**Advanced Partial Differential Equations**
**Professor Doctor Kaushik Bal**
**Department of Mathematics and Statistics**
**Indian Institute of Technology, Kanpur**
**Lecture 20**
**Heat Equation: Homogeneous Problem**

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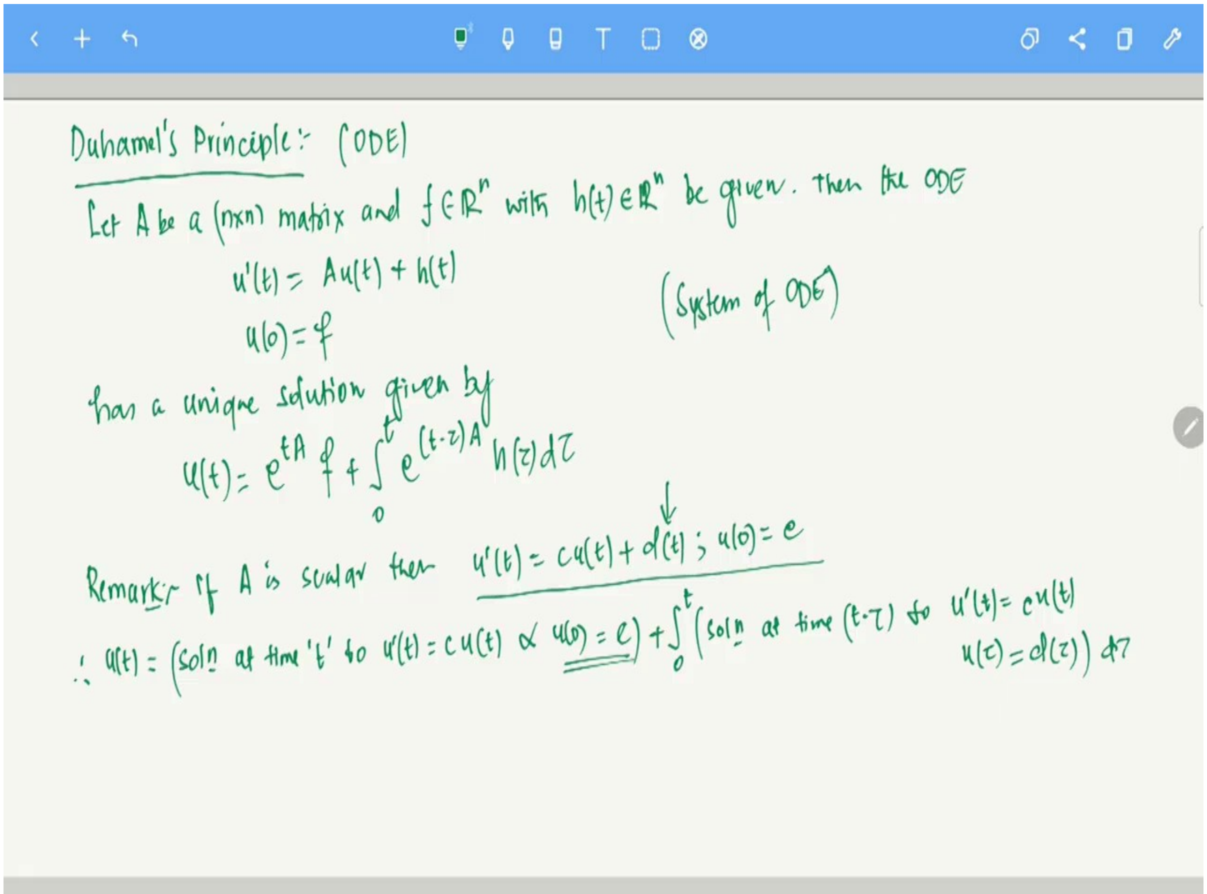

