

**Advanced Partial Differential Equations**  
**Professor Doctor Kaushik Bal**  
**Department of Mathematics and Statistics**  
**Indian Institute of Technology, Kanpur**  
**Lecture 24**

**Wave Equation: Physical Interpretation and Uniqueness**

(Refer Slide Time: 00:13)

Wave Equation  $\Omega$  is open in  $\mathbb{R}^n$ .

$$u_{tt} - \Delta u = g, \quad (1)$$

- If  $g=0$  then (1) is homogeneous.
- If  $g \neq 0$  then (1) is non-homogeneous.
- The unknown function  $u: \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$ ,  $u = u(x, t)$
- The Laplacian  $\Delta$  is w.r.t the spatial variable
- $\square u := u_{tt} - \Delta u$

"Vague Idea" - In terms of behaviour of solution  
 Heat Eqn & Laplace Eqn are very similar.  
 whereas wave eqn has a very different attitude.

Heat  
 $u_t - \Delta u = g$

In Calculus  
 $u = u(x, t)$   
 $\Delta u = u_{xx} + u_{tt}$   
 $x \in \mathbb{R}$

but in PDE (Wave)  
 $u_{tt} - \Delta u = 0$   
 For  $n=1$   
 $u_{tt} - u_{xx} = 0$

Welcome, students. This week we are going to talk about wave equation. So, this is the second important equation which we study in this course online once really and essentially this is also an evolution equation. So, let me write down the equation.

(Refer Slide Time: 08:21)

$Lu = 0$  - (1) (2<sup>nd</sup> order Linear D.E)

"The behaviour of (1) depends on 'higher order'."

$Lu = Au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + f = 0$

After Canonical transform,  
H/E/P

Laplace - Elliptic Eqn  
Wave - Hyperbolic Eqn  
Heat - Parabolic Eqn

$w_{\xi\xi} + w_{\eta\eta} = L(w_{\xi}, w_{\eta}, w) \leftarrow$  Elliptic Eqn  
 $w_{\xi\xi} = L(\dots) \leftarrow$  Hyperbolic'

$\underline{\underline{Lu=0}}$  - (1) (2<sup>nd</sup> order Linear D.E)

"The behaviour of (1) depends on 'higher order'."

$$Lu = au_{xx} + bu_{xy} + cu_{yy} + dx + ey + f \text{ for } g=0$$

After Canonical transform,

H/E/P

Laplace - Elliptic Eqn

wave - Hyperbolic Eqn

heat - Parabolic Eqn

$w_{\xi\xi} + w_{\eta\eta} = L(w_{\xi}, w_{\eta}, w) \leftarrow$  Elliptic Eqn

$w_{\xi\xi} = L(\dots) \leftarrow$  Hyperbolic

(Refer Slide Time: 14:08)

The image shows a digital whiteboard with handwritten notes comparing Heat/Laplace and Wave (Homogeneous Problem). The notes are organized into two columns separated by a vertical line. The left column is titled 'Heat/Laplace' and the right column is titled 'Wave (Homogeneous Problem)'. Below these columns, there is a section titled 'Physical Interpretation:-' with two entries: 'For n=1; Vibrating String.' and 'For n=2; Vibrating Membrane.'

Heat/Laplace

- (i)  $u \in C^\infty$ .
- (ii) For bounded domain, MVT works.
- (iii) Heat exhibits infinite speed of propagation.

Wave (Homogeneous Problem)

- (i)  $u \notin C^\infty$ . (In general).
- (ii) MVT does not work.
- (iii) exhibits finite speed of propagation.

Physical Interpretation:-

- For  $n=1$ ; Vibrating String.
- For  $n=2$ ; Vibrating Membrane.

(Refer Slide Time: 19:16)

" $u(x,t)$ " represents the displacement in some direction of the pt ' $x$ ' at time  $t \geq 0$ .

Let  $V$  represents any smooth subregion of  $\Omega$ .

Now, the acceleration of the wave within  $V$  is given by

$$\frac{d^2}{dt^2} \left( \int_V u \, dx \right) = \int_V u_{tt} \, dx$$

and, force is  $-\int_{\partial V} F \cdot \nu \, dS$ . ( $\nu$ : unit outward normal)

$F$  = Force acting on  $V$  through  $\partial V$ .

$$\begin{aligned} \therefore \text{Newton's law: } \int_V u_{tt} \, dx &= - \int_{\partial V} F \cdot \nu \, dS \stackrel{\text{G-D-T}}{=} - \int_V \operatorname{div}(F) \, dx \\ &\Rightarrow u_{tt} + \operatorname{div}(F) = 0 \end{aligned}$$



(Refer Slide Time: 25:59)

In general  $F$  is assumed to be proportional to the displacement gradient:

$$F(\nabla u) \approx -C \nabla u.$$

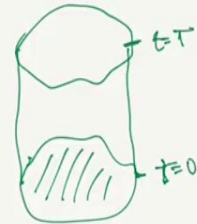
$$\therefore u_{tt} - c \Delta u = 0; \quad (\text{Wave Eq}).$$

\* Common Strategy to study wave is look the well-posed.

Energy Method :- (Uniqueness of solution)

Let  $\Omega \in \mathbb{R}^n$  be open, bounded domain with a smooth boundary.

also,  $\partial_T = \Omega \times (0, T] \Rightarrow \Gamma_T = \overline{\partial_T} \setminus \Omega_T ; T > 0.$



(Refer Slide Time: 31:55)

Initial/Boundary Value Problem  $\varphi$

$$\left. \begin{aligned} u_{tt} - \Delta u &= f \text{ in } \Omega_T \\ u &= g \text{ on } \Gamma_T \\ u_t &= h \text{ on } \Omega \times \{t=0\} \end{aligned} \right\} \text{--- (1)}$$

$\left. \begin{array}{l} \text{Heat} \\ u \in C^2_1 \end{array} \right\} \left| \begin{array}{l} \text{Wave} \\ u \in C^2 \end{array} \right.$

Th<sup>o</sup> (Uniqueness for wave Equation) :- There exists at most one function  $u \in C^2(\bar{\Omega}_T)$  which solves (1).

