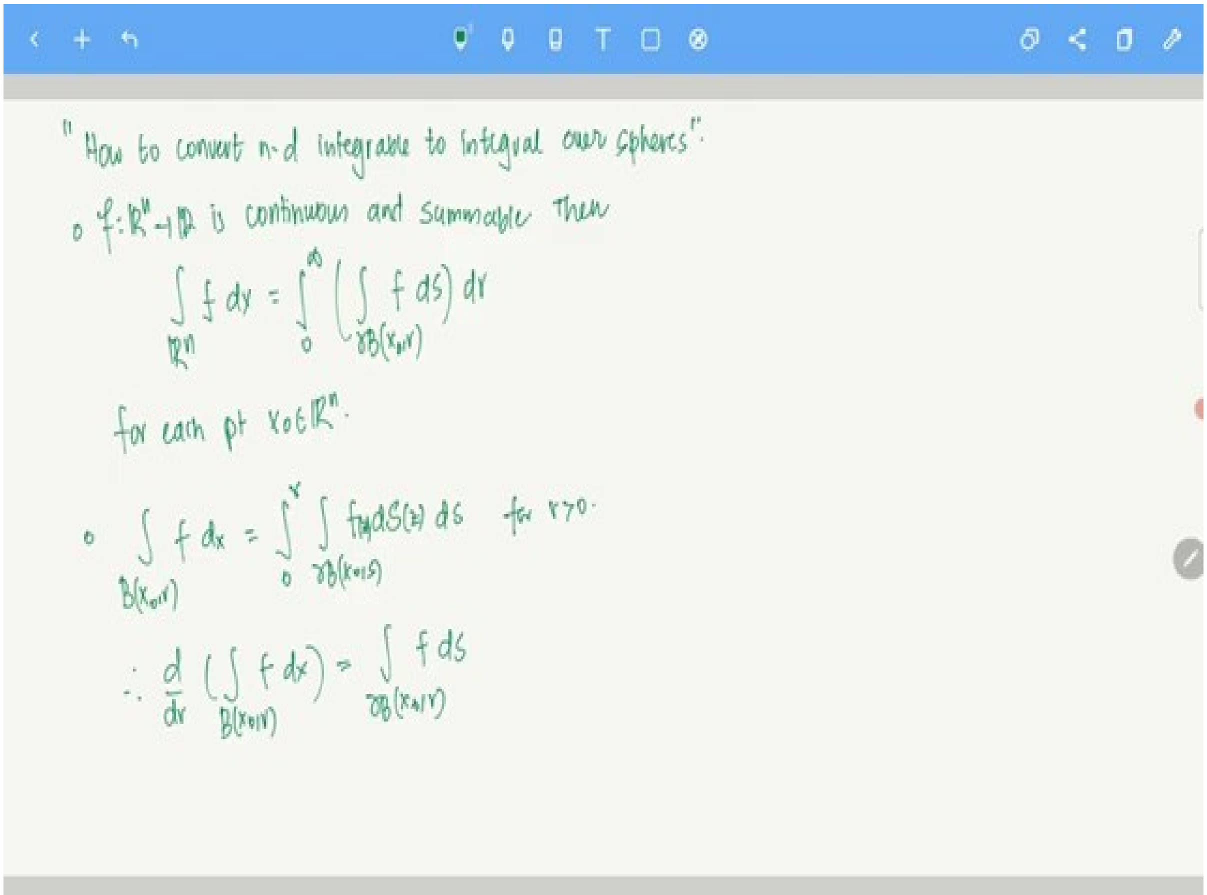

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$$B := \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\, dy$$

$$B_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} \Delta u(y,t)\, dS(y)$$

$$\therefore U_{rr}(x;r,t) = \fint_{\partial B(x,r)} \Delta u\, dS + \left(\frac{1}{n} - 1\right) \fint_{B(x,r)} \Delta u\, dy \quad ; r > 0 \quad \text{— (IV)}$$

and $\lim_{r \to 0^+} U_{rr}(x;r,t) = \frac{1}{n} \Delta u(x,t)$

Now, using (IV) one can calculate $U_{rrr} \cdots$

$\therefore U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$

"How to convert n-d integrals to integral over spheres".

