Advanced Partial Differential Equations
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Lecture 25
D'Alembert Formula

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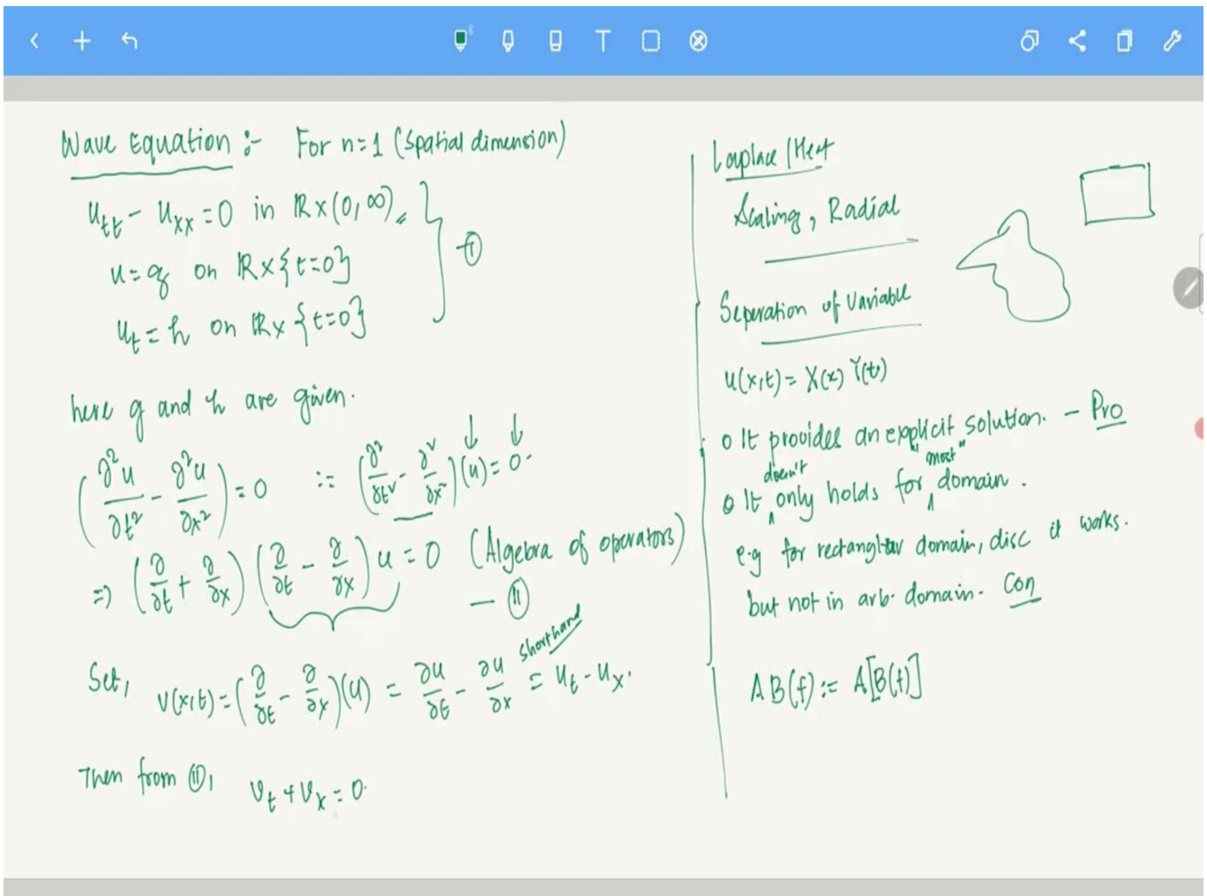

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$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R},\ t > 0) \quad \text{— (III)}$$

(III) is a transport equation and the solution of (III) is given

$$v(x,t) = a(x-t) \text{ where } a(x) := v(x,0)\ ;\ a \in C^1(\mathbb{R})$$

Again since, $v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)$

$$u_t - u_x = a(x-t) \text{ in } \mathbb{R} \times (0,\infty) \quad \text{— (IV)}$$

$$u(x,t) = \int_0^t a(x+t-2s)\,ds + b(x+t)$$

$$= \frac{1}{2}\int_{x-t}^{x+t} a(\xi)\,d\xi + b(x+t) \quad \text{where } b(x) := u(x,0)\ ;\ b \in C^1(\mathbb{R})$$

