Advanced Partial Differential Equations
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Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Lecture 23
Maximum Principle for Heat Equation

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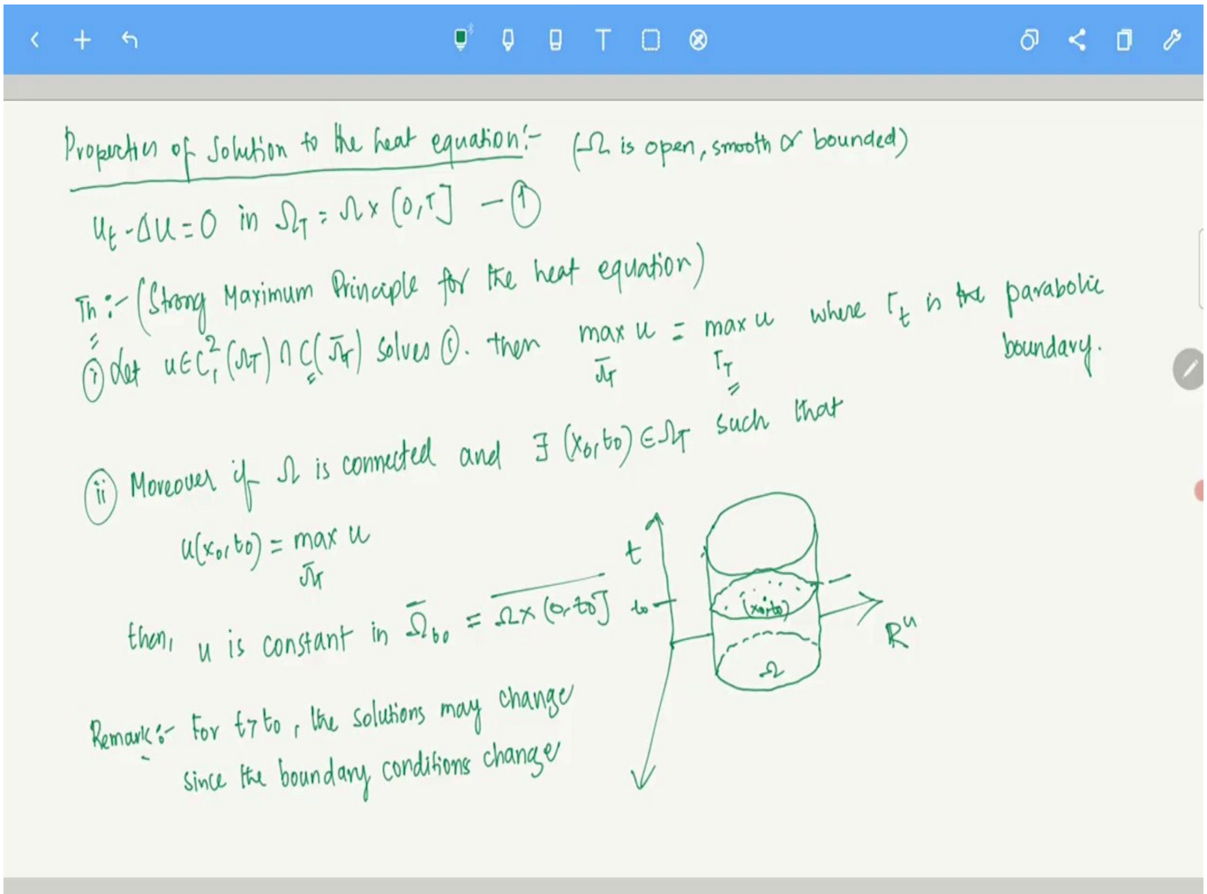

Welcome, students. In today’s class or in this video, essentially, we are going to talk about the consequences of mean value property for the heat operator. So, essentially, what we are going to do is look at some properties of the heat equation based on the mean value theorem.