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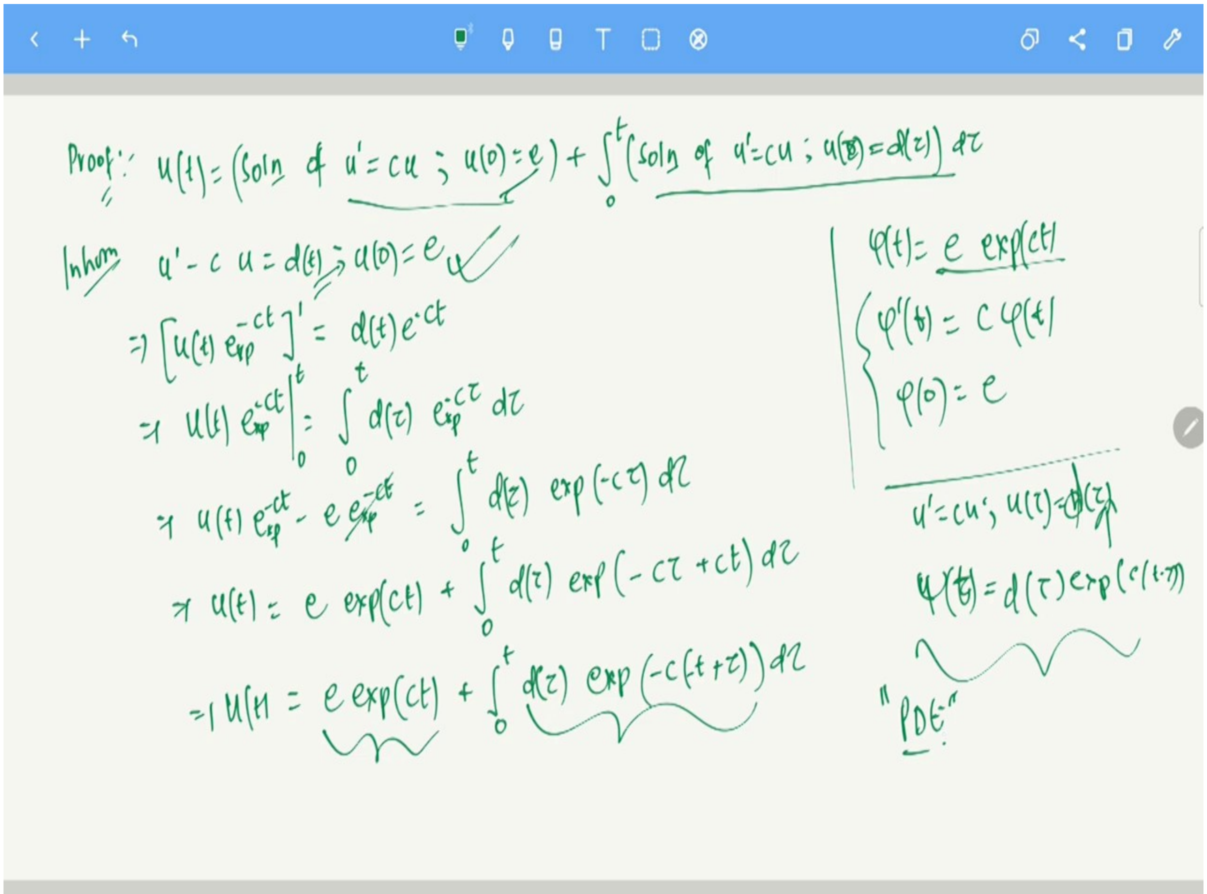

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# $u_t - \Delta u = f$ in $\mathbb{R}^n \times (0,\infty)$
$u = g$ on $\mathbb{R}^n \times \{t=0\}$ $\Bigg\}$ — (1)

Inhomogeneous heat equation.


"From Duhamel's Principle for ODE, If one can find a formula to solve the homogeneous problem then some "Integration" type operation may provide a soln of (1)"

Homogeneous Heat Equation :- (Initial value)

$$u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty)$$
$$u = g \text{ on } \mathbb{R}^n \times (0,\infty)$$

Recall, Fundamental Soln to the heat operator is given by $\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t>0, x \in \mathbb{R}^n \\ 0 & o/w. \end{cases}$

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$(x,t) \mapsto \Phi(x,t)$ solves the heat eqn away from $(0,0)$

$\therefore (x,t) \mapsto \Phi(x-y,t)$ solves the heat eqn for each fixed $y \in \mathbb{R}^n$.

$$\therefore u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

Motivation

should also be a soln.

Th: (Solution of IVP) Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$

Then (i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta_x u(x,t) = 0 \quad (x \in \mathbb{R}^n,\ t > 0)$

(iii) $\lim_{\substack{(x,t) \to (x^0,0) \\ x \in \mathbb{R}^n,\ t>0}} u(x,t) = g(x^0)$ for each $x^0 \in \mathbb{R}^n$.

