Advanced Partial Differential Equations
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Lecture 24
Wave Equation: Physical Interpretation and Uniqueness

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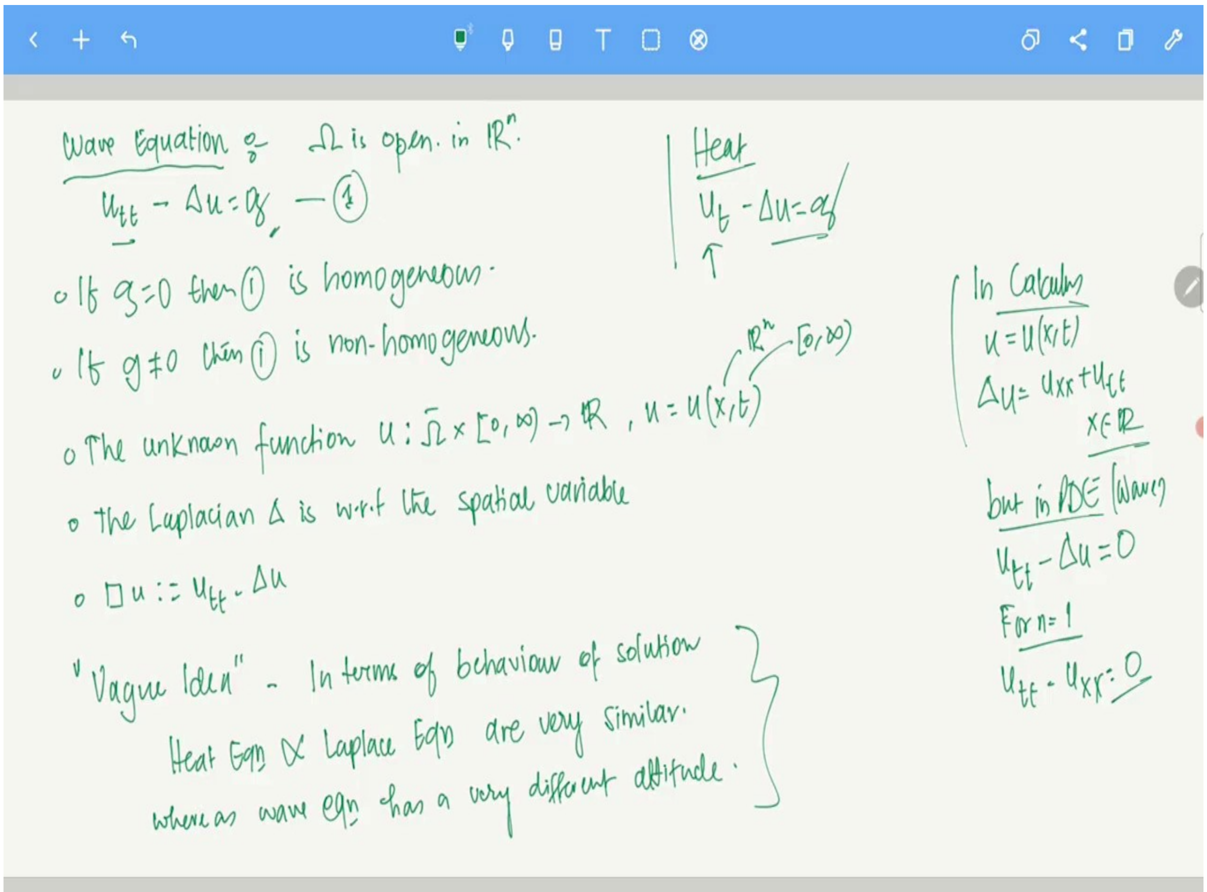

Welcome, students. This week we are going to talk about wave equation. So, this is the second important equation which we study in this course online once really and essentially this is also an evolution equation. So, let me write down the equation.