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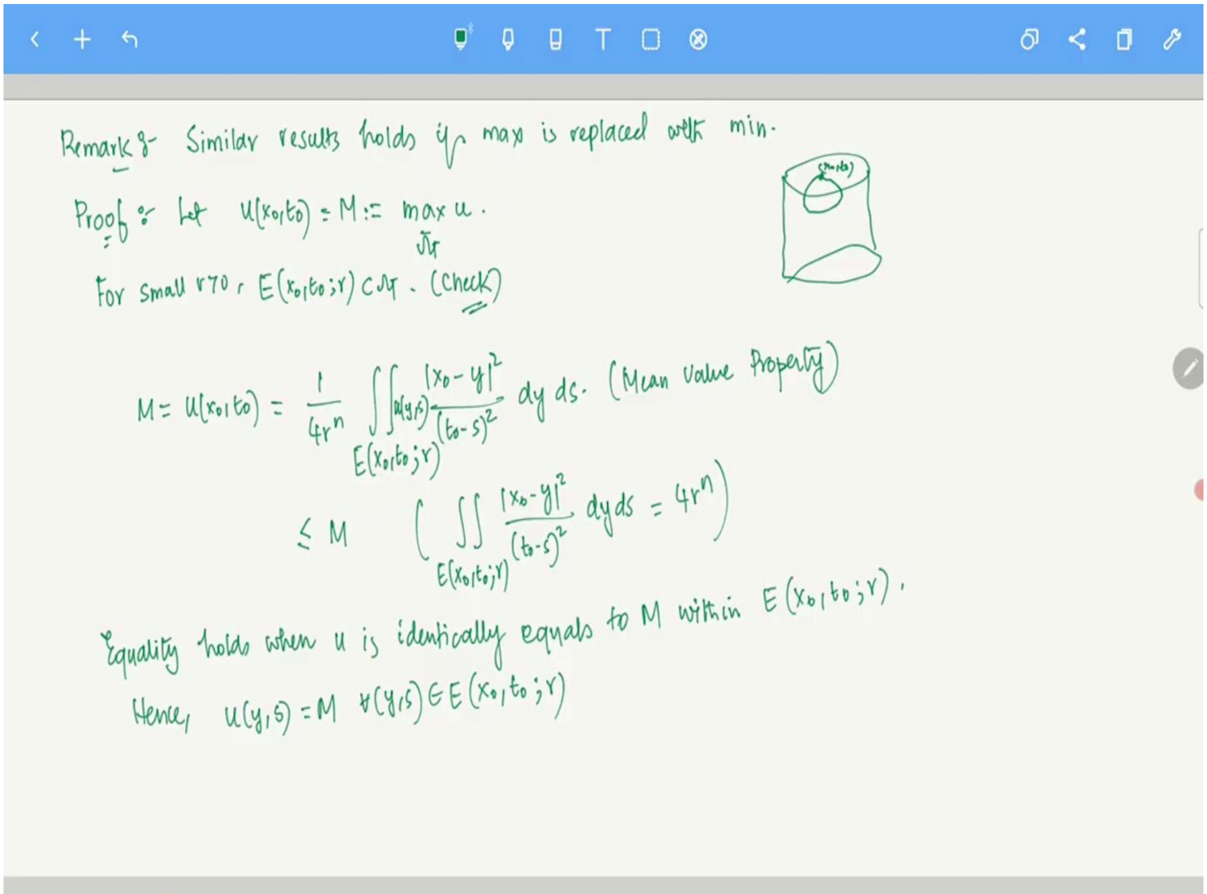


Properties of Solution to the heat equation:- (Ω is open, smooth or bounded)

$u_t - \Delta u = 0$ in $\Omega_T = \Omega \times (0, T]$ — (1)

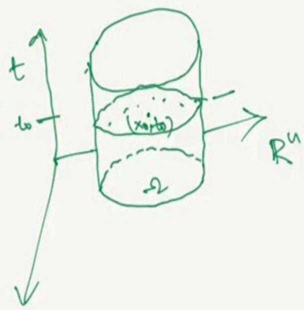
Th:- (Strong Maximum Principle for the heat equation)

(i) Let $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ solves (1). Then $\max\limits_{\bar{\Omega}_T} u = \max\limits_{\Gamma_T} u$ where Γ_t is the parabolic boundary.

