Advanced Partial Differential Equations

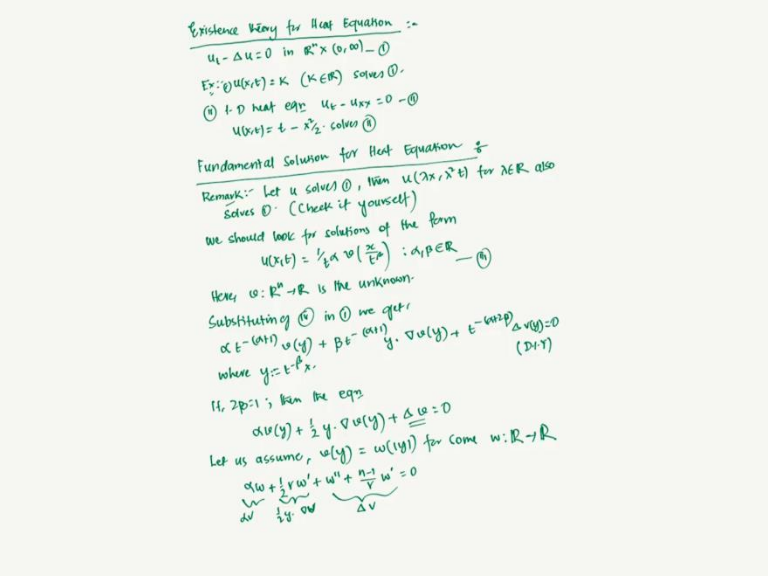
Professor Doctor Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture 19

Fundamental Solution

So, in this part of the video, we are going to talk about the existence theory for heat equation.

(Refer Slide Time: 0:25)



If you remember, for heat equation up till now, we want to talk about well pose this. And what did we derive? we derived that there is a unique solution. If omega is bounded, we have a rare unique solution and the backward uniqueness too. But now, we want to find the heat equation in solution, I mean existence. So, let us say you have are given this equation ut minus Laplacian of e equals to 0.

And this is for now let us just assume this is in our Rn cross, this is the time variables 0 infinity. So, this is time this is x. So, this is your equation. Now, what sort of functions can you think of we satisfy this equation. So, for example, let us say u of xt equals to a constant, any constant k, whatever is k, k is in r let us say. Solves one, so lets us say that 01 solves 1, is still not very difficult to see.

And of course, number 2, let us just assume that one dimensional and heat equation 1D heat equation given by ut minus ux, x equals to 0. That is one dimension in heat equation. And what is the solution of this thing? u of xt, how does it look like, any solution of one dimensional and heat equation, it will look like this.

So, essentially, it will look like u of xt is see, I mean, I am just showing you one way of doing this thing. I want to u such that who is one derivative with respect to t is the twice derivative with respect to x. So, essentially, let us say u of t is t, u of xt, I am starting out with t.

So, if you are taking ut here is basically 1 and minus this has to be uxx. So basically, you are looking for u such that uxx is 1. So, uxx is 1 is essentially x square by 2. So, you take 1 derivative is 2x and twice derivative is 2. So, that is 1 and 1. So, this is 1 is a solution of this. Let us say that is 2 solves 2. Of course, you can construct in such a way you can of course construct a solution like this which satisfy the 2 dimensional or any dimensional equation. That is of course can be done. But we are interested in much more.

If you remember from the Laplace equation, we can derive something from the fundamental solution which we use to solve the Laplace equation, which we use to solve the Laplace equation, in homogeneous Laplace equation. So, we are also here also, we are going to do exactly the same kind of thing. We are going to find some kind of fundamental solution of heat equation.

So, fundamental solutions are you know I do not want to give you the exact definition of fundamental solution, is not very difficult. It is basically this thing equals to the direct delta. Then any solution which satisfies that, but the thing is, I am not quite sure whether you guys know direct delta because that is basically a distribution. So, I am going to skip all the jargons associated with fundamental solutions.

So, for now, is a very, very special kind of solution, you can think of it like that. So, what is the fundamental solution? Fundamental solution for heat equation. So, you see for Laplace equation, we know what is the fundamental solution. I want to find it for the heat equation.

So, one small remark is to see the remark while finding fundamental solution. If you remember for Laplacian, we actually use the fact that it is rotational invariant, Laplace equation. And hence, we look for regular solutions. Here, no such thing can be done because the variable t is a problem. When t varies between 0 and infinity that we have some problem.

So, what we are going to do is we are going to only look at some scaling factors. So, once more remark, this I want you guys to check. So, let u solves 1. If u is small as 1 then if you scale this u. So, as that u of lambda x lambda square of t, you do it like this for lambda ER, then these also solves 1. Is this clear?