- $f:\mathbb{R}^n\to\mathbb{R}$ is continuous and summable. Then

$$\int_{\mathbb{R}^n} f\,dx = \int_0^\infty \left(\int_{\partial B(x_0,r)} f\,dS\right) dr$$

for each pt $x_0\in\mathbb{R}^n$.

- $$\int_{B(x_0,r)} f\,dx = \int_0^r \int_{\partial B(x_0,s)} f(z)\,dS(z)\,ds \quad \text{for } r>0.$$

$$\therefore \frac{d}{dr}\left(\int_{B(x_0,r)} f\,dx\right) = \int_{\partial B(x_0,r)} f\,dS$$

$$\lim_{r\to 0} U_r(x;r,t) = 0.$$

Again, $U_{rr}(x;r,t) = ?$

$$U_r = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y,t)\, dy$$

$$\Rightarrow r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\, dy \quad - \quad A = B$$

$$B := \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\, dy$$

$$A = r^{n-1} U_r$$

$$\therefore A_r = (n-1) r^{n-2} U_r + r^{n-1} \underline{U_{rr}}$$

$$= r^{n-1} U_{rr} + \left(1 - \frac{1}{n}\right) \frac{1}{n\alpha(n) r} \int_{B(x,r)} \Delta u(y,t)\, dy$$

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$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\, dS(y)$$

$$\therefore U_{tt}(x;r,t) = \fint_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) \quad \text{——} \text{(V)}$$

$$U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = \fint_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) - \left(\frac{1}{n}-1\right) \fint_{B(x,r)} \Delta u(y,t)\, dy - \frac{n-1}{\cancel{r}} \cdot \frac{\cancel{r}}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy - \fint_{\partial B(x,r)} \Delta u(y,t)\, dS(y)$$

$$= \fint_{\partial B(x,r)} \left[u_{tt}(y,t)\, dS(y) - \Delta u(y,t)\right] dS(y)$$

$$= 0$$

also, $U = G$ and $U_t = H$.

$$B := \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y,t)\, dy$$

$$B_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} \Delta u(y,t)\, dS(y)$$

$$\therefore U_{rr}(x;r,t) = \fint_{\partial B(x,r)} \Delta u\, dS + \left(\frac{1}{n}-1\right) \fint_{B(x,r)} \Delta u\, dy \quad ; r>0. \quad \text{— (IV)}$$

and $\lim_{r \to 0^+} U_{rr}(x;r,t) = \frac{1}{n} \Delta u(x,t)$

Now, using (IV) one can calculate $U_{rrr} \cdots$

$\therefore U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$

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Solution for $n=3$:- Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves (1). Let U, G and H are defined as (1) and (R) resp. set

$$\tilde{U} = rU, \quad \tilde{G} = rG \quad \text{and} \quad \tilde{H} = rH$$

Aim: $\tilde{U}_{tt} - \tilde{U}_{rr} = 0$ in $\mathbb{R}_+ \times (0,\infty)$ ✓

$\tilde{U} = \tilde{G}$ and $\tilde{U}_t = \tilde{H}$ on $\mathbb{R}_+ \times \{t=0\}$

$\tilde{U} = 0$ on $\{r=0\} \times (0,\infty)$

Note, $\tilde{U}_{tt} = rU_{tt} = r\left[U_{rr} + \frac{2}{r}U_r\right]$ ($\because U$ satisfy E.P.D eqn)

$$= rU_{rr} + 2U_r$$

$$= (U + rU_r)_r = \tilde{U}_{rr}.$$

Also, $\tilde{G}_{rr}(0) = 0$ (D.L.Y)

Wave Equation :-

Let $n \geq 2$ & $m \geq 2$ and $u \in C^m(\mathbb{R}^n \times [0,\infty))$ solve the I.V.P

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty)$$
$$u = g \,;\; u_t = h \text{ on } \mathbb{R}^n \times \{t=0\} \qquad \text{—(1)}$$

(Spherical Mean)

Using '(1)' we define a new fn

E-P-D eqn

$\Downarrow$

1-D wave eqn (n = odd)

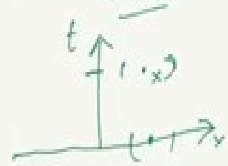
$\Downarrow$

n = even.