(ii) Moreover if Ω is connected and $\exists\, (x_0, t_0) \in \Omega_T$ such that

$$u(x_0, t_0) = \max_{\bar{\Omega}_T} u$$

then, u is constant in $\bar{\Omega}_{t_0} = \overline{\Omega \times (0, t_0]}$



Remark:- For $t > t_0$, the solutions may change since the boundary conditions change

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In Ω_T, draw any line segment connecting (x_0, t_0) with some $(y_0, s_0) \in \Omega_T$ for $s_0 < t_0$.

Set, $r_0 := \min\{ s \geq s_0 \mid u(x,t) = M$ for all points $(x,t) \in L$ & $s \leq t \leq t_0 \}$.

($\because$ u is continuous hence the minimum exists).

Let $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point (z_0, r_0) in $L \cap \Omega_T$

and so, $u \equiv M$ on $E(z_0, r_0; r)$ for small $r > 0$.

$\because$ $E(z_0, r_0; r)$ contains $L \cap \{ r_0 - \delta \leq t \leq r_0 \}$ for some small $\delta > 0$.

- which is a contradiction.

Hence, $r_0 = s_0$. and so $u \equiv M$ on L.

Now, fix $x \in \Omega$ and $0 \leq t < t_0$. $\exists \{x_0, x_1, \ldots, x_m = x\}$ s.t the line segment joining x_{i-1} to x_i must lie on Ω. Select time $t_0 > t_1 > \ldots > t_m = t$. Then the line segments in $\mathbb{R}^{n+1}$ connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) must lie in Ω_T.

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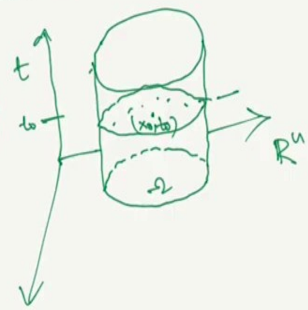

Remark:- For $t > t_0$, the solutions may change since the boundary conditions change.

Remark :- Similar results holds if max is replaced with min.

Proof :- Let $u(x_0,t_0) = M := \max\limits_{\Omega_T} u$.

For small $r > 0$, $E(x_0,t_0;r) \subset \Omega_T$. (Check)

$$M = u(x_0,t_0) = \frac{1}{4r^n} \iint\limits_{E(x_0,t_0;r)} u(y,s) \frac{|x_0-y|^2}{(t_0-s)^2}\, dy\, ds. \quad \text{(Mean value Property)}$$

$$\leq M \quad \left(\iint\limits_{E(x_0,t_0;r)} \frac{|x_0-y|^2}{(t_0-s)^2}\, dy\, ds = 4r^n \right)$$

Equality holds when u is identically equals to M within $E(x_0,t_0;r)$.

