**Advanced Partial Differential Equations**

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**Lecture 21**

**Heat Equation: Inhomogeneous Problem**

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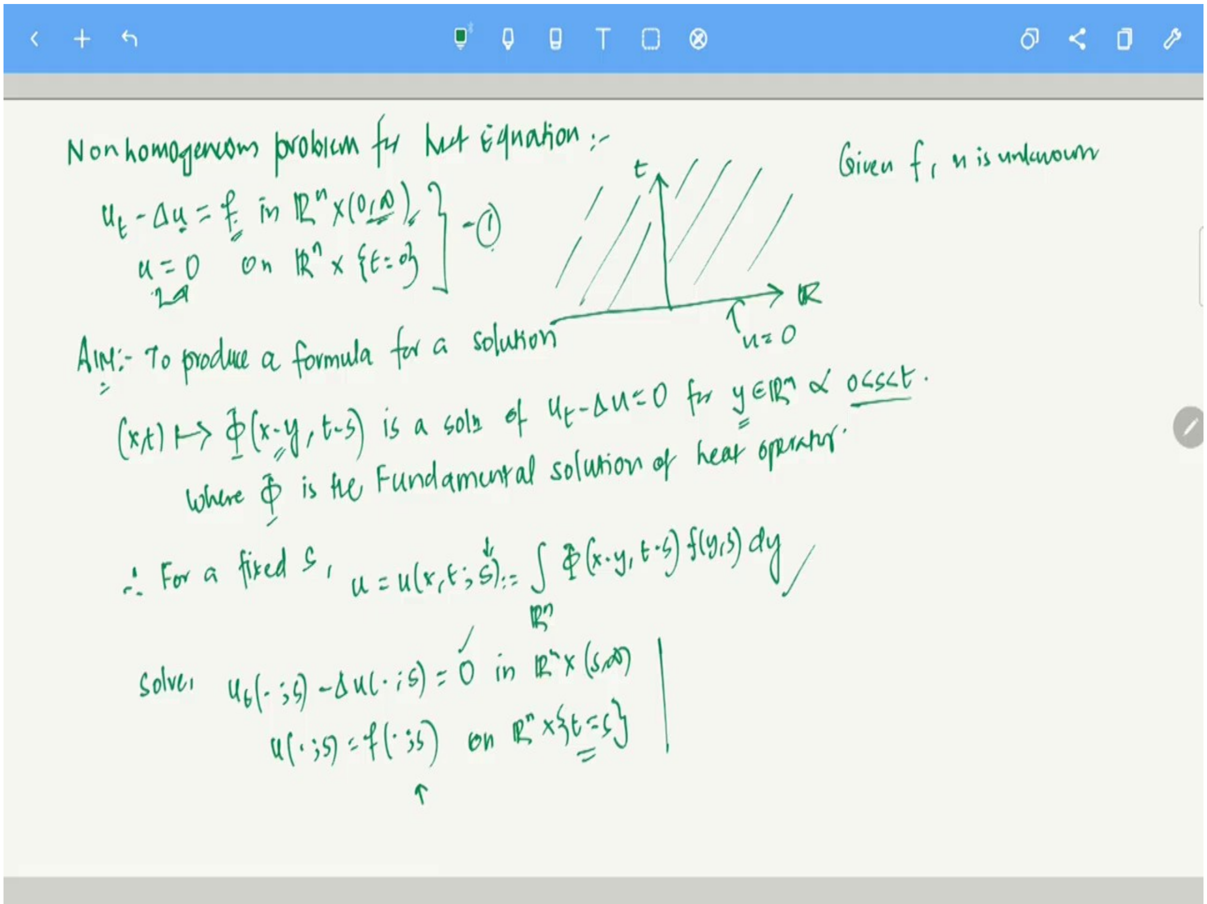


So, in today's video, what we are going to do is look at nonhomogeneous problems for heat equation.

