Advanced Partial Differential Equations
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Lecture 22
Mean Value Property of Heat Equation

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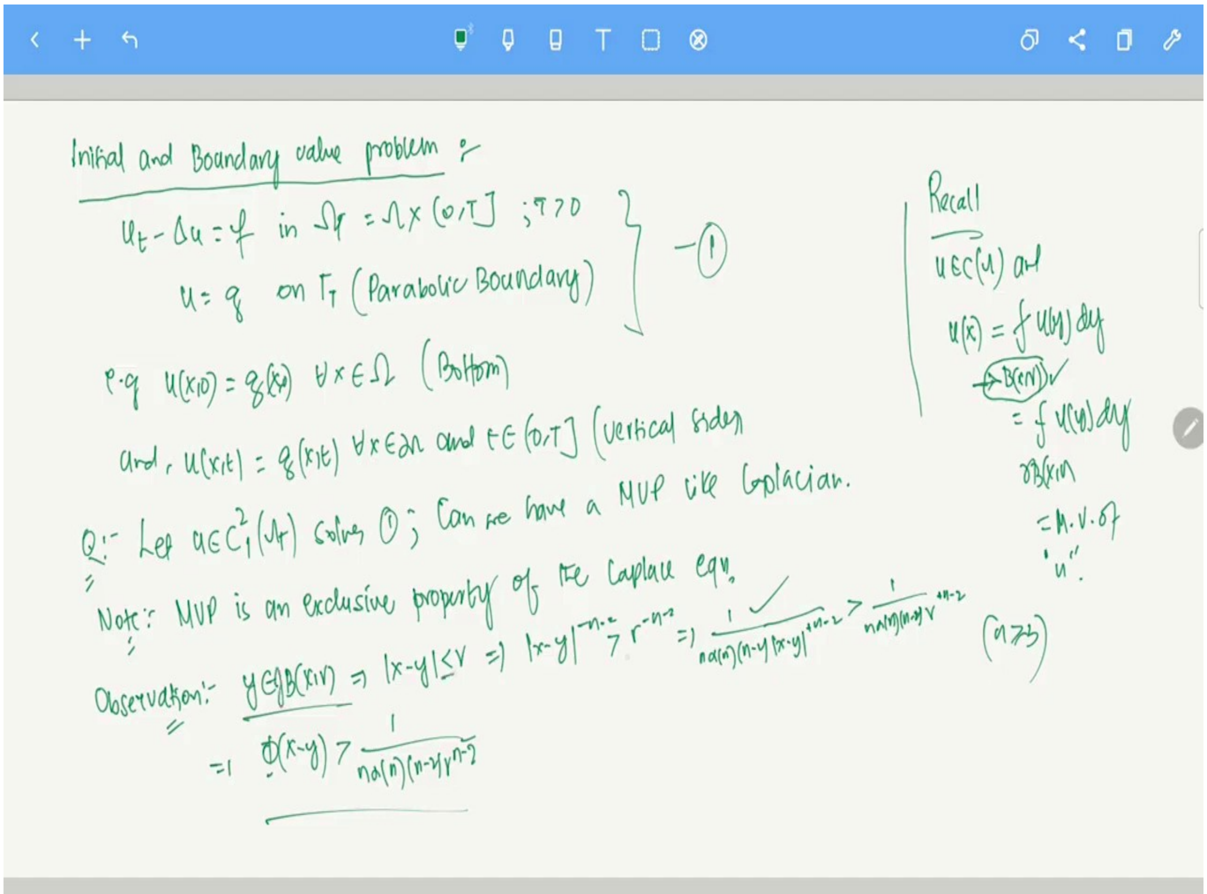

Welcome students. In today's class and in this week specifically we are going to talk about initial and boundary value problem.

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$\left(\Phi(x-y) = c \Rightarrow \partial B(x,r) \right)$.

This suggests; the level sets of the Fundamental Solution $\Phi(x-y, t-s)$ may be used in case of Heat equation.

Defn: For fixed $x \in \mathbb{R}^n$ & $t > 0$, $r > 0$ we define

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^{n+1} \mid s \leq t \text{ and } \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}$$

This set $E(x,t;r)$ is called the Heat Ball and is a region in space time, the boundary of which is a level set of $\Phi(x-y, t-s)$. e.g. $\partial E(x,t;r) = \{ \Phi(x-y, t-s) = c \}$

