Advanced Partial Differential Equations Professor Doctor Kaushik Bal Department of Mathematics and Statistics Indian Institute of Technology Kanpur Lecture 23 Maximum Principle for Heat Equation

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Welcome, students. In today's class or in this video, essentially, we are going to talk about the consequences of mean value property for the heat operator. So, essentially, what we are going to do is look at some properties of the heat equation based on the mean value theorem.

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$\left(\begin{array}{c}\n e & + & 5 \\ \hline\n 0 & 0 & \fline \\ 1 & 0 & \fline \\ 2 & 0 & \fline \\ 3 & 0 & \fline \\ 4 & 0 & \fline \\ 5 & 0 & \fline \\ 6 & 0 & \fline \\ 7 & 0 & 0 \\ 7 & 0 & \fline \\ 7 & 0 & 0 \\ 8 & 0 & \fline \\ 7 & 0 & 0 \\ 9 & 0 & 0 \\ 1 & 0 & \fline \\ 8 & 0 & 0 \\ 1 & 0 & \fline \\ 9 & 0 & 0 \\ 1 & 0 & \fline \\ 1 & 0 & 0 \\ 1 & 0 & \fline \\ 1 & 0 & 0 \\ 0 & 0 & 0$

$\begin{array}{ccccccccccccccccc} \mathbb{Q}^1 & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{T} & \mathbb{Q} & \mathbf{0} \end{array}$

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\sigma \prec \sigma \not
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Python to Jolution to the had equation:	(1. is open, smooth of bounded)
\n $u_t - \Delta u = 0$ in $\Omega_t = \Omega \times (0, \tau) - \Omega$ \n	
\n π = (Ω from the binacable by the left equation)	
\n π = $\frac{1}{\pi}$ in the parabolic of the $ueC_t^2(\Omega_T) \Omega(\overline{\Omega_T})$ (slow a 0: from $\frac{1}{\pi}$ in the right of the $ueC_t^2(\Omega_T) \Omega(\overline{\Omega_T})$ (slow a 0): $\frac{1}{\pi}$ in the right $\frac{1}{\pi}$ from the right-hand $\frac{1}{\pi}$ in the right-hand $\frac{1}{\$	

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k + 5
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ln 10\tau, draw any line argument connecting (k_0t_0) with some (y_0, y_0) $ln \tau$ is given.
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So, r_0r = min \{5.760 |u(x_0) = N \text{ for all points } (k, t) \in L \text{ if } S \le t \le t_0\}.
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c : u \text{ is continuous, } k \le t
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ln 10\tau, draw any line. The minimum (k_0t_0) with the maximum (k_0t_0). The maximum (k_0t_0) is given by the maximum (k_0t_0). The maximum (k_0t_0) with the maximum (k_0) with the maximum (k
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$\begin{array}{cccccccccccccc} \mathbb{Q}^{\dagger} & \mathbb{Q} & \mathbb{Q} & \mathbb{T} & \mathbb{C} & \mathbb{Q} & \mathbb{Q} \end{array}$

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Property 10 of Solution to the fact equation:	(1. is open, smooth or bounded)
\n $u_f - \Delta U = 0$ in $\Omega_f = \Omega \times (0, f) - \Omega$ \n	\n π : (Strong Maximum Principle, for the text equation)
\n π : (Strong Maximum Principle, for the text equation)	\n π to the next value of f to the boundary of Ω and Ω to the first value of f to the boundary of Ω and Ω .\n
\n π to the second law of Ω to the second law of Ω and Ω .\n	\n π to the second law of Ω and Ω .\n
\n $u(x_0, b_0) = max$ to the solution of Ω to the second law of Ω .\n	\n π to the solution of Ω and Ω .\n
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 $\begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{array}$ Q^* Q Q T Q Q Then from Step done above we have u=M on each segment and so $U(x,\epsilon)$ = M. Remark 8 Let ue C, (11) n C (Jr) solves $-4 - 0$ u_{t-} auto in π
 $u=0$ on θ or $[0.7]$ $M = G$ \overline{a} We can come on 970.5 to 10^{17} $u=0$ on 92×617
where 970.5 to 9 Koepwelk $9(k_0) > 0$.
Then, SMP says that $u = 0$ everywhere in 94 .
Then, SMP says that $u = 0$ everywhere in 94 .
"Infinite speed of Propoogatio

$\mathbf{U}^{\dagger} \quad \mathbf{U} \quad \mathbf{U} \quad \mathbf{T} \quad \mathbf{U} \quad \mathbf{0}$