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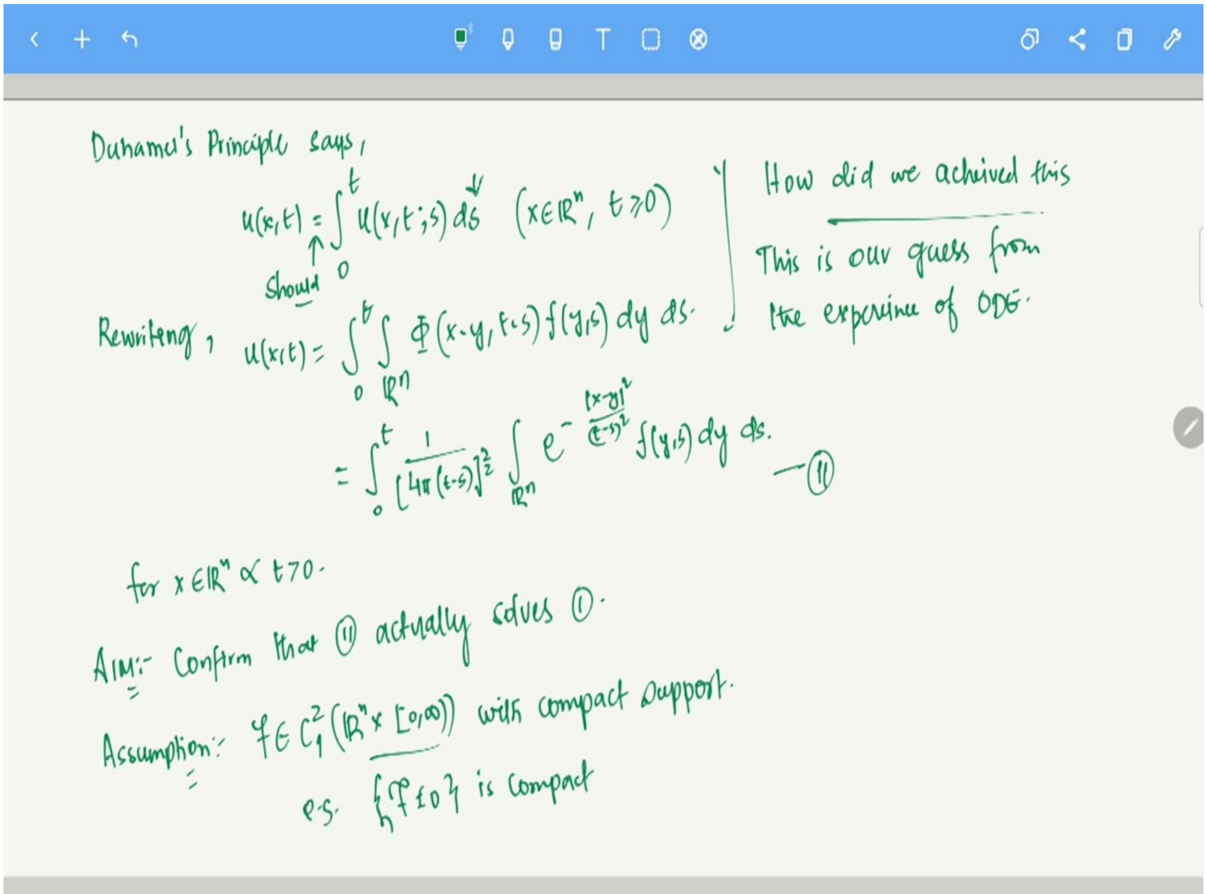
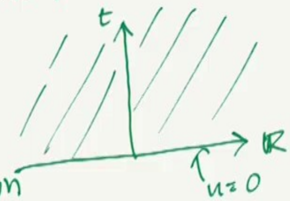


Nonhomogeneous problem for heat equation :-

Given $f$, $u$ is unknown

$$\left.\begin{aligned} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= 0 \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}\right\} - (1)$$




AIM:- To produce a formula for a solution

$(x,t) \mapsto \Phi(x-y, t-s)$ is a soln of $u_t - \Delta u = 0$ for $y \in \mathbb{R}^n$ & $0<s<t$.

where $\Phi$ is the Fundamental solution of heat operator.