AIM :- Find an explicit formula for u in terms of g and h.

Define, For $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. Define

$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\, dS(y) \qquad \text{—(A)}$$



The average of $u(\cdot,t)$ over the sphere $\partial B(x,r)$.

||ly, $G(x;r) := \fint_{\partial B(x,r)} g(y)\, dS(y)$ & $H(x;r) := \fint_{\partial B(x,r)} h(y)\, dS(y)$.

Th: (Euler-Poisson-Darboux equation) Fix $x \in \mathbb{R}^n$ and let 'u' satisfy

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$$\left.\begin{aligned} &U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty) \\ &U = G,\ U_t = H \quad \text{on } \mathbb{R}^+ \times \{t=0\} \end{aligned}\right| \text{— (III)}$$

Note: ⓐ $U_{rr} - \left(\frac{n-1}{r}\right) U_r$ represents the radial ~~part~~ form of Laplacian

ⓑ (III) is called the E-P-D eqn.

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For, $0 \leq r \leq t$,

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy$$

now, $u(x,t) = \lim_{r\to 0^+} \frac{\tilde{U}(x,r,t)}{r}$. $\left(u(x,t) = \lim_{r\to 0^+} U(x;r,t)\right)$

$$= \lim_{r\to 0^+}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy\right]$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

$$\therefore u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g\,dS\right) + t \fint_{\partial B(x,t)} h\,dS$$

Solution for $n=3$:- Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves ①. Let U, G and H are def

① and ⊛ resp. set

$$\tilde{U} = rU, \quad \tilde{G} = rG \quad \text{and} \quad \tilde{H} = rH$$

Aim:
$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty)$$
$$\tilde{U} = \tilde{G} \quad \text{and} \quad \tilde{U}_t = \tilde{H} \quad \text{on } \mathbb{R}_+ \times \{t=0\}$$
$$\tilde{U} = 0 \quad \text{on } \{r=0\} \times (0,\infty)$$

Note, $\tilde{U}_{tt} = rU_{tt} = r\left[U_{rr} + \frac{2}{r}U_r\right]$ ($\because U$ satisfy E.P.D eqn)

$$= rU_{rr} + 2U_r$$
$$= (U + rU_r)_r = \tilde{U}_{rr}.$$

Also, $\tilde{G}_{rr}(0) = 0$ (D.I.Y)

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$$\therefore \fint_{\partial B(x,t)} g(y)\, dS(y) = \fint_{\partial B(0,1)} g(x+tz)\, dS(z)$$

$$\therefore \frac{\partial}{\partial t}\left(\fint_{\partial B(x,t)} g(y)\, dS(y) \right) = \fint_{\partial B(0,1)} \nabla g(x+tz)\cdot z\, dS(z)$$

$$= \fint_{\partial B(x,t)} \nabla g(y)\cdot \left(\frac{y-x}{t}\right) dS(y)$$

Hence $$u(x,t) = \fint_{\partial B(x,t)} t h(y) + g(y) + \nabla g(y)\cdot(y-x)\, dS(y) \quad (x\in\mathbb{R}^3 \text{ \& } t>0) \quad \text{---} \boxed{A}$$

This formula $\boxed{A}$ is called the Kirchhoff's formula for (1) in $n=3$.

For, $0 \le r \le t$,

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy$$

now, $u(x,t) = \lim_{r \to 0^+} \dfrac{\tilde{U}(x,r,t)}{r}$. $\left(u(x,t) = \lim_{r \to 0^+} U(x;r,t) \right)$

$$= \lim_{r \to 0^+} \left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy \right]$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

$$\therefore\ u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g\,dS \right) + t \fint_{\partial B(x,t)} h\,dS$$