Note that, $\Phi(x-y, t-s) \geq r^{-n}$ is equivalent to saying

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} : t - \frac{r^2}{4\pi} \leq s \leq t \text{ and } \psi(y-x, s-t) \geq 0 \right\}$$

where $\psi(y,s) = n \ln r + \frac{|y|^2}{4s} - \frac{n}{2} \ln(-4\pi s)$; $s < 0$, Initial and Boundary value problem :-

$$\left.\begin{aligned} u_t - \Delta u &= f \text{ in } \Omega_T = \Omega \times (0,T] \; ; T>0 \\ u &= g \text{ on } \Gamma_T \text{ (Parabolic Boundary)} \end{aligned}\right\} \quad -(1)$$

e.g. $u(x,0) = g(x,0) \; \forall x \in \Omega$ (Bottom)

and, $u(x,t) = g(x,t) \; \forall x \in \partial\Omega$ and $t \in (0,T]$ (vertical sides)

Q:- Let $u \in C^2_1(\Omega_T)$ solves (1); Can we have a MVP like Laplacian.

Note:- MVP is an exclusive property of the Laplace eqn.

Observation:- $y \in \partial B(x,r) \Rightarrow |x-y| \leq r \Rightarrow |x-y|^{-n-2} \geq r^{-n-2} \Rightarrow \frac{1}{n\alpha(n)(n-2)|x-y|^{n-2}} \geq \frac{1}{n\alpha(n)(n-2)r^{n-2}}$ $(n \geq 3)$

$\Rightarrow \Phi(x-y) \geq \frac{1}{n\alpha(n)(n-2)r^{n-2}}$ "

Recall

$u \in C(\Omega)$ and

$u(x) = ⨍_{B(x,r)} u(y)\,dy$

$= ⨍_{\partial B(x,r)} u(y)\,dy$

$=$ M.V. of "u".

Φ

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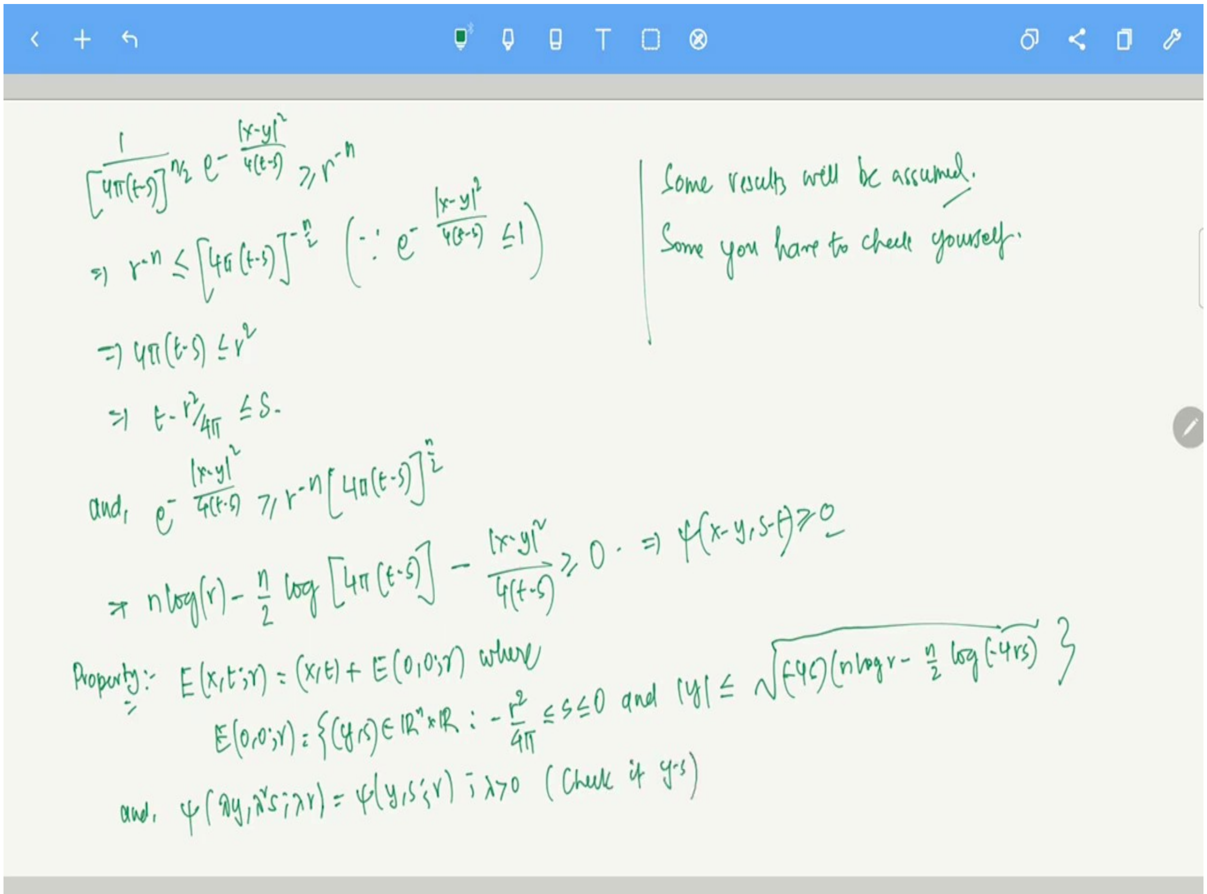

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where $\psi(y,s) = n \ln r + \frac{|y|^2}{4s} - \frac{n}{2} \ln(-4\pi s)$; $s<0$.