So, if u solves 1, then u of lambda t lambda square t for lambda ER also solved 1. Now, this actually gives us some idea about what sort of so this you guys have to check it yourself. Not very difficult thing to do, I am absolutely certain that you guys can do it yourself, you just need some chain rule, application of chain rule. Let us do that.

Now, once you have this, so you see essentially what that says is we basically should look for solutions of the form u of xt as 1 by t power alpha. Solid form of function V of x by t power beta. This is what enter alpha beta it is R. Here V from Rn to R is the unknown. See, here this says that if you scale the x variable as lambda x and t as lambda square t, so scaling is with respect to t it has to be lambda square, with respect to x it is just lambda.

With this scaling, what we can show is, if u solves this equation that scaling if you scale the u like that, then it also that particular function, this new function that also satisfies the heat equation. So, from here what you get is you are basically getting that, I mean, there is a scaling going on between x and t. And we want to exploit that particular idea. That is why we are just taking it like v of x by t to the power beta.

So, essentially if you think of it like this, it is basically x by root t it should be, x by root t lambda is 1. So, here this should be square, I mean, if you want to scale it properly, so basically, v of x by lambda root t should be there. But we are that should actually working, but we are not doing that we are just writing it as a arbitrary beta and we want to find what beta is.

Now, let us say this is 3. Now, substituting 3 in 1, in this we get as I am just writing it down. Alpha times t power minus alpha plus 1, v of y plus beta times t power minus alpha plus 1 y dot gradient of vy plus t power minus alpha plus 2 beta. Laplacian of v of y equals to 0. How will you get this? I am just putting this thing in this equation. So, basically you have this u, write down one ut is, and after that write down what uxx is, and after that you just put it together.

So, this also you have to do it yourself, you have to cross check in this part. It is very, very easy thing to do and not a problem. So, here, where y is defined by minus beta times x scale. So, with this definition, we are just writing it like this. Now, you see here, this beta is a problem. Why it is a problem, because if 2 beta is 1, then all of this expression is basically the same so I can throw out t2 minus alpha plus 1. See if 2 beta is 1, then see, alpha beta is on our hands.

If I am taking 2 beta 2 be one, so, this will be from minus alpha plus 1. This is minus alpha plus 1, this minus alpha 1 plus 1. I can throw it out. If I throw it out, what will happen is the equation transform into alpha times v, v of y of course, plus half y dot gradient of v of y plus Laplacian of v is 0. This is what we are getting. And if I want to simplify this further, so let us assume, see again v and all these things is on our hands. So, we can assume whatever we want. Of course, it has to be in line with whatever the equation says.

So, we cannot just assume any arbitrary thing you want, but essentially what all you are going to do is, we are going to look for radial v. So, we are going to assume v of y should look like some function w of mod y. For some w from R-to-R. So essentially I am looking for v which is radial.

So, if that is happening, again you can just take, this thing we calculated flapper. If v of y is radial function of w then Laplacian of V is w double (())(10:46) n minus 1 by l w. This we need to why we did Laplacian. So, let us write it down off so plus half rw prime, because mod y is r here. So, this is why r and gradient of v is basically w prime and this one Laplacian of v is w double prime plus n minus 1 by r W prime is 0.

Let me put it this way, this is for your Laplacian of v, this is for your half y gradient W sorry gradient V, and this is of course alpha times v. See, gradient of v, v is radial. So, gradient of v is just the derivative of v. I mean, v is w of mod y, so that was the derivative of w. So, which is w. Why is r why is basically mod y? So, that is your rw (())(11:48) with r.

(Refer Slide Time: 11:57)

$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0 \quad (\alpha = \frac{n}{2}) \quad (D.1.7)$

$\Rightarrow r^{n-1}w' + \frac{1}{2}r^n w = c$ (c is some constant)

Assuming, $\lim_{r\to\infty} w = \lim_{r\to\infty} w' = 0$.

$\Rightarrow c = 0$.

$\therefore w' = -\frac{1}{2} r w$

$\therefore w = d e^{-r^2/4}$. ($d$ is some constant)

Our required function, $u(x,t) = \frac{d}{t^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$

Definition:- The function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

is called the Fundamental Solution of the Heat equation.

Properties of Fundamental Solutions:-

(i) $\forall\, x, y \in \mathbb{R}^n$ & $t > 0$, $\Phi(x-y,t) > 0$.

