

**Advanced Partial Differential Equations**  
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**Lecture 25**  
**D'Alembert Formula**

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Wave Equation :- For  $n=1$  (Spatial dimension)

$$\left. \begin{aligned} u_{tt} - u_{xx} &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u &= g \text{ on } \mathbb{R} \times \{t=0\} \\ u_t &= h \text{ on } \mathbb{R} \times \{t=0\} \end{aligned} \right\} \textcircled{1}$$

here  $g$  and  $h$  are given.

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u = 0 \quad \because \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) (u) = 0$$

$\Rightarrow \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u = 0$  (Algebra of operators) —  $\textcircled{11}$

Set,  $v(x,t) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) (u) = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} \stackrel{\text{shorthand}}{=} u_t - u_x$ .

Then from  $\textcircled{11}$ ,  $v_t + v_x = 0$ .

Laplace (Heat)

Scaling, Radial

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Separation of Variable

$u(x,t) = X(x) \cdot T(t)$

- o It provides an explicit solution. — Pro
- o It <sup>doesn't</sup> only holds for <sup>most</sup> domain.
- e.g. for rectangular domain, disc it works.
- but not in arb. domain. — Con

$AB(f) := A[B(f)]$

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$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R}, t > 0) \quad \text{--- (iii)}$$

(iii) is a transport equation and the solution of (iii) is given

$$v(x,t) = a(x-t) \quad \text{where } a(x) := v(x,0) ; a \in C^1(\mathbb{R})$$

Again since,  $v(x,t) := \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) u(x,t)$

$$u_t - u_x = a(x-t) \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{--- (iv)}$$

$$u(x,t) = \int_0^t a(x+t-2s) ds + b(x+t)$$

$$= \frac{1}{2} \int_{x-t}^{x+t} a(\xi) d\xi + b(x+t) \quad \text{where } b(x) := u(x,0) ; b \in C^1(\mathbb{R})$$

The initial condition gives

$$b(x) = g(x) \quad (x \in \mathbb{R})$$

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$$a(x) = v(x|0) = u_t(x,0) - u_x(x|0) \\ = h(x) - g'(x)$$

Substituting a, b in (v) we have,

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} \{ h(s) - \underbrace{g'(s)}_{=} \} ds + \underline{g(x+t)}$$

and hence,

$$u(x,t) = \frac{1}{2} [ \underline{g(x+t)} + g(x-t) ] + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds \quad (x \in \mathbb{R}, t > 0) \quad \text{(Fundamental Th of Calculus)} \\ \rightarrow \text{(vi)}$$

(vi) is called the d'Alembert's formula.

$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R}, t > 0) \quad \text{--- (iii)}$$

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--- (v)

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$$b(x) = g(x) \quad (x \in \mathbb{R})$$

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Check :-  $u_{tt} - u_{xx} = 0$  in  $\mathbb{R} \times (0, \infty) \rightarrow (*)$  "Another way of deducing d'Alembert's Formula".  
 $u(x,0) = g; x \in \mathbb{R}$   
 $u_t(x,0) = h; x \in \mathbb{R}$ .

#  
 $a u_{xx} + b u_{xy} + c u_{yy} + L(u_x, u_y, u) = 0 \rightarrow (1)$  "Canonical Transformation"  
 $(x,y) \mapsto (\xi, \eta)$   
Define,  $w(\xi, \eta) := u(x,y)$ . Assuming (1) is hyperbolic e.g.  $b^2 - 4ac > 0$ . then (\*) can be written as  $w_{\xi\eta} = 0$  where  
 $\xi(x,t) = x+t$  ||  
 $\eta(x,t) = x-t$  ||

$\therefore w(\xi, \eta) = A(\xi) + B(\eta)$  For some  $A, B \in C^2(\mathbb{R})$   
hence,  $u(x,t) = A(x+t) + B(x-t)$  # Put the i.c on  $u(x,t)$  and deduce d'Alembert's Formula.

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In Assume  $g \in C^2(\mathbb{R})$  and  $h \in C^1(\mathbb{R})$  and define  
$$u(x,t) = \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(\xi) d\xi \quad (x \in \mathbb{R}, t \geq 0)$$

then (i)  $u \in C^2(\mathbb{R} \times [0, \infty))$

(ii)  $u_{tt} - u_{xx} = 0$  in  $\mathbb{R} \times (0, \infty)$

(iii)  $\lim_{\substack{(x,t) \rightarrow (x^*, 0) \\ t > 0}} u(x,t) = g(x^*) \quad \& \quad \lim_{\substack{(x,t) \rightarrow (x^*, 0) \\ t > 0}} u_t(x,t) = h(x^*)$

Remark: In contrast with a Laplace (Heat Equation, <sup>the solution</sup> wave eqn) is given by  
 $u \in C^k$  provided  $g \in C^k \quad \& \quad h \in C^{k-1}$ .

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Reflection Method :- ( $n=1$ )

$$u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty)$$

$$u = g, u_t = h \text{ on } \mathbb{R}_+ \times \{t=0\}$$

$$u = 0 \text{ on the line } x=0$$

where  $g$  and  $h$  are smooth with  $g(0) = h(0) = 0$ .

Odd Reflection (Extending the function smoothly)

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & (x \geq 0, t \geq 0) \\ -u(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

$$\tilde{g}(x) := \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x \leq 0 \end{cases}$$

$$\tilde{h}(x) := \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x \leq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} \tilde{u}(x) = - \lim_{x \rightarrow 0^-} \tilde{u}(x)$$

$$\tilde{u}(0) = 0$$

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Note:  $\tilde{u}_{tt} - \tilde{u}_{xx} = 0$  in  $\mathbb{R} \times (0, \infty)$  ] Please verify.  $(u_{0t} - c^2 u_{xx} = 0)$   
 $\tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h}$  on  $\mathbb{R} \times \{t=0\}$

$$\tilde{u}(x,t) = \frac{1}{2} [\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(s) ds$$
$$u(x,t) = \begin{cases} \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds & \text{if } x \geq b \gg 0. \\ \frac{1}{2} [g(x+t) - g(t-x)] + \frac{1}{2} \int_{-x+t}^{x+t} h(s) ds & \text{if } 0 \leq x \leq b. \end{cases} \quad \text{--- (VII)}$$

Remark:  $g''(0) = 0$  for  $u \in C^2$ .



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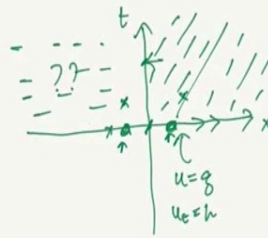
where  $g$  and  $h$  are smooth with  $g(0) = h(0) = 0$ .

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$$\lim_{x \rightarrow 0^+} g(x) = - \lim_{x \rightarrow 0^-} g(-x)$$

$$\underline{g(0) = 0}$$