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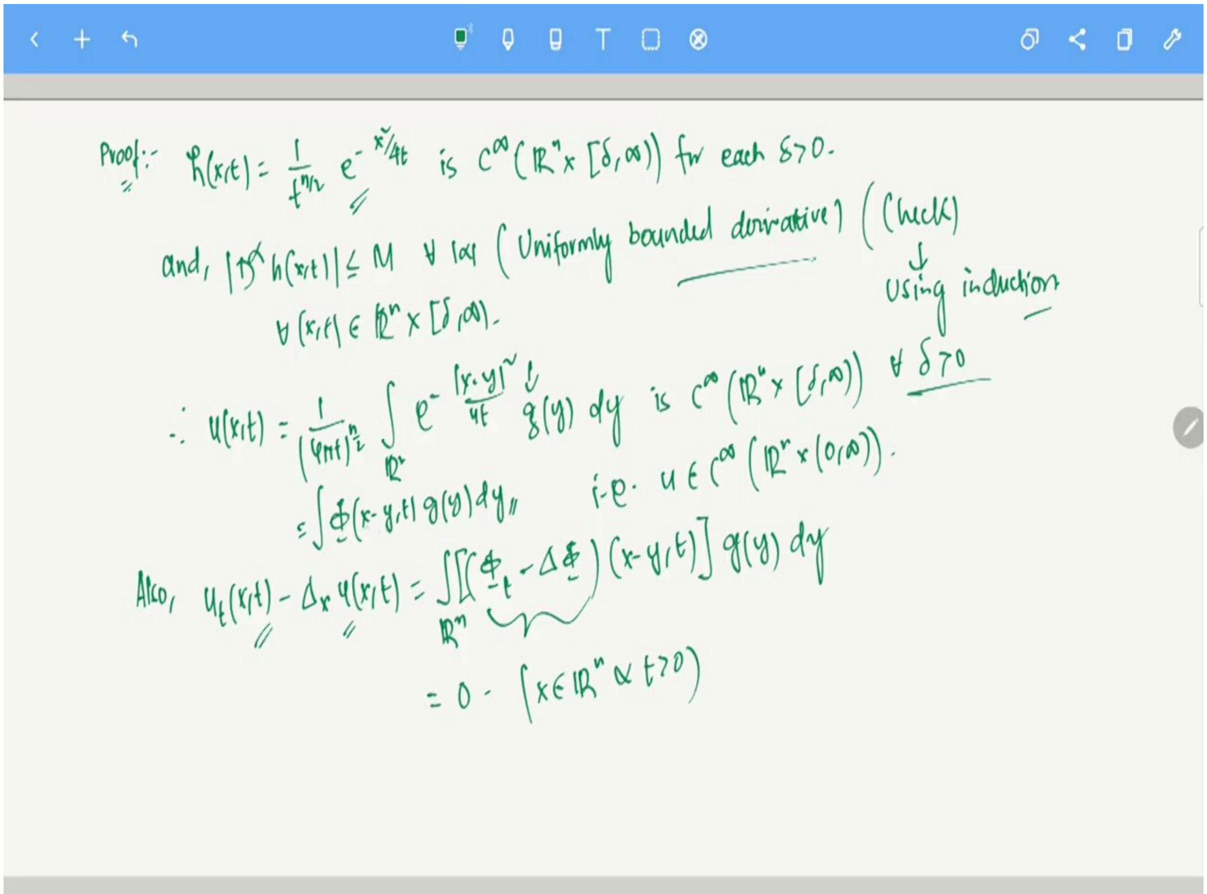