(ii) $\forall\, x, y \in \mathbb{R}^n$ & $t > 0$, $(\partial_t - \Delta)\Phi(x-y,t) = 0$

(iii) $\forall\, x \in \mathbb{R}^n$ & $t > 0$, $\int_{\mathbb{R}^n} \Phi(x-y,t)\, dy = 1$.

(iv) For any $\delta > 0$ and for all $x \in \mathbb{R}^n$, $\lim_{t\to 0^+} \int_{|y-x|>\delta} \Phi(x-y,t)\, dy = 0$.

So, once you do all of these what we can do is I can write this equation as r power n minus 1 w prime, whole prime plus half r power n w prime whole prime this is equals to 0. This is after setting alpha as n by 2. This is see, once it is now this alpha is also in our hand. So, if I am replacing alpha to be n by 2, then that equation transforming to this, this also please check it yourself.

So, there are 3 things these are very easy thing to check. So, it is not a problem. And thus, from here, what do we get we get r power n minus 1 w prime plus half r power n, w this is equals to some constant C. C is some constant. Now, you see what happens as I mean, I want to find that what happens to the limit of w as r tends to infinity. So, we are assuming, see, this w is on our hands, so we are assuming that the we are looking for a radial function such that limit r tends to infinity w equals to limit r tends to infinity w prime, which is equals to 0.

This is what we are assuming. Why we are assuming this because this will actually give us C equals to 0. So, if C equals to 0, therefore, what we will have see, what we are trying to do is we are going to try to find some solutions of that. Alpha, beta is on our hand. So, we chose beta in such a way that the equation becomes more easy. Now, after that, we chose w, which is a radial function. So essentially, we chose V in such a way that it becomes a radial function and we got this.

Now, I want to remove the C to and of course, alpha equals to n by 2, we are getting these things. So, I chose alpha also. Now, I want remove this C, to remove this C, I am assuming that the radial function which I am choosing we will do this thing. That will give you C equals to 0. Now, if C equals to 0, what does that gives you, that gives me w prime equals to minus half rw. From here, I can just throw out r1 n minus 1.

So, you understand this is on our hands. We are manipulating. That is what we are manipulating the solution. And therefore, what do we have, if you want to solve this thing, we have basically w is some constant. I do not know what do I put a, b, c whatever I put everything. Let say d, d exponential of minus r square by 4. Let us just choose. I mean, the d is sub constant.

So, I hope you understand. This is all manipulation. We are manipulating our solution to look like this, that is all. So, if you combine all of these, so if you combine these w, w is what v and if you combine all of these then what do we conclude, we can conclude that the function which you are choosing, so our required function let us say u of xt, how does it look like, it looks like some constant d by t to the power n by 2 exponential, this is exponential minus mod x square by this e exponential, by 4t.

So, this is what we are going to get. So, from here we are going to define the fundamental solution. So, the function f of xt or C of xt, let us just call say c of xt, which is given 1 by 4 by t whole power n by 2 exponential minus mod x square by 4t. And x is in Rn and t positive and 0 for x is in Rn and t negative, is called the fundamental solution of heat equation.

I hope this is clear to all of you. See, why it was happening is because we have chose, we chose our u to be 1 by t to the power alpha. Alpha we chose to be n by 2. So, 1 by t2 n by 2 only we are getting the solution. So, alpha we chosen to be n by 2 that is why 1 by alpha which is n by 2, and v is x by t to the power beta.

So, that is why v is w. So, basically it is exponential of minus mod x square, r is mod x, mod x square by 4t. Because beta we just put the value of beta and this is what we are getting and this is the fundamental solution of equation. So, please remember this is a very, very important property and very so, this I mean, this is our basis of all harmonic functions sorry, not harmonic function, this is a basis of many different analysis in harmonic theory.

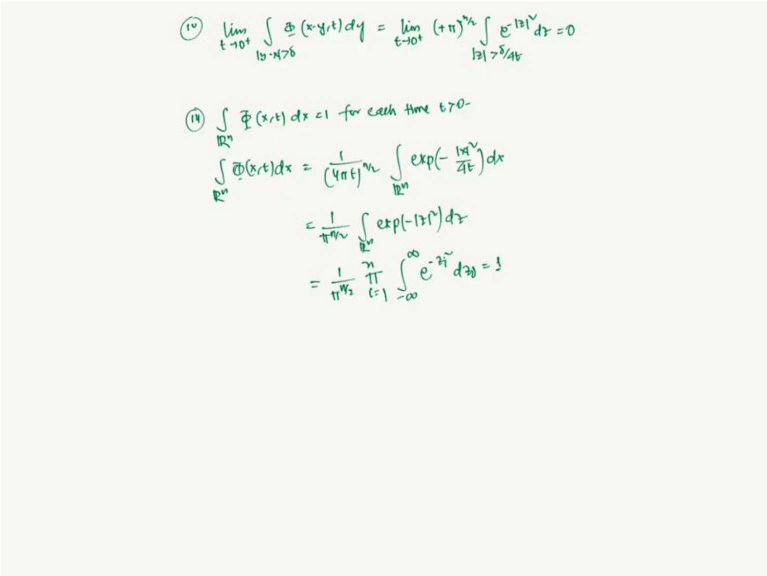
So, let us write down some properties of fundamental solution. So, some properties of fundamental solution. So, number 1, for all xy in rn and t positive, fundamental solution of x minus y times t, if we just write it like this, this is always positive. I hope this is fine. Exponential function is positive, this is a positive function as there is nothing to do here.