In Jt, draw any line segment connecting (X0, to) with Rome (
$$
Y_{0}.x_{0}
$$
) $6.9F$ from.
\nSet. $Y_{0}.x$ min² \$750 | $U(x,0)$: M, for all points $(x,t) \in L$ K $5 \le t \le t_{0}$ }.
\n $(\because u$ is continuous than the the minimum exists).
\nLet $Y_{0}.750$. Then $U(20, Y_{0}) = M$ for some point $(Z_{0}.Y_{0})$ in L.0.91
\nand a0, $U \equiv M$ on $E(E_{0}.Y_{0}^{0.5})$ for small 970.
\n $\therefore E(Z_{0}.Y_{0}^{0.5})$ contains L. $1 \leq Y_{0} - \frac{1}{2} \leq t \leq Y_{0}$ for some small \$770 -
\n \therefore Which is a contradiction.
\nHence, $Y_{0} = S_{0}$. and a0 $U \neq M$ on L;
\nNow, $\lim_{x \to 1} X \in \Omega$ and $0 \le t \le b_{0}$. If $\frac{1}{2} X_{0}.Y_{1} \dots Y_{m} = X^{2}$ $S \neq b_{0}$ *In the argument* joining
\n $X_{0} = 1$ and $0 \le t \le b_{0}$. If $\frac{1}{2} X_{0}.Y_{1} \dots Y_{m} = X^{2}$ *S \neq b_{0} In the argument*

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 $Q^* Q Q T Q Q Q$ $\begin{array}{ccccccc}\n\circ & & & \circ & & \circ & & \circ\n\end{array}$ Uniqueness of the Heat equation:

The gc C(Ft) and $f \in C(M)$. Then $\frac{3}{5}$ at most one solution $u \in C^2(W) \cap C(M)$

Let gc C(Ft) and $f \in C(M)$. Then $\frac{3}{5}$ Uniqueness of the Heat equation: we generally $u_t = \Delta u = \frac{4}{v}$ in v_t . $\frac{2}{v} = 0$
of the problem $u_t = \frac{4}{v}$ on Γ_t . $Proofs$ $D.19$. Proof's D.1.4.
Question: Maximum Principle for the Cauchy Problem in R" ??
Question: Maximum Principle for the Cauchy or aoths of ue-au=0 Question: Maximum Principle for the Cauchy Problem in the i.
Remark 8 Of course $u=0$ is always a poly of u_0 and u_0 on $\mathbb{R}^n \times (0,5)$
Remark 8 Of course $u=0$ is always a poly of u_0 on $\mathbb{R}^n \times \{e=0\}$
but, mayk of course $u=0$ is arrowing of rows rapidly as $(x \rightarrow \infty)$
 $=$ but, one can show all other pointions of rows rapidly as $(x \rightarrow \infty)$.

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$\begin{array}{ccccccccccccccccc} \mathbb{Q}^1 & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{T} & \mathbb{C} & \mathbf{0} & \mathbf{0} \end{array}$

 $\begin{array}{ccccccc}\n\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0}\n\end{array}$

Uniformov of the Heat equation:

\nLet
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q \in C(F)
$$
 and $f \in C(\sqrt{r})$. Then \exists at most one solution $u \in C^2(\sqrt{r}) \cap C(\sqrt{r})$

\nLet $q \in C(F)$ and $f \in C(\sqrt{r})$. Then \exists at most one solution $u \in C^2(\sqrt{r}) \cap C(\sqrt{r})$

\nof the problem $u \in \Delta u = \oint_{u \in Q} \int_{v \in Q}$

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8 + 5
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800 \div 100
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800 \div 1000
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800 \div 1000
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1000 \div 1000
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v(x,t) = u(x,t) - \frac{u}{(T+4-t)^{n}h} e^{-\frac{h^{n}}{h(T+4-t)}} \\
= Ce^{x/h^{2}} - \frac{u}{(T+4-t)^{n}h} e^{-\frac{h^{2}}{h(T+4-t)}} \\
= Ce^{x/h^{2}} - \frac{u}{(T+4-t)^{n}h} e^{-\frac{h^{2}}{h(T+4-t)}} \\
= C e^{x/h^{2}} - \frac{u}{(T+6)^{n}h} e^{x/h^{2}}.
$$

$\mathbf{U}^{\dagger} \quad \mathbf{U} \quad \mathbf{U} \quad \mathbf{T} \quad \mathbf{U} \quad \mathbf{0}$

 $\begin{array}{ccccccc}\n\circ & & & \circ & & \circ & & \circ\n\end{array}$

Proof	Assame, $4dT \angle 1$ then $3630 \text{ s} + 4d(T+6) \angle 1$.
For $geR^n \times N \ge 0$ and define $9(x, \theta):= u(x, \theta) - \frac{M}{(T+e-t)^{n/2}} e^{-\frac{1}{4(T+e-t)}} \text{ (see } r \text{ if } r \ge 0)$	
Clearly, $0t - \Delta u = 0$ (Using $\text{Linearity of } \text{the } \text{heat} \text{ of } \text{beta})$	
First 170 and $0dt$ $0L = B(y, \theta)$ from $4\pi = B(y, \theta) \times (0, \pi)$	
Here, $mgx - y = max \times 1$	
How $mgx - y = max \times 1$	
How $g = \frac{1}{4} \pi$	
Now $g = \frac{1}{4} \pi$	
Now $g = \frac{1}{4} \pi$	
Now $g = \frac{1}{4} \pi$	
but, $g = \frac{1}{4} [x - g] = 0$	
but, $g = \frac{1}{4} [x - g] = 0$	