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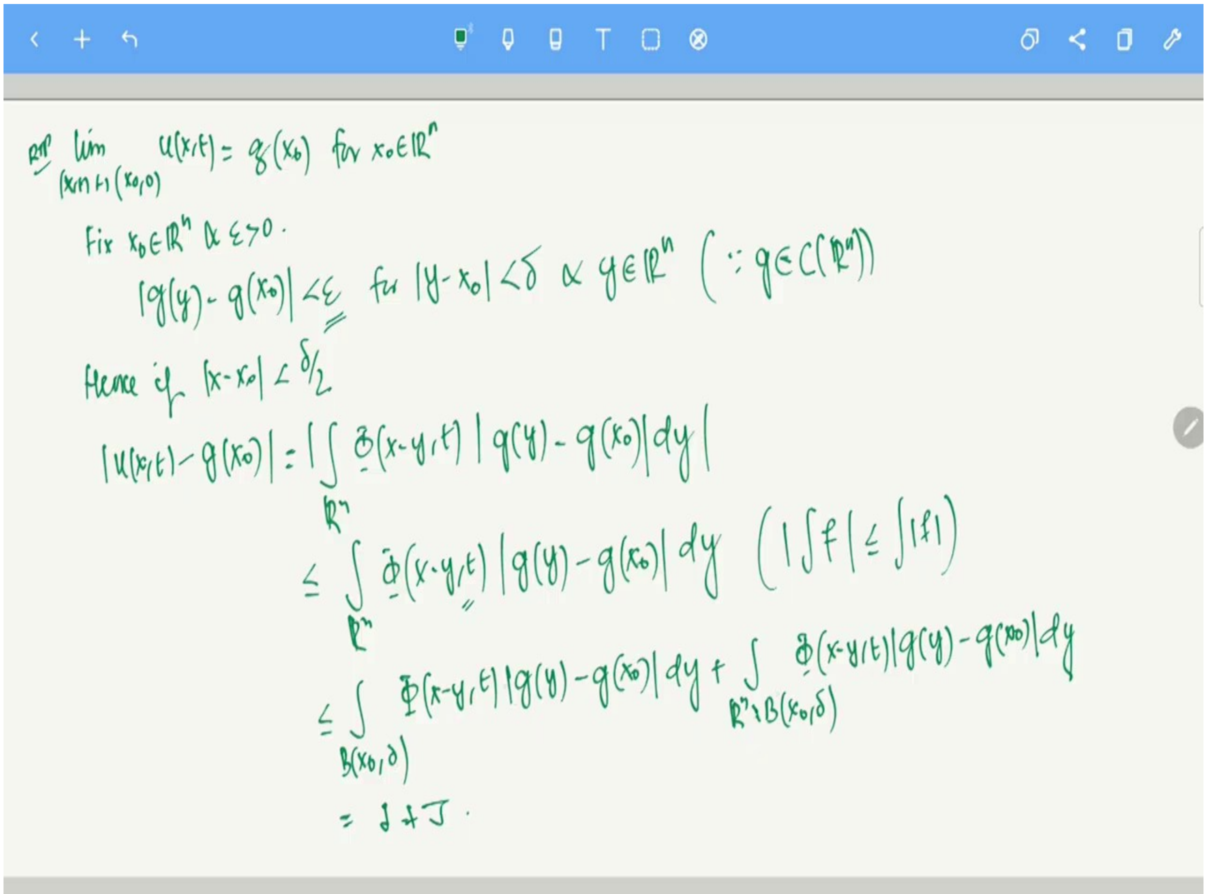

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If $|x-x_0| < \delta/2$ & $|y-x_0| \geq \delta$ then

$$|y-x_0| \leq |y-x| + |x-x_0|$$
$$= |y-x| + \delta/2$$
$$\leq |y-x| + \frac{1}{2}|y-x_0|$$

$\therefore |y-x| \geq \frac{1}{2}|y-x_0|$

(Continuity of $g$ is required to estimate $I$)

(Bdd of $g$ is required to estimate $J$)

$$\therefore J = \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(x-y,t)\,|g(y)-g(x_0)|\,dy \leq 2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\,dy$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|y-x_0|^2}{4t}}\,dy$$

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$$= \frac{C}{t^{n/2}} \int_{B(x_0,\delta)^c} e^{-\frac{|x_0-y|^2}{4t\cdot 4}}\, dy = \frac{C}{t^{n/2}} \int_{\delta}^{\infty} e^{-r^2/16t}\, r^{n-1}\, dr \quad \text{(Integration of radial function)}$$

$\to 0$ as $t \to 0^+$ (The growth of $\int_{\delta}^{\infty} e^{-r^2/16t}\, r^{n-1}\, dr$ is faster than $t^{n/2}$) (Check) (L'Hospital)

$\therefore$ If $|x-x_0| < \delta/2$ and $t>0$ small enough

$$|u(x,t) - g(x_0)| < 2\epsilon$$

Remark:- If $g$ is bounded and continuous s.t. $g \geq 0$ ($g \not\equiv 0$) then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, g(y)\, dy$$

$\therefore$ $u(x,t)$ is positive for all $x \in \mathbb{R}^n$ & $t>0$.