$\therefore$ For a fixed $s$, $u = u(x,t;s) := \int_{\mathbb{R}^n} \Phi(x-y, t-s)\, f(y,s)\, dy$

solve: $u_t(\cdot;s) - \Delta u(\cdot;s) = 0$ in $\mathbb{R}^n \times (s,\infty)$

$u(\cdot;s) = f(\cdot;s)$ on $\mathbb{R}^n \times \{t=s\}$

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Th: Let $u$ be given by (II). Then

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t)$ $(x \in \mathbb{R}^n \ \& \ t>0)$

(iii) $\lim\limits_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n,\, t>0}} u(x,t) = 0$

$u_t - \Delta u = f \in C_1^2$

Proof: $u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy\, ds$. (Change of variable is required in order to take the derivative inside)

$\because$ $f \in C_1^2$ with cpt spt. and $\Phi$ is smooth near $s=t>0$,

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, f_t(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy.$$

(Diff under the sign of integration)

Duhamel's Principle says,

$$u(x,t) = \int_0^t u(x,t;s)\, ds \quad (x \in \mathbb{R}^n,\ t \geq 0)$$

Should

How did we achived this

This is our guess from the experience of ODE.

Rewriting,
$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds.$$

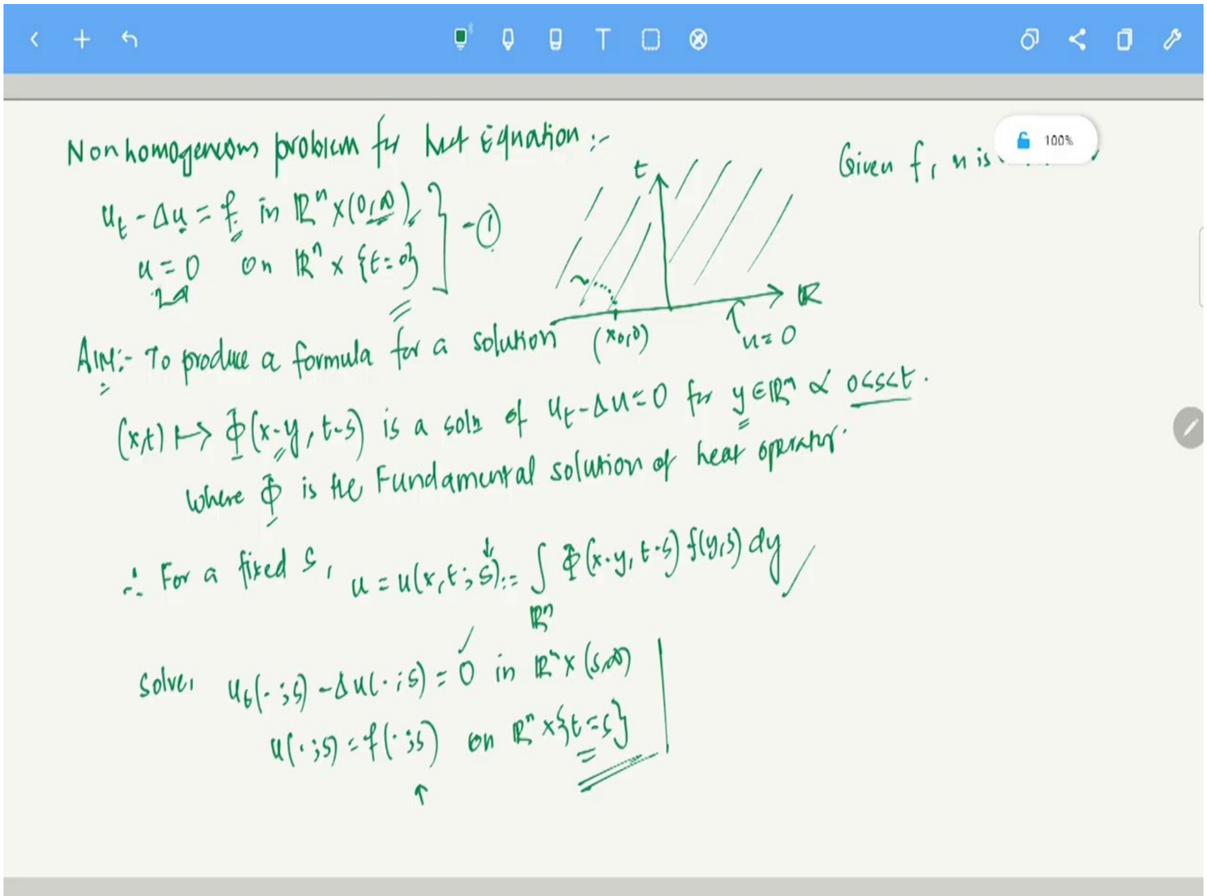
$$= \int_0^t \frac{1}{[4\pi(t-s)]^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{(t-s)^2}} f(y,s)\, dy\, ds. \quad \text{— (11)}$$