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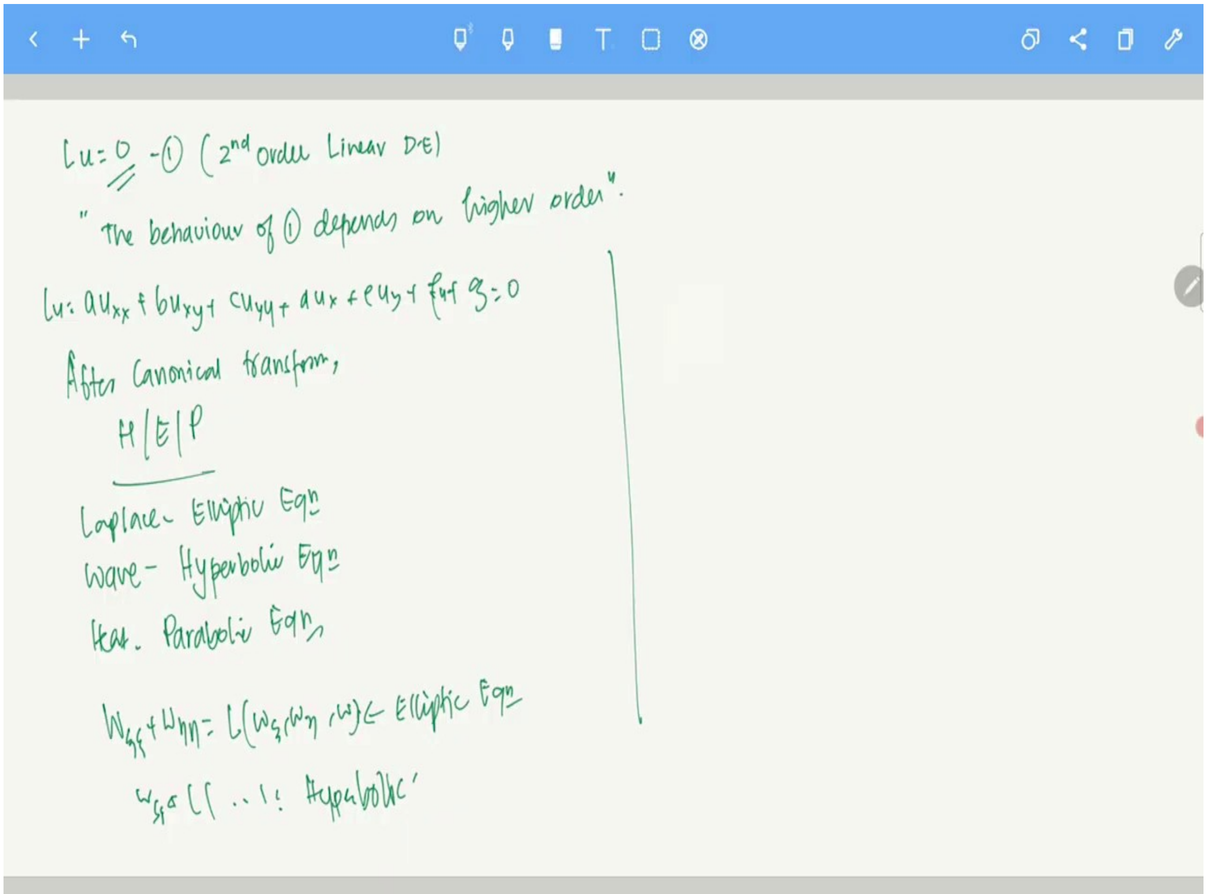

$Lu = 0$ — ① (2^{nd} order Linear D.E)

" The behaviour of ① depends on higher order".