A function  $u \in C^2(\bar{\Omega}_T)$  is said to be a solution of (1) if it satisfies (1) for all  $(x,t) \in \bar{\Omega}_T$ .

In general  $F$  is assumed to be proportional to the displacement gradient:

$$F(\nabla u) \approx -c \nabla u.$$

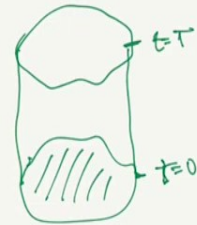
$$\therefore u_{tt} - c \Delta u = 0; \quad (\text{Wave Eq}).$$

\* Common Strategy to study wave is look the well-posed.

Energy Method :- (Uniqueness of Solution)

Let  $\Omega \in \mathbb{R}^n$  be open, bounded domain with a smooth boundary.

$$\text{also, } \partial_T = \Omega \times (0, T] \Rightarrow \Gamma_T = \overline{\partial_T} \setminus \partial_T; \quad T > 0.$$





(Refer Slide Time: 36:56)

Proof:- let  $\tilde{u}$  be any other solution of (1).

Set,  $w = u - \tilde{u}$ . then  $w$  solves

$w_{tt} - \Delta w = 0$  in  $\Omega_T$

$w = 0$  on  $\Gamma_T$

$w_t = 0$  on  $\Omega \times \{t=0\}$ .

[Calculus of Variations -  
Evan's (10e) Ch. 8/10.]

Define the energy:  $E(t) := \frac{1}{2} \int_{\Omega} w_t^2(x,t) + |\nabla w(x,t)|^2 dx \quad (0 \leq t \leq T)$

Now,  $u$  &  $\tilde{u}$  are  $C^2(\Omega_T) \Rightarrow w \in C^2(\Omega_T)$

Clearly  $E$  is differentiable:

$$E'(t) = \int_{\Omega} w_t(x,t) w_{tt}(x,t) dx + \int_{\Omega} \nabla w_t \cdot \nabla w_t dx \stackrel{\text{IBP}}{=} \int_{\Omega} (w_t w_{tt} - w_t \Delta w) dx$$
$$= \int_{\Omega} w_t (w_{tt} - \Delta w) dx = 0.$$

Initial/Boundary Value Problem of

$$\left. \begin{aligned} u_{tt} - \Delta u &= f \text{ in } \Omega_T \\ u &= g \text{ on } \Gamma_T \\ u_t &= h \text{ on } \Omega \times \{t=0\} \end{aligned} \right\} \text{--- (1)}$$

Heat  $u \in C^2$  | Wave  $u \in C^2$

Thm (Uniqueness for wave Equation):- There exists at most one function  $u \in C^2(\bar{\Omega}_T)$  which solves (1).

A function  $u \in C^2(\bar{\Omega}_T)$  is said to be a solution of (1) if it satisfies (1) for all  $(x,t) \in \bar{\Omega}_T$ .

Now, let us talk about uniqueness. Proof, what I am going to do here, for the proof of uniqueness, I am going to use something called a energy estimates which we already did in other cases also, in the case of Laplace, in the case of heat, here also we are going to do something similar.

(Refer Slide Time: 46:11)

The image shows a digital notepad with a blue header bar containing navigation icons. The main area is a light green background with handwritten text in green ink. The text is as follows:

$\because w = 0$  on  $\Gamma_f \Rightarrow w_t = 0$  on  $\partial\Omega \times [0, T]$ .

Thus for all  $0 \leq t \leq T$ ,  $E(t) = E(0) = 0$

and hence,  $w_t$  and  $\nabla w = 0$  within  $\Omega$ .

$\because w = 0$  on  $\Omega \times \{t=0\} \Rightarrow w = 0$  in  $\Omega \Rightarrow w = u - \tilde{u}$  in  $\Omega$ .

$= 0$

$\Rightarrow u = \tilde{u}$  in  $\Omega$ .

Proof: let  $\tilde{u}$  be any other solution of (1).

Set,  $w = u - \tilde{u}$ . Then  $w$  solves

$$w_{tt} - \Delta w = 0 \text{ in } \Omega_T$$

$$w = 0 \text{ on } \Gamma_T$$

$$w_t = 0 \text{ on } \Omega \times \{t=0\}.$$

[Calculus of Variations]  
Evan's (10e) Ch. 8/10.]

Define the energy:  $E(t) := \frac{1}{2} \int_{\Omega} w_t^2(x,t) + |\nabla w(x,t)|^2 dx \quad (0 \leq t \leq T)$

$E(0) = \frac{1}{2} \int_{\Omega} 0^2 + 0^2 dx = 0$

Now,  $u$  &  $\tilde{u}$  are  $C^2(\Omega_T) \Rightarrow w \in C^2(\Omega_T)$

Clearly  $E$  is differentiable:

$$E'(t) = \int_{\Omega} w_t(x,t) w_{tt}(x,t) dx + \int_{\Omega} \nabla w_t \cdot \nabla w_t dx \stackrel{\text{I.B.P}}{=} \int_{\Omega} (w_t w_{tt} - w_t \Delta w) dx$$

$$= \int_{\Omega} w_t (w_{tt} - \Delta w) dx = 0.$$

And hence you have this, what we call, uniqueness. So, with this we are going to end this lecture.