Hence, $u(y,s) = M \quad \forall (y,s) \in E(x_0,t_0;r)$

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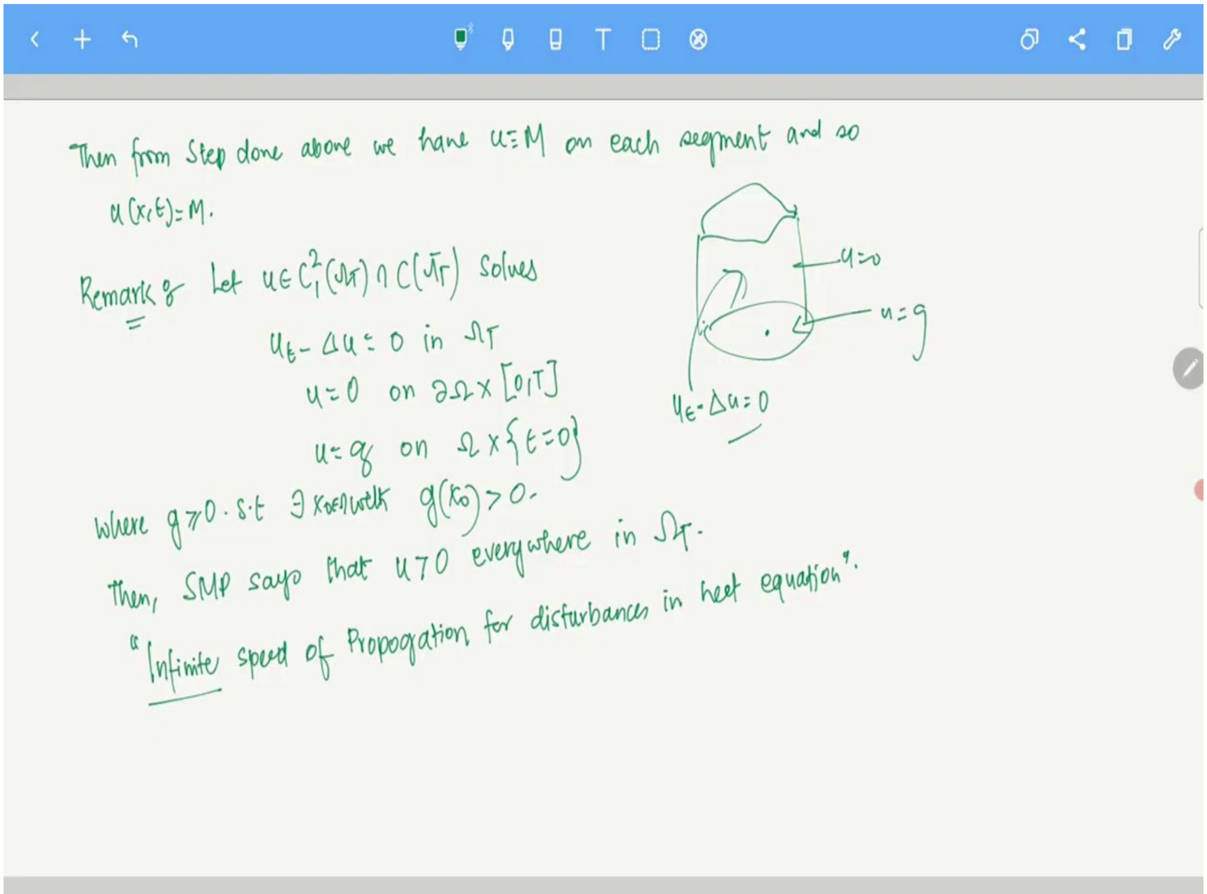

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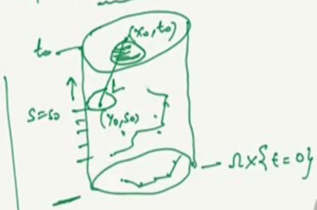

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Uniqueness of the Heat equation:-

Let $g \in C(\Gamma_T)$ and $f \in C(\Omega_T)$. Then $\exists$ at most one solution $u \in C^2_1(\Omega_T) \cap C(\bar{\Omega}_T)$ of the problem

$$\left.\begin{aligned} u_t - \Delta u &= f \text{ in } \Omega_T. \\ u &= g \text{ on } \Gamma_T. \end{aligned}\right\} - (1)$$

Proof:- D.I.Y.

Question:- Maximum Principle for the Cauchy Problem in $\mathbb{R}^n$??

Remark:- Of course $u = 0$ is always a soln of $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,T)$

$u = 0$ on $\mathbb{R}^n \times \{t = 0\}$

but, one can show all other solutions grows rapidly as $|x| \to \infty$.