for $x \in \mathbb{R}^n$ & $t > 0$.

AIM:- Confirm that (11) actually solves (1).

Assumption:- $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ with compact support.

e.g. $\{f \neq 0\}$ is compact




So, let me write down the theorem first, so the theorem. So, this is the existence theorem.

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also, $\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y,t-s)\, dy\, ds$ $(i,j = 1,2,\dots,n)$ (Please verify)

$\therefore u_t, u_{x_i x_j}$ are continuous in $\mathbb{R}^n \times (0,\infty)$.

By Defn.

$\therefore u \in C_1^2(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta_x u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \left[ \left( \frac{\partial}{\partial t} - \Delta_x \right) f(x-y,t-s)\, dy\, ds \right] + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$

$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[ \left( -\frac{\partial}{\partial s} - \Delta_y \right) f(x-y,t-s) \right] dy\, ds$ (Please Check this)

$+ \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[ \left( -\frac{\partial}{\partial s} - \Delta_y \right) f(x-y,t-s) \right] dy\, ds$

$+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$

"Some Calculations are intentionally left."

$:= I_\varepsilon + J_\varepsilon + K.$

Th: Let $u$ be given by (11). Then

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n \ \& \ t>0)$

(iii) $\lim\limits_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$

$u_t - \Delta u = f \in C_1^2$

Proof: $u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy\, ds.$ (Change of variable is required in order to take the derivative inside)

$\because$ $f \in C_1^2$ with cpt spt and $\Phi$ is smooth near $s=t>0$,

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy.$$

(Diff under the sign of integration)

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$$|J_\varepsilon| = \left| \int_0^\varepsilon \int_{\mathbb{R}^n} \left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y, t-s) \cdot \Phi(y,s)\, dy\, ds \right|$$

$$\le \underbrace{\left(\|f_t\|_\infty + \|D^2 f\|_\infty\right)} \int_0^\varepsilon \left( \int_{\mathbb{R}^n} \Phi(y,s)\, dy \right) ds$$

$$\left[\because \int_{\mathbb{R}^n} \Phi(y,s)\, dy = 1 \text{ for each } t > 0 \text{ hence} \right]$$

$$= C\varepsilon$$

$f \in C_1^2$ with cpt spt

$f_t$ must be bdd

$\|f_t\|_\infty$

$$\left| \left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y, t-s) \right|$$

$$\le \left| \frac{\partial f}{\partial s} \right| + |\Delta_y f|$$

$$\le \|f_t\|_\infty + \|D^2 u\|_\infty$$

Again, $$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) \Phi(y,s)\right] f(x-y, t-s)\, dy\, ds$$

I.B.P

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \left[\left(\frac{\partial}{\partial s} - \Delta_y\right) \Phi(y,s)\right] f(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy$$

$$- \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0)\, dy$$

K

also, $\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y,t-s)\, dy\, ds$ $(i,j=1,2,\dots,n)$ (Please verify this)

$\therefore u_t,\ u_{x_i x_j}$ are continuous in $\mathbb{R}^n \times (0,\infty)$.