If $|x-x_0| < \delta/2$ & $|y-x_0| \geq \delta$ then

$$|y-x_0| \leq |y-x| + |x-x_0|$$
$$= |y-x| + \delta/2$$
$$\leq |y-x| + \frac{1}{2}|y-x_0|$$

$$\therefore |y-x| \geq \frac{1}{2}|y-x_0|$$

(Continuity of $g$ is required to estimate $I$)

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$$\therefore J = \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(x-y,t)\,|g(y)-g(x_0)|\,dy \leq 2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\,dy$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|y-x_0|^2}{4t\cdot 4}}\,dy$$

RTP $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$ for $x_0 \in \mathbb{R}^n$

Fix $x_0 \in \mathbb{R}^n$ & $\varepsilon > 0$.

$$|g(y) - g(x_0)| < \varepsilon \quad \text{for } |y - x_0| < \delta \ \& \ y \in \mathbb{R}^n \quad (\because g \in C(\mathbb{R}^n))$$

Hence if $|x - x_0| < \delta/2$

$$|u(x,t) - g(x_0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t) \, |g(y) - g(x_0)| \, dy \right|$$

$$\leq \int_{\mathbb{R}^n} \Phi(x-y,t) \, |g(y) - g(x_0)| \, dy \quad \left( \left|\int f\right| \leq \int |f| \right)$$

$$\leq \int_{B(x_0,\delta)} \Phi(x-y,t) \, |g(y) - g(x_0)| \, dy + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t) \, |g(y) - g(x_0)| \, dy$$

$$:= I + J.$$

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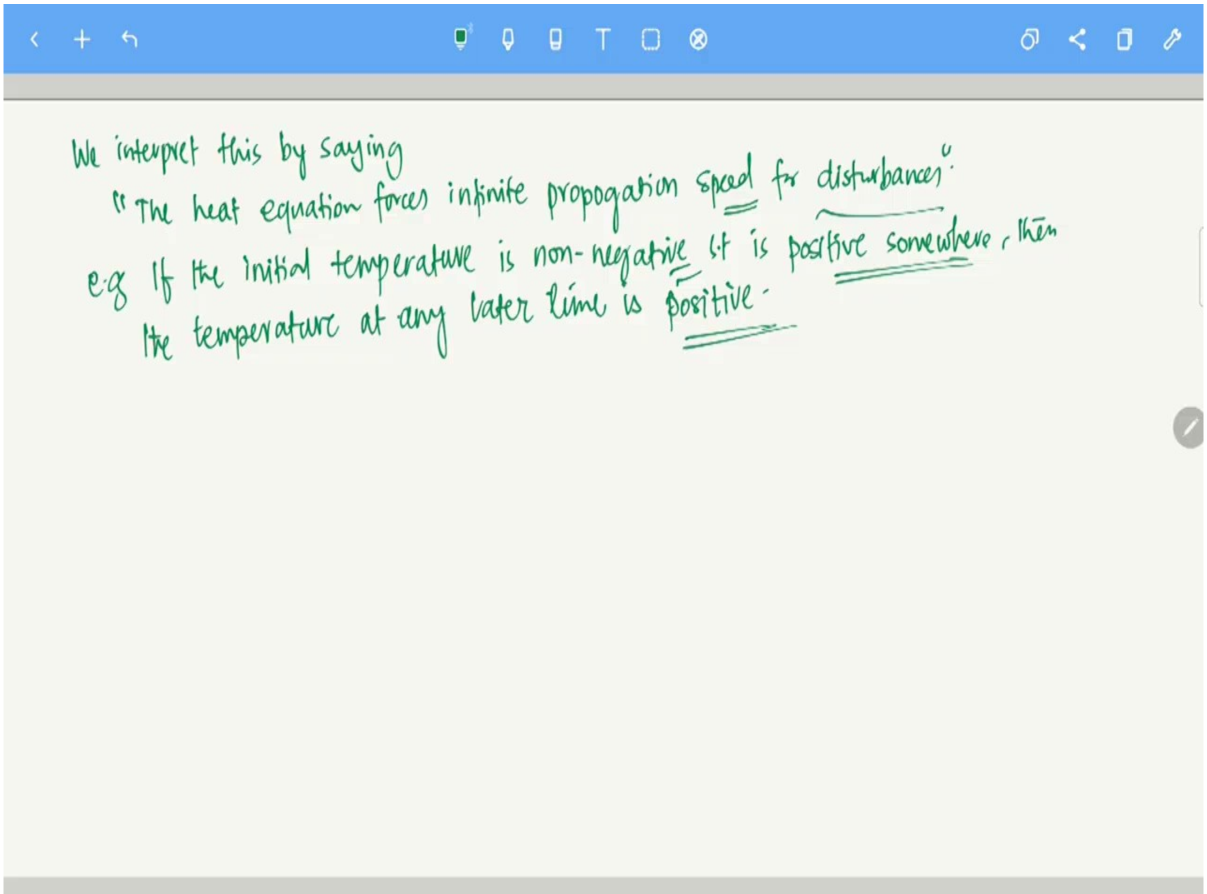