$Lu: au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$

After Canonical transform,

H | E | P

Laplace - Elliptic Eqn

wave - Hyperbolic Eqn

Heat - Parabolic Eqn,

$w_{\xi\xi} + w_{\eta\eta} = L(w_\xi, w_\eta, w) \leftarrow$ Elliptic Eqn

$w_{\xi\eta} = L(\ldots)$: Hyperbolic

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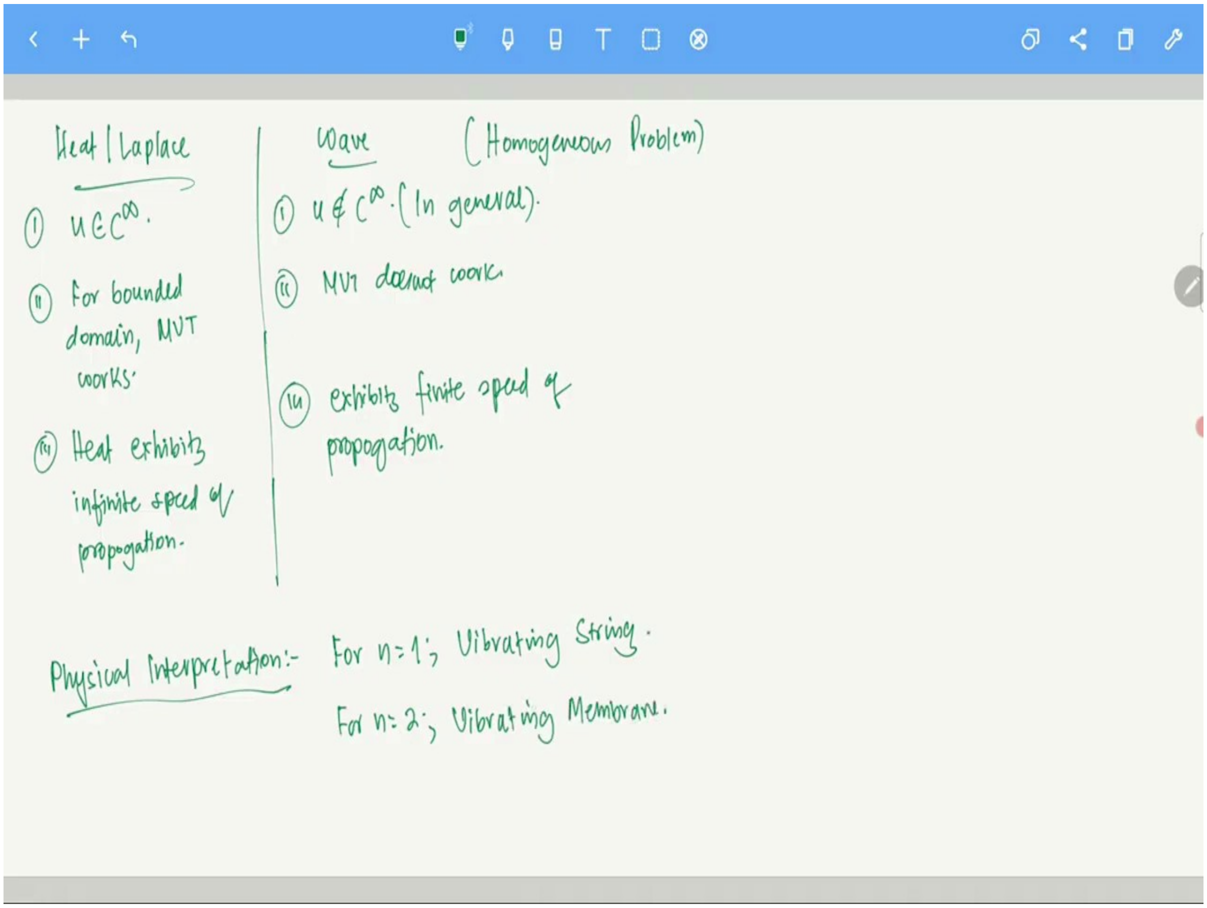

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"$u(x,t)$" represents the displacement in some direction of the pt 'x' at time $t \geq 0$.

Let V represents any smooth subregion of Ω.

Now, the acceleration of the wave within V is given by

$$\frac{d^2}{dt^2}\left(\int_V u \, dx\right) = \int_V u_{tt} \, dx$$

and, net force is $-\int_{\partial V} F \cdot \nu \, dS$. ($\nu$: unit outward normal)

F : Force acting on V through ∂V.

$\therefore$ Newton's law: $\int_V u_{tt} \, dx = -\int_{\partial V} F \cdot \nu \, dS \overset{G.D.T}{=} -\int_V \operatorname{div}(F) \, dx$

$\Rightarrow u_{tt} + \operatorname{div}(F) = 0$

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In general F is assumed to be proportional to the displacement gradient.

$F(\nabla u) \simeq -c\nabla u$.

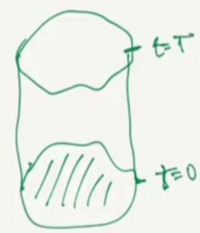
$\therefore u_{tt} - c\Delta u = 0$. (wave Eqn).

Common Strategy to study wave is look the well-posed.

Energy Method :- (Uniqueness of Solution)

Let $\Omega \, (\subseteq \mathbb{R}^n)$ be open, bounded domain with a smooth boundary.

also, $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \bar{\Omega}_T \setminus \Omega_T$; $T > 0$.