The initial condition gives

$$b(x) = g(x) \quad (x \in \mathbb{R})$$

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$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0)$$
$$= h(x) - g'(x)$$

Substituting a, b in (V) we have,

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \{h(\xi) - g'(\xi)\}\, d\xi + g(x+t)$$

and hence,

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\, d\xi \quad (x \in \mathbb{R}, t > 0) \quad \text{(Fundamental Th of Calculus)} \quad \to \text{(VI)}$$

(VI) is called the d'Alembert's formula.

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Check :- $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0,\infty)$ — (*) "Another way of deducing d'Alembert's Formulae".

$u(x,0) = g$; $x \in \mathbb{R}$

$u_t(x,0) = h$; $x \in \mathbb{R}$.

#

$a u_{xx} + b u_{xy} + c u_{yy} + L(u_x, u_y, u) = 0$ — (i) "Canonical Transformation"

$(x,y) \overset{\text{C.O.V}}{\mapsto} (\xi,\eta)$

Define, $w(\xi,\eta) := u(x,y)$. Assuming (i) is hyperbolic e.g $b^2 - 4ac > 0$. then (*) can be written as $w_{\xi\eta} = 0$ where

$\xi(x,t) = x + t$

$\eta(x,t) = x - t$

$\therefore$ $w(\xi,\eta) = A(\xi) + B(\eta)$ For some $A, B \in C^2(\mathbb{R})$

hence, $u(x,t) = A(x+t) + B(x-t)$

Put the I.C on $u(x,t)$ and deduce d' Alembert's Formulae.

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Th: Assume $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$ and define

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\, d\xi \quad (x \in \mathbb{R}, t > 0)$$

Then

(i) $u \in C^2(\mathbb{R} \times [0,\infty))$

(ii) $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0,\infty)$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u(x,t) = g(x^0)$ & $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t) = h(x^0)$.

Remark: In contrast with a Laplace / Heat Equation, the solution wave eqn is given by $u \in C^k$ provided $g \in C^k$ & $h \in C^{k-1}$.

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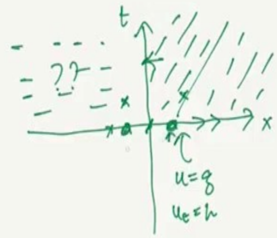
Reflection Method :- $(n=1)$

$$u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty)$$

$$u = g,\ u_t = h \quad \text{on } \mathbb{R}_+ \times \{t=0\}$$

$$u = 0 \quad \text{on the line } x = 0$$

where g and h are smooth with $g(0) = h(0) = 0$.



Odd Reflection (Extending the function smoothly)

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & (x \geq 0,\ t \geq 0) \\ -u(-x,t) & (x \leq 0,\ t \geq 0) \end{cases}$$

$$\lim_{x\to 0^+} g(x) = -\lim_{x\to 0^-} g(-x)$$

$$g(0) = 0$$

$$\tilde{g}(x) := \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x \leq 0. \end{cases}$$

$$\tilde{h}(x) := \begin{cases} h(x), & x \geq 0 \\ -h(-x); & x \leq 0. \end{cases}$$

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Note: $\tilde{u}_{tt} - \tilde{u}_{xx} = 0$ in $\mathbb{R}\times(0,\infty)$
$\tilde{u} = \tilde{g}, \ \tilde{u}_t = \tilde{h}$ on $\mathbb{R}\times\{t=0\}$] Please Verify. $(u_{tt} - c^2 u_{xx} = 0)$

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(\xi)\, d\xi$$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\, d\xi & \text{if } x \geq t \geq 0. \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(\xi)\, d\xi & \text{if } 0 \leq x \leq t. \end{cases} \quad \text{(VII)}$$

Remark: $g''(0) = 0$ for $u \in C^2$.

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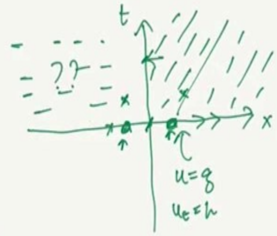

where g and h are smooth with $g(0)=h(0)=0$.

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