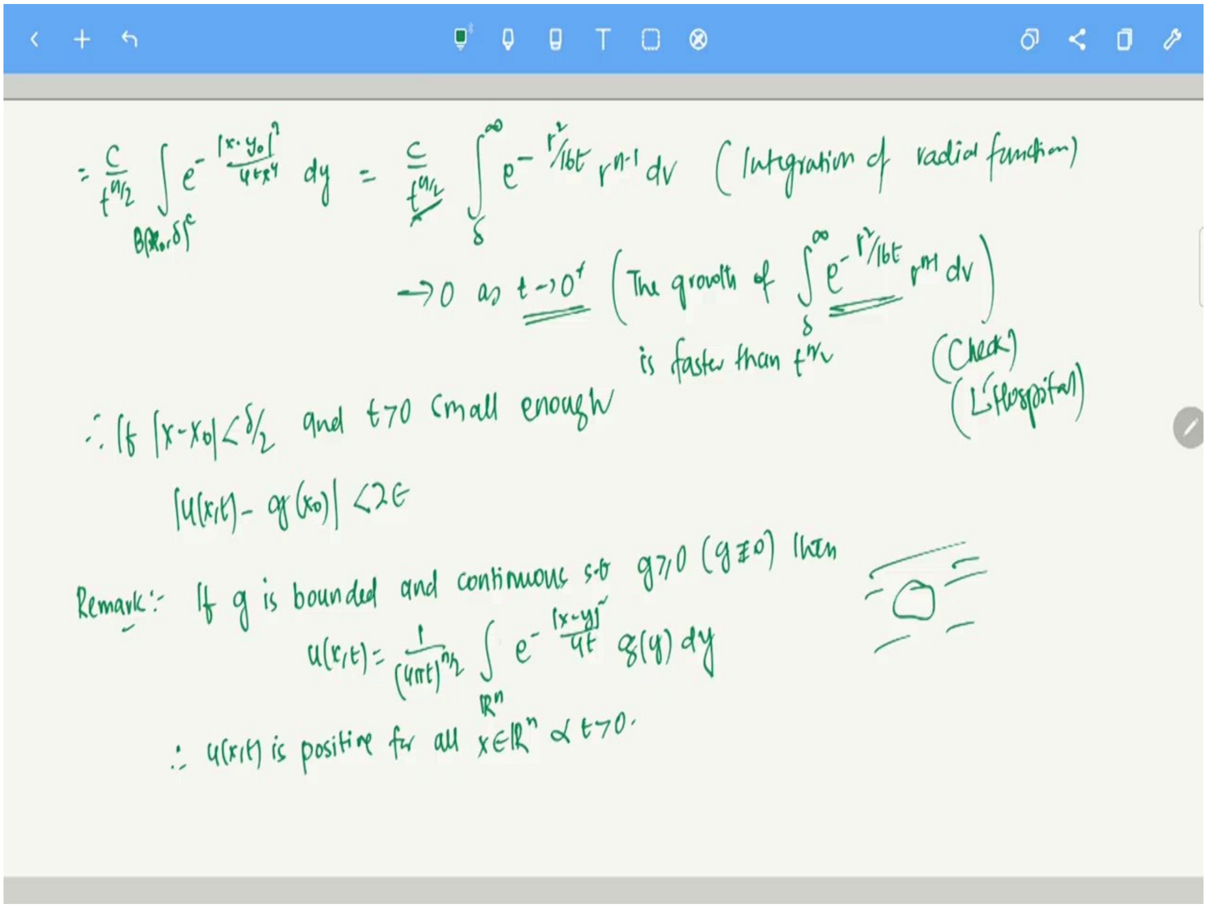


So, please remember that heat equation, this is called the speed of propagation. How does the disturbances spread over time? Disturbances means, initially there is a disturbance. We want to see how this disturbance distributes itself and that is actually the truth. If it is non-negative, then the temperature is positive everywhere, so basically infinite speed of propagation. This is what we call it as a infinite speed of propagation.

What does that mean? It means that it does not die out. Is it fine? It does not die out. So, once it is positive at one point initially it will remain positive everywhere for all later time. So, it does not die. So, that is what infinite speed of propagation does. So, with this we are going to end this lecture.