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Maximum Principle in $\mathbb{R}^n$:-

Let $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

and satisfies the growth estimate,

$$u(x,t) \leq C e^{\alpha |x|^2} \quad (x \in \mathbb{R}^n \;\&\; 0 \leq t \leq T)$$

for some constants $C, \alpha > 0$.

Then, $\displaystyle \sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$.

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but, one can show all other solutions grows rapidly as $|x| \to \infty$.

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Proof:- Assume, $4\alpha T < 1$ then $\exists \epsilon > 0$ s.t $4\alpha(T+\epsilon) < 1$.

Fix $y \in \mathbb{R}^n$ & $\mu > 0$ and define

$$v(x,t) := u(x,t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{-\frac{|x-y|^2}{4(T+\epsilon-t)}} \quad (x \in \mathbb{R}^n, t > 0)$$

Clearly, $v_t - \Delta v = 0$ (Using linearity of the heat operator)

Fix $r > 0$ and set $\Omega := B(y,r)$ then $\Omega_T = B(y,r) \times (0,T]$.

Hence, $\max_{\bar{\Omega}_T} v = \max_{\Gamma_T} v$.

Now if $x \in \mathbb{R}^n$, $v(x,0) = u(x,0) - \frac{\mu}{(T+\epsilon)^{n/2}} e^{-\frac{|x-y|^2}{4(T+\epsilon)}} < u(x,0) = g(x)$

and, if $|x-y| = r$, $0 \le t \le T$ then,

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$$v(x,t) = u(x,t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{-\frac{r^2}{4(T+\epsilon-t)}}$$

$$\leq Ce^{\alpha|x|^2} - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{-\frac{r^2}{4(T+\epsilon-t)}}$$

$$\leq C e^{\alpha(|y|+r)^2} - \frac{\mu}{(T+\epsilon)^{n/2}} e^{-\frac{r^2}{4(T+\epsilon)}}$$

Now, $4\alpha(T+\epsilon) < 1$ hence, $\frac{1}{4(T+\epsilon)} = \alpha + \gamma$ for some $\gamma > 0$.

$$\therefore v(x,t) \leq Ce^{\alpha(|y|+r)^2} - \mu\left[4(\alpha+\gamma)\right]^{\frac{n}{2}} e^{(\alpha+\gamma)r^2}.$$

$$\leq \sup_{\mathbb{R}^n} g. \quad \text{(If } r \text{ is selected sufficiently large)} \quad \text{IV}$$

Maximum Principle in $\mathbb{R}^n$:-

Let $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

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and satisfies the growth estimate,

$$u(x,t) \leq C e^{\alpha|x|^2} \quad (x \in \mathbb{R}^n, 0 \leq t \leq T)$$

for some constants $C, \alpha > 0$.

Then, $\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$

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Clearly, $v_t - \Delta v = 0$ (Using Linearity of the heat operator)

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Hence, $\max\limits_{\bar{\Omega}_T} v = \max\limits_{\Gamma_T} v$. —— (11)

Now if $x \in \mathbb{R}^n$, $v(x,0) = u(x,0) - \frac{\mu}{(T+\epsilon)^{n/2}} e^{-\frac{|x-y|^2}{4(T+\epsilon)}} < u(x,0) = g(x)$ —— (14)

and, if $|x-y| = r$, $0 \le t \le T$ then,

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By (II), (III) & (IV) one has,

$$v(y,t) \le \sup_{\mathbb{R}^n} g \quad \forall\, y \in \mathbb{R}^n \;\&\; t \in [0,T].$$

Let, $\mu \to 0$; $\sup\limits_{\mathbb{R}^n \times [0,T]} u = \sup\limits_{\mathbb{R}^n} g$.

If $4aT \not< 1$ then, $\left[0, \frac{1}{8a}\right], \left[\frac{1}{8a}, \frac{1}{4a}\right], \dots$

Apply the above result in each interval

As a corollary, $u_t - \Delta u = f$ in $\mathbb{R}^n \times (0,T]$ (Uniqueness in $\mathbb{R}^n$)

$u = g$ on $\mathbb{R}^n \times \{t = 0\}$

has atmost one solution, $|u(x,t)| \le Ce^{a|x|^2}$ $\forall\, x \in \mathbb{R}^n$ & $t \in (0,T]$.