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Mean Value Theorem for Heat Equation :-

Let $u \in C_1^2(\Omega_T)$ solves the Heat equation. Then

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds$$

for all $E(x,t;r) \subset \Omega_T$.

Proof :- Define, $$\varphi(r) := \frac{1}{r^n} \iint_{(\xi,\tau) \in E(x,t;r)} u(\xi,\tau) \frac{|x-\xi|^2}{(t-\tau)^2}\, d\xi\, d\tau$$

$$\therefore \quad \varphi(r) = \frac{1}{r^n} \iint_{(\xi,\tau) \in E(0,0;r)} u(x+\xi, t+\tau) \frac{|\xi|^2}{\tau^2}\, d\xi\, d\tau$$

$$= \iint_{(\xi,\tau) \in E(0,0;1)} u(x+r\xi, t+r^2\tau) \cdot \frac{|\xi|^2}{\tau^2}\, d\xi\, d\tau \quad \text{(Change of Variable)}$$

$$\frac{1}{[4\pi(t-s)]^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \geq r^{-n}$$

$$\Rightarrow r^{-n} \leq [4\pi(t-s)]^{-\frac{n}{2}} \quad \left(\because e^{-\frac{|x-y|^2}{4(t-s)}} \leq 1\right)$$

$$\Rightarrow 4\pi(t-s) \leq r^2$$

$$\Rightarrow t - r^2/4\pi \leq s.$$

and, $$e^{-\frac{|x-y|^2}{4(t-s)}} \geq r^{-n}[4\pi(t-s)]^{\frac{n}{2}}$$

$$\Rightarrow n\log(r) - \frac{n}{2}\log[4\pi(t-s)] - \frac{|x-y|^2}{4(t-s)} \geq 0. \Rightarrow \psi(x-y, s-t) \geq 0$$

Some results will be assumed.

Some you have to check yourself.

Property:- $E(x,t;r) = (x,t) + E(0,0;r)$ where

$$E(0,0;r) = \left\{(y,s) \in \mathbb{R}^n \times \mathbb{R} : -\frac{r^2}{4\pi} \leq s \leq 0 \text{ and } |y| \leq \sqrt{(-4s)\left(n\log r - \frac{n}{2}\log(-4\pi s)\right)}\right\}$$

and, $\psi(\lambda y, \lambda^2 s; \lambda r) = \psi(y, s; r)$; $\lambda > 0$ (Check it yrs)

So, now that we have some idea of what the heat ball is, I mean, in this case, in our specific case what the set is. Let us write down the heat equation, so mean value property for heat equation, mean value theorem for heat equation.

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Differentiating,

$$\varphi'(r) = \iint_{(\xi,\tau)\in E(0,0;1)} \left[\nabla u(x+r\xi, t+r^2\tau)\cdot \xi \frac{|\xi|^2}{\tau^2} + u_t(x+r\xi, t+r^2\tau)\frac{|\xi|^2}{\tau}\cdot 2r \right] d\xi\, d\tau \quad \text{(Chain Rule).}$$

$$= \frac{1}{r^n} \iint_{(\xi,\tau)\in E(0,0;r)} \nabla u(x+\xi, t+\tau)\cdot \xi \frac{|\xi|^2}{\tau^2} + \frac{2}{r^{n+1}} \int u_t(x+\xi, t+\tau)\frac{|\xi|^2}{\tau} = A+B.$$

Focusing on B,

$$\frac{2}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} u_t(x+\xi, t+\tau)\frac{|\xi|^2}{r} = \frac{4}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} u_t(x+\xi, t+\tau)\cdot \nabla\psi(\xi,\tau)\cdot \xi$$

$$\left(\because \nabla\psi(\xi,\tau) = \xi/2\tau\right)$$

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where $\psi(y,s) = n \ln r + \dfrac{|y|^2}{4s} - \dfrac{n}{2} \ln(-4\pi s)$; $s<0$,

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$$\frac{a}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} u_t(x+\xi, t+\tau) \frac{|\xi|^2}{\tau^2} = \frac{4}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} u_t(x+\xi, t+\tau) \cdot \nabla\psi(\xi,\tau)\cdot\xi$$