$\therefore u \in C_1^2(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta_x u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(\frac{\partial}{\partial t} - \Delta_x\right) f(x-y,t-s)\, dy\, ds\right] + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$

$= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\right] dy\, ds$ (Please Check this)

$+ \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\right] dy\, ds$

$+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$

$: I_\epsilon + J_\epsilon + K.$

"Some Calculations are intentionally left."

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$$\therefore I_\varepsilon + K = \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy$$

$$\therefore u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy$$

$$= f(x,t) \quad (x \in \mathbb{R}^n \ \& \ t > 0)$$

$$\left| \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy - f(x,t) \right|$$

$$= \int_{\mathbb{R}^n} \left[ \Phi(y,\varepsilon) f(x-y, t-\varepsilon) - \Phi(y,\varepsilon) f(x,t) \right] dy$$

$$= \int_{\mathbb{R}^n} \Phi(y,\varepsilon) \left| f(x-y, t-\varepsilon) - f(x,t) \right| dy$$

$< \delta \cdot 1 = \delta$ ($f$ is cont w.r.t $x$ & $t$)

For (iii); $\lim\limits_{\substack{(x,t) \to (x_0,0) \\ x \in \mathbb{R}^n,\, t>0}} u(x,t) = 0$ for each $x_0 \in \mathbb{R}^n$.

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy\, ds$$

$f \in C_1^2$ with cpt spt

$$\Rightarrow \|u(\cdot,t)\|_{L^\infty} \le \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, dy\, ds$$

$$= t \|f\|_{L^\infty} \to 0 \text{ as } t \to 0$$

$$|J_\varepsilon| = \left|\int_0^\varepsilon \int_{\mathbb{R}^n} \left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y, t-s) \cdot \Phi(y,s)\, dy\, ds\right|$$

$$\leq \underbrace{\left(\|f_t\|_\infty + \|D^2 f\|_\infty\right)}\int_0^\varepsilon\left(\int_{\mathbb{R}^n} \Phi(y,s)\, dy\right) ds$$

$$\left[\because \int_{\mathbb{R}^n} \Phi(y,s)\, dy = 1 \text{ for each } t>0 \text{ hence}\right]$$

$$= C\varepsilon$$

$f \in C_1^2$ with cpt spt

$f_t$ must be bdd

$\|f_t\|_\infty$

$$\left|\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y, t-s)\right|$$

$$\leq \left|\frac{\partial f}{\partial s}\right| + |\Delta_y f|$$

$$\leq \|f_t\|_\infty + \|D^2 u\|_\infty$$

Again, $$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)\right] f(x-y, t-s)\, dy\, ds$$

$$\overset{\text{I.B.P}}{=} \int_\varepsilon^t \int_{\mathbb{R}^n} \left[\underbrace{\left(\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)}_{=0}\right] f(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy$$

$$- \int_{\mathbb{R}^n} \Phi(y,t) f(x-y, 0)\, dy$$

K

also, $\dfrac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \displaystyle\int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y,t-s)\, dy\, ds$ $(i,j = 1,2,\dots,n)$ (Please verify this)

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$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\right] dy\, ds \quad \text{(Please Check this)}$$

$$+ \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\right] dy\, ds$$

$$+ \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$$

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$$:= I_\varepsilon + J_\varepsilon + K.$$