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{-\frac{r^2}{4(T+\varepsilon-t)}}$$

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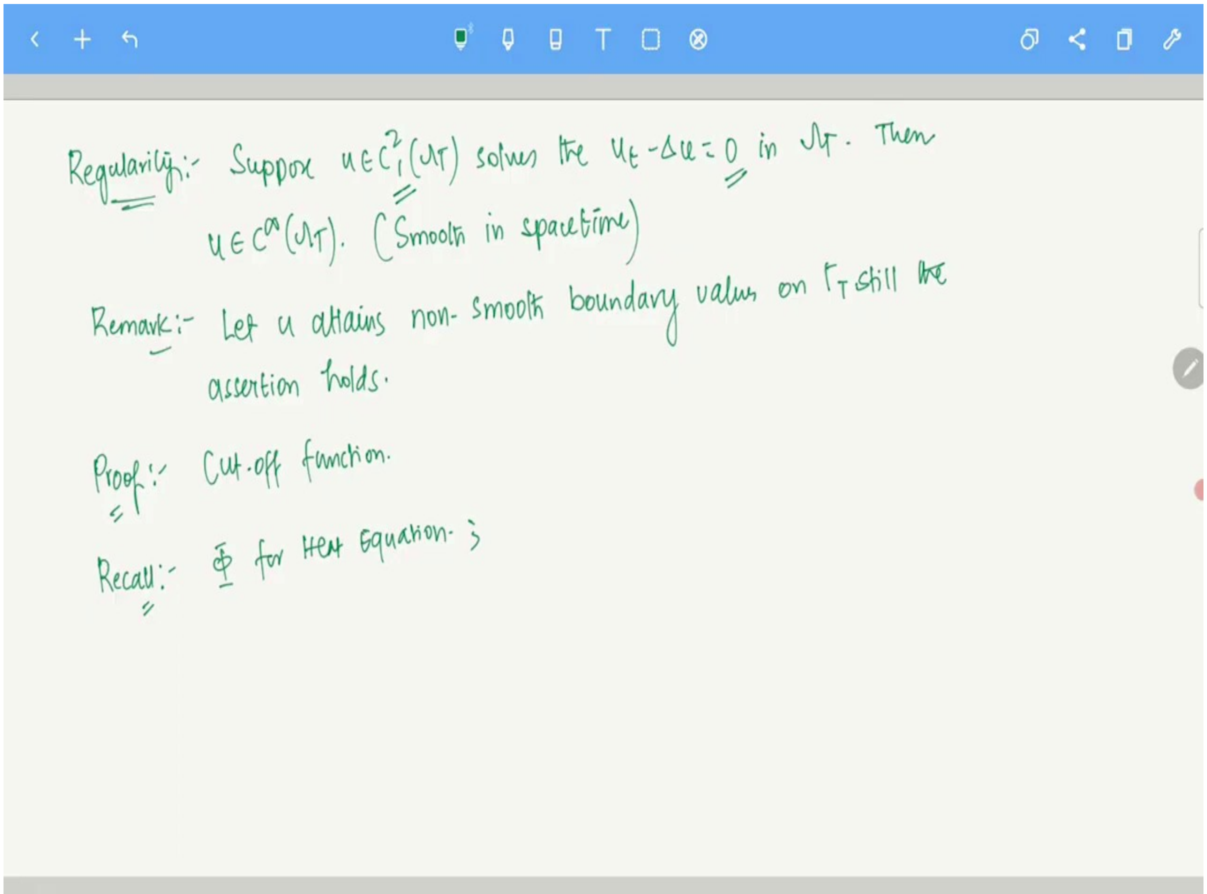