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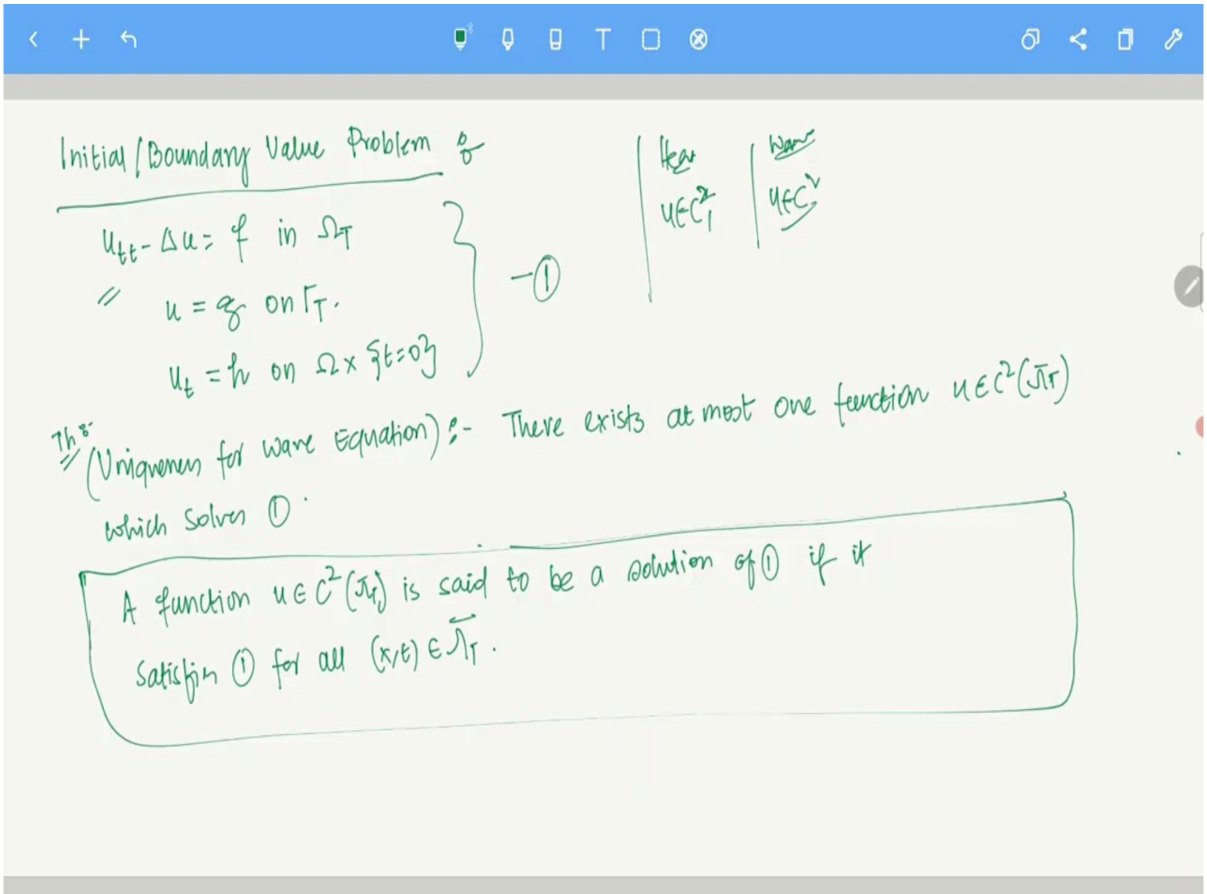

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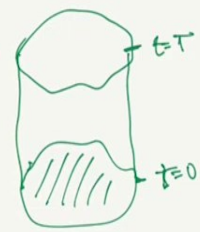

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Proof: Let $\tilde{u}$ be any other solution of (1).