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k + 5
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b \downarrow 0
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$\begin{array}{cccccccccccccc} \mathbb{Q}^{\dagger} & \mathbb{Q} & \mathbb{Q} & \mathbb{T} & \mathbb{T} & \mathbb{C} & \mathbf{0} \end{array}$

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U(k,t) = U(x,t) - \frac{M}{(T+\alpha-t)^{n/2}} e^{-\frac{W^{2}}{4(T+\alpha-t)}}
$$
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$$
\leq C e^{\alpha |x|^{2}} - \frac{M}{(T+\alpha-t)^{n/2}} e^{-\frac{W^{2}}{4(T+\alpha-t)}}
$$
\n
$$
\leq C e^{\alpha (|y|+r)^{2}} - \frac{M}{(T+\alpha)t^{n/2}} e^{-\frac{W^{2}}{4(T+\alpha-t)}}
$$
\n
$$
\leq C e^{\alpha (|y|+r)^{2}} - \frac{M}{(T+\alpha)t^{n/2}} e^{-\frac{W^{2}}{4(T+\alpha)t^{n/2}}}
$$
\nNow, $U_{d}(T+t) \leq 1$ from $U_{d}(T+t) = -\alpha + \frac{1}{2} \int_{0}^{T} (\alpha + t)^{n/2}.$
\n
$$
\leq U_{d}(x,t) \leq C e^{\alpha (10/1+t)^{2}} - \frac{1}{2} \int_{0}^{T} [\alpha + t)^{n/2} e^{(\alpha + t)^{n/2}+1}.
$$
\n
$$
\leq U_{d}(x,t) \leq C e^{\alpha (10/1+t)^{2}} - \frac{1}{2} \int_{0}^{T} [\alpha + t)^{n/2} e^{(\alpha + t)^{n/2}+1}.
$$

$\begin{array}{ccccccccccccccccc} \mathbb{Q}^{\dagger} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{T} & \mathbb{C} & \mathbf{0} & \mathbf{0} \end{array}$

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Proof:	Assame, $l_{dT} \angle 1$ then $\exists \epsilon > 0$ s+ $4d(T+\epsilon) \angle 1$.	
For $g \in \mathbb{R}^{n} \times N \gg 0$ and define	$\frac{1}{2} \times r^{2}$	$(\kappa \in \mathbb{R}^{n}, t \Rightarrow 0)$
$v(\kappa, t): u(\kappa, t) - \frac{M}{(T+\epsilon - t)^{n/2}} e^{-\frac{1}{2}(T+\epsilon - t)}$ $(\kappa \in \mathbb{R}^{n}, t \Rightarrow 0)$		
Clearly, $v_{t} - \Delta v = 0$ (Using $\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(\kappa - \lambda) \cdot \frac{1}{T} \int_{0}^{T} f(\kappa - \lambda) \$		

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Requiredality:	Support	ue C ² (Urt) solwa	tr \leftarrow U \leftarrow -U \leftarrow Q in U \leftarrow Then				
U \leftarrow C ⁰ (Urt)	($\frac{C}{C}$ modn in spauchiivu)						
Remark::	Left U dilations non-smoolk boundary value on Trcbill he						
Qiseution holds.							
Proof:	(Ut-off (function)						
et	g for Heut equation:						
Recall:	g for Heut equation:						

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By 1 (iii) w(ii) one has, V(V/E) = super V y GERN & EE [ON]. Let, $\mu \rightarrow 0$; $\sup_{\mathbb{R}^n \times [0,1]} u = \sup_{\mathbb{R}^n} \mathcal{F}$. If $4aT f1$ shen, $[0, \frac{1}{8}aT, \frac{1}{8}aT, \frac{1}{1}aT, \cdots]$
kpply the above result in each interval \mathbb{D} Apply the above result in each models
then corollary, $u_{\epsilon-2}u_{\epsilon} = \frac{1}{2} \ln \mathbb{R}^n \times (0.13)$ (Uniqueness is \mathbb{R}^n)
then others to solution, $|u(x,t)| \leq Ce^{x|x|^2}$ $\forall x \in \mathbb{R}^n \vee t \in (0.13)$.

Then we proved a very important, but kind of difficult mean value property, which is not essentially your mean value, but we defined a kind of heat ball and we showed that in that ball you can do all sort of thing that the mean value holds in a different set. And then today we proved that strong maximum principle holds, uniqueness holds and you can use it to even settle the well-poseness problem, just, these ideas are exactly the same as we did for Laplacian. So, I am not really going deep into all that. So, all of this is done. So, with this we are going to finish the heat equation part of this course. Thank you very much.