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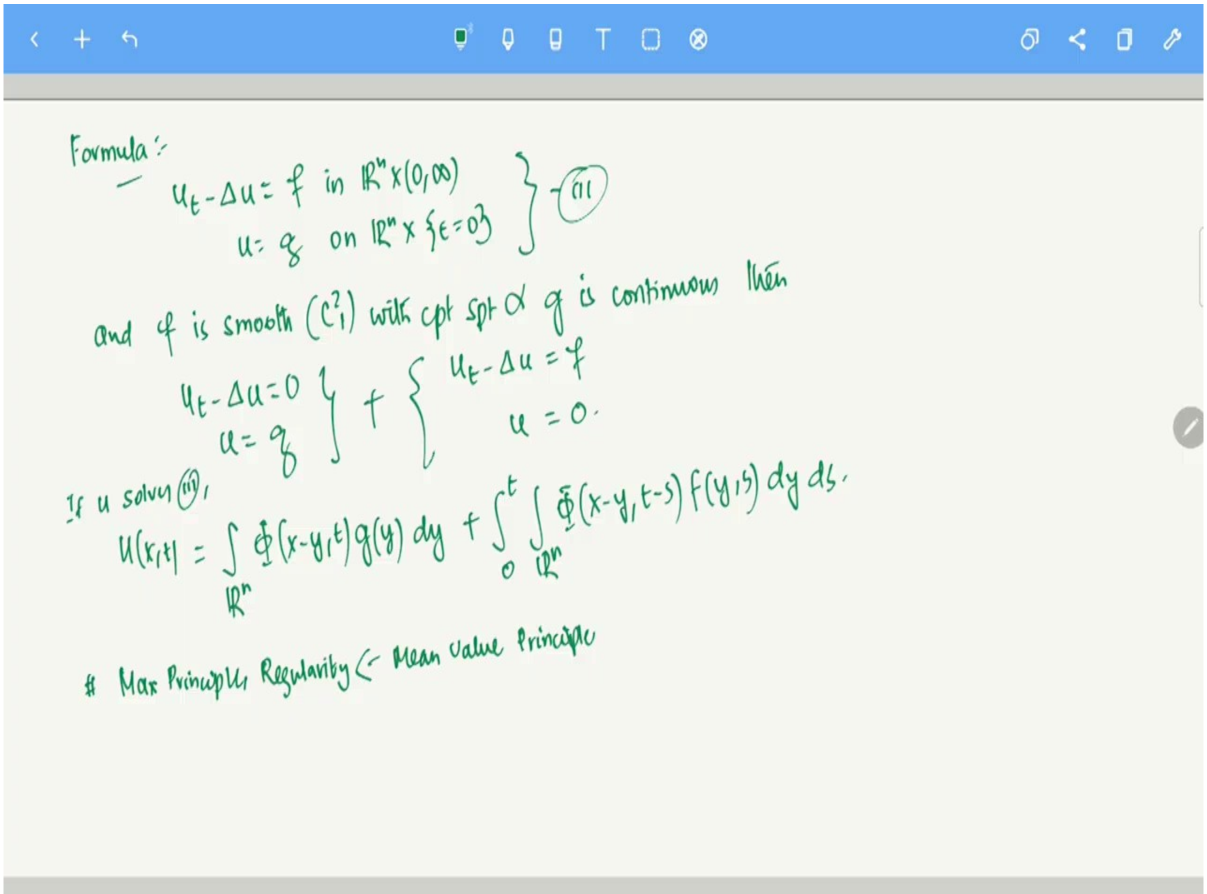


So, this is how you solve the inhomogeneous heat equation. So, in next lecture, next week, what we are going to do is we are going to end up heat equation part with looking at the maximal principle, regularity of solution and of course how do you prove maximal principle and regularity this is proved using a mean value theorem, just as Laplacian. But the thing is this, for the heat thing or mean value theorem, maximal principle and regularity are quite complicated as opposed to that Laplacian. So, we will do that in next week video. Thank you.