Set, $w = u - \tilde{u}$. Then w solves

$$w_{tt} - \Delta w = 0 \quad \text{in } \Omega_T$$
$$w = 0 \quad \text{on } \Gamma_T$$
$$w_t = 0 \quad \text{on } \Omega \times \{t = 0\}.$$

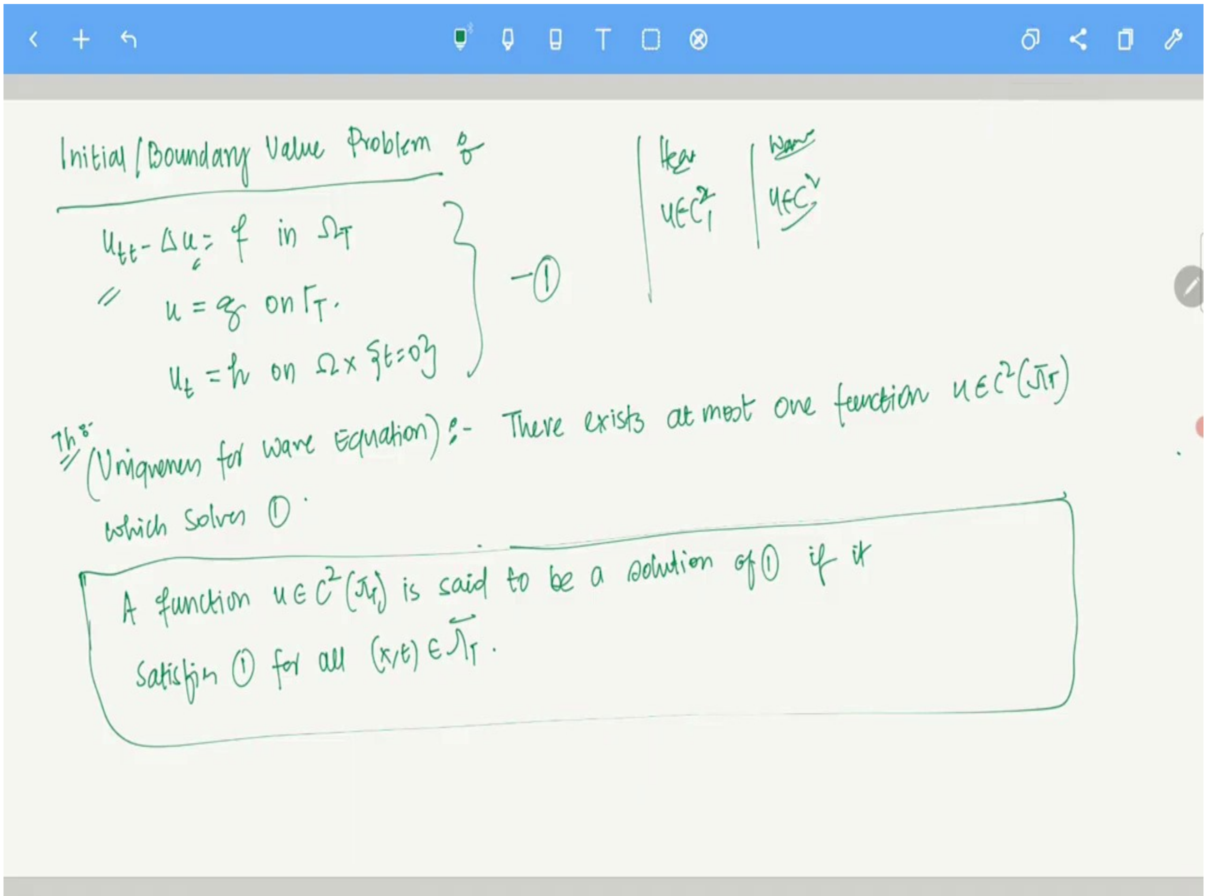
[Calculus of Variation: Evan's (PDE) Ch. 8/10.]

Define the energy: $E(t) := \frac{1}{2} \int_\Omega w_t^2(x,t) + |\nabla w(x,t)|^2 \, dx \quad (0 \le t \le T)$

Now, u & $\tilde{u}$ are $C^2(\Omega_T) \Rightarrow w \in C^2(\Omega_T)$

Clearly E is differentiable.

$$E'(t) = \int_\Omega w_t(x,t)\, w_{tt}(x,t)\, dx + \int_\Omega \nabla w \cdot \nabla w_t \, dx \overset{\text{I.B.P}}{=} \int_\Omega (w_t w_{tt} - w_t \Delta w)\, dx$$
$$= \int_\Omega w_t (w_{tt} - \Delta w)\, dx = 0.$$



Now, let us talk about uniqueness. Proof, what I am going to do here, for the proof of uniqueness, I am going to use something called a energy estimates which we already did in other cases also, in the case of Laplace, in the case of heat, here also we are going to do something similar.