$\left(\operatorname{div}(u v n) = u v \operatorname{div} n + \cdots \right)$ Leibnitz Rule

$$= \frac{4}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} \operatorname{div}\left[u_t(x+\xi, t+\tau)\psi(\xi,\tau)\xi \right] - \psi(\xi,\tau)\nabla u_t(x+\xi,t+\tau)\cdot\xi - n\psi(\xi,\tau)u_t(x+\xi,t+\tau)$$

Chain Rule

$$= -\frac{4n}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} \psi(\xi,\tau)u_t(x+\xi,t+\tau) - \frac{4}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} \psi(\xi,\tau)\cdot\frac{\partial}{\partial\tau}\left[Du(x+\xi,t+\tau)\cdot\xi \right].$$

$$\left[\because \psi\big|_{\partial E(0,0;r)} \overset{\text{Check}}{=} 0 \right]. \qquad \left[\overset{\text{G.D.T}}{\int_{U}} \operatorname{div}(u) = \int_{\partial U} u\cdot\nu \, dS \right]$$

Considering,

$$\int_{(\xi,\tau)\in E(0,0;r)} \psi(\xi,\tau)\frac{\partial}{\partial\tau}\left[Du(x+\xi,t+\tau)\cdot\xi \right] \overset{\text{I.B.P}}{=} -\int_{(\xi,\tau)\in E(0,0;r)} \psi_\tau(\xi,\tau)\cdot Du(x+\xi,t+\tau)\cdot\xi + \cancel{\int_{\text{Bdry}}}.$$

Differentiating,

$$\varphi'(r) = \iint_{(\xi,\tau)\in E(0,0;1)} \left[\nabla u(x+r\xi, t+r^2\tau)\cdot \xi \frac{|\xi|^2}{\tau^2} + u_t(x+r\xi, t+r^2\tau) \frac{|\xi|^2}{\tau}\cdot 2r\right] d\xi\, d\tau \quad \text{(Chain Rule)}.$$

$$= \frac{1}{r^n} \iint_{(\xi,\tau)\in E(0,0;r)} \nabla u(x+\xi, t+\tau)\cdot \xi \frac{|\xi|^2}{\tau^2} + \frac{2}{r^{n+1}} \int u_t(x+\xi, t+\tau) \frac{|\xi|^2}{\tau} = A + B.$$

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$$= -\int_{(\xi,\tau)\in E(0,0;r)} \left(-\frac{n}{2r} - \frac{|\xi|^2}{4\tau^2}\right) \nabla u(x+\xi, t+\tau)\cdot\xi \quad \text{(Check)}$$

Rearranging all the expressions,

$$\varphi(r) = -\frac{4n}{r^{n+1}} \int_{\dots} \psi(\xi,\tau)\, u_t(x+\xi,t+\tau) - \frac{2n}{r^{n+1}} \int \frac{\nabla u(x+\xi,t+\tau)\cdot\xi}{\tau}$$

$$\underset{\text{(Check)}}{\overset{\text{I.B.P}}{=}} +\frac{4n}{r^{n+1}}\int \nabla\psi(\xi,\tau)\cdot\nabla u(x+\xi,t+\tau) - \frac{2n}{r^{n+1}}\int\frac{\nabla u(x+\xi,t+\tau)\cdot\xi}{\tau} = 0.$$

$\therefore$ φ is constant.

(Lemma due to Fulks)

$$\therefore \varphi(r) = \lim_{r\to 0}\int_{E(0,0;r)} u(x+r\xi, t+r^2\tau)\frac{|\xi|^2}{\tau^2} = u(x,t)\int_{E(0,0;1)}\frac{|\xi|^2}{\tau^2}\,d\xi\, d\tau = 4r^n\cdot u(x,t)$$

$$\frac{\partial}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} u_t(x+\xi, t+\tau) \frac{|\xi|^2}{\tau} = \frac{4}{r^{n+1}} \int_{(\xi,\tau)\in E(0,0;r)} u_t(x+\xi, t+\tau) \cdot \nabla\psi(\xi,\tau)\cdot\xi$$

$\left(\begin{array}{l} \mathrm{div}(u v n) = u v \,\mathrm{div}\, n + \dots \\ \text{Leibnitz Rule} \end{array} \right)$

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$$\left(\because \nabla\psi(\xi,\tau) = \xi/2\tau\right)$$

This integral I am not calculating this integral, but I mean this goes way back to 1961 or something, there is a paper by Fulks where he has proved this property. This is not a very easy thing to prove. So, I am going to assume this. So, with this, we are going to end this lecture. So, we have proved what we wanted. Thank you very much.