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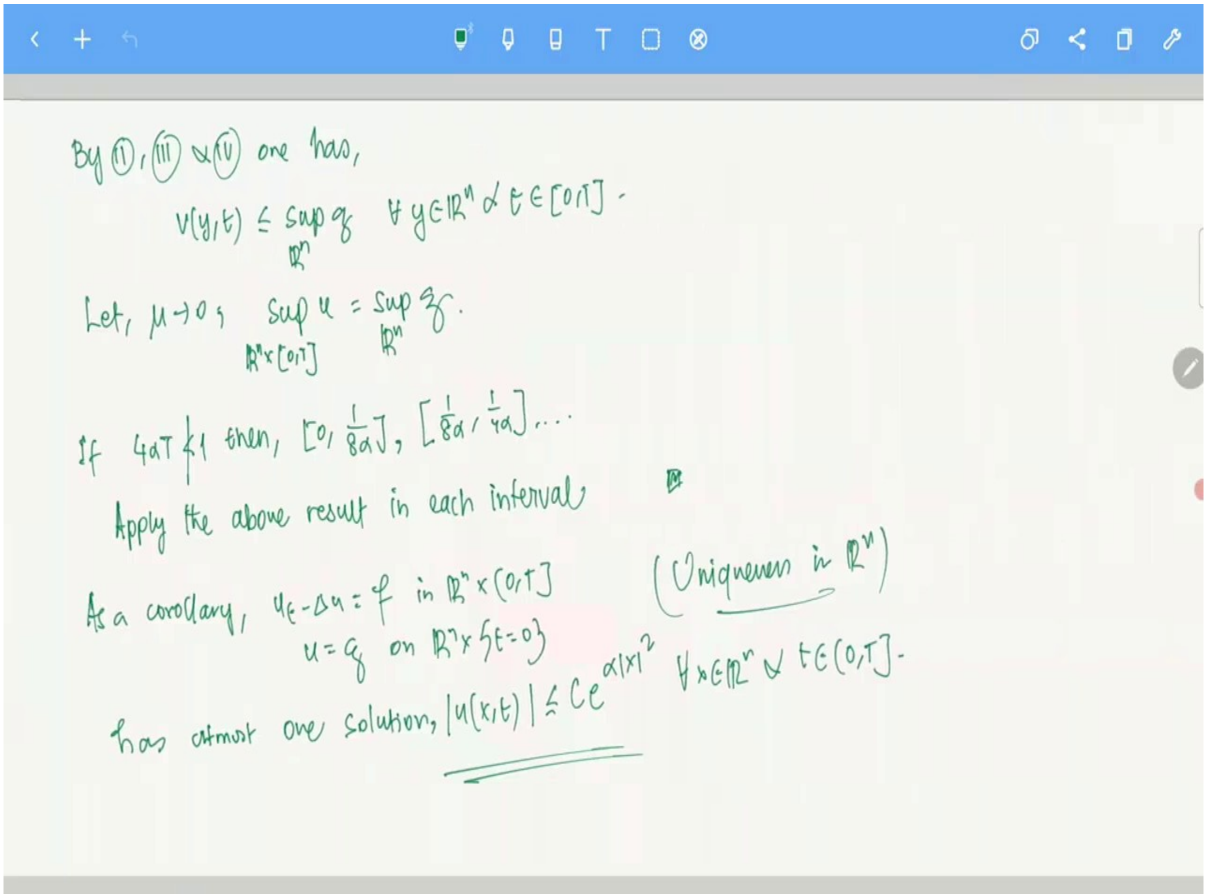

Then we proved a very important, but kind of difficult mean value property, which is not essentially your mean value, but we defined a kind of heat ball and we showed that in that ball you can do all sort of thing that the mean value holds in a different set. And then today we proved that strong maximum principle holds, uniqueness holds and you can use it to even settle the well-poseness problem, just, these ideas are exactly the same as we did for Laplacian. So, I am not really going deep into all that. So, all of this is done. So, with this we are going to finish the heat equation part of this course. Thank you very much.