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$\because w = 0$ on $\Gamma_T \Rightarrow w_t = 0$ on $\partial\Omega \times [0,T]$.

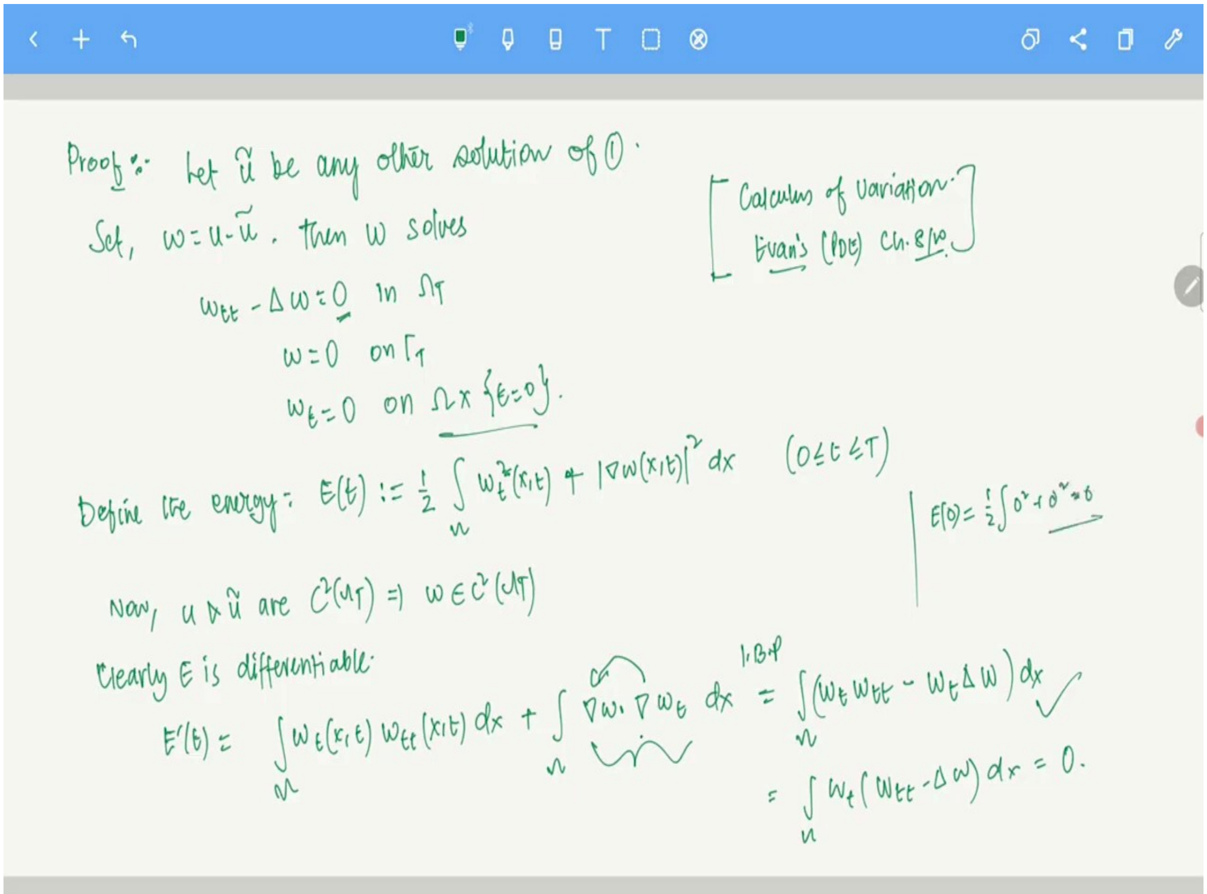
Thus for all $0 \leq t \leq T$, $E(t) = E(0) = 0$

and hence, w_t and $\nabla w \equiv 0$ within Ω_T.

$\because w = 0$ on $\Omega \times \{t = 0\} \Rightarrow w \equiv 0$ in $\Omega_T \Rightarrow w = u - \tilde{u}$ in Ω_T.

$= 0$

$\Rightarrow u = \tilde{u}$ in Ω_T.



And hence you have this, what we call, uniqueness. So, with this we are going to end this lecture.