Number 2, again for all xy in Rn and t positive, see, if you write this thing like delta t, so which is ut minus Laplacian of fi of x minus y times t. Can you guess what it is, this is going to be 0, because fundamental solution this actually solve the integration. So, it has to be this. I hope do, you understand, what is the delta of e this vt.

And for all x in Rn and t positive, this is true, integral over Rn fi of x minus y times t dy, this is going to be 1. Why this is see, this is the relation which you want. So, basically y, the integral of this fundamental solution has to be 1, just like a equation and we just because of that we took these d to be 1 by 4 by whole power n by 2. Do you understand? That is the reason why we took on this is a constant.

And number 4, the last property, is for any delta positive and for all x in Rn limit t tends to 0 plus and integrals. So basically, if you are looking at a fundamental solution above I mean, you are outside the exterior domain fi of x minus y times delta sorry t, dy is 0.

(Refer Slide Time: 20:45)



So, these are the properties. So, let me proof one and two are just like I mean, the easy one. Let us prove the fourth property first and then we are going to the third property. So, the four property is limit t tends to 0 plus integral over the exterior domain sorry, this is greater than delta, x greater than delta.

So, essentially, what it is saying is if you are taking the fundamental solution of the heat equation dy outside the ball with radius delta in the exterior domain, and as you push t towards 0, this is becoming 0. How do you show that? So, to show something like this, what we are going to do is we are going to write this as limit t tends to 0 plus, this is some calculation. Let us write it like minus pi times n by 2 minus n by 2, sorry.

So, I am just writing out what phi of t is and after that I am using a change of variable. So, I hope you guys can check this yourself, this is not very difficult thing to do. So, for mod z greater than delta by 4, that is why that I mean, there is 1 by 4 term is there, that is all. Because I am just choosing mod delta to be greater than delta y by 4t, this change of variable, this turns out to be mod z square dz, and that is becoming.

So, I am just using a change of variable here and you write in phi to be mod z greater than delta by 2. On this domain, I am just writing fi, which is exponential minus mod z square.

Once we have does this, this constant will look like this and we can show that as limit t tends to 0, this particular thing goes to 0. Because this is becoming smaller and smaller.

See, as t goes to 0, this particular thing for delta by 4t is going towards infinity. So, basically you are taking mod z which is greater than extremely big, and mod z is extremely big. So, and this integral sensitivity summable this has to be concealed. Now, what about the third property, the third property is also a change of variable.

So, essentially it says that, I mean, you see, essentially third property you can write it like this, there is nothing wrong in writing this thing. fi of t, lets us say t x, fi of I mean, you can take phi x minus y to be I mean, we can just write it like, fi of xt. This is our fundamental solution.

What these equations, the third equations says is integral over Rn fi of xt or fi of x minus yt, whatever you want to write it here, that is not a problem, you can use that change of variable. So, fi of xt dx, this is going to be 1, for each time t positive. Now, my question is how would you calculate this thing, this is very easy. Let us write integral of Rn fi of xt, dx, this is 1 by 4 pi t whole power n by 2 integral over Rn exponential minus mod x square by 4t, d of x.

And this is 1 by pi 2 power n by 2 integral over Rn exponential minus mod z square, dz I hope this is fine. Exactly the same thing which you did here, just a change of variables. Once you have these, then this is 1 by pi to the power n by 2. This you can write is as x1 square, x2 square, xn square exponential that will break it up so that will be the product of n equals to sorry, i equals to 1 to n minus infinity-to-infinity exponential minus Zi square dx and that is going to be 1 sorry, this is tzi and that is going to be 1.

So, this is the properties of fundamental solution. Now, in the next portion or next week what we are going to do is, we are going to use fundamental solution to find out the solution of the initial value problem of the heat equation. So, with this, we are going to end the lecture.