Advanced Partial Differential Equations
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Lecture 18
Backward Uniqueness

So, in this part, let us talk about something a little different from uniqueness. So, essentially this is the uniqueness, but this is called a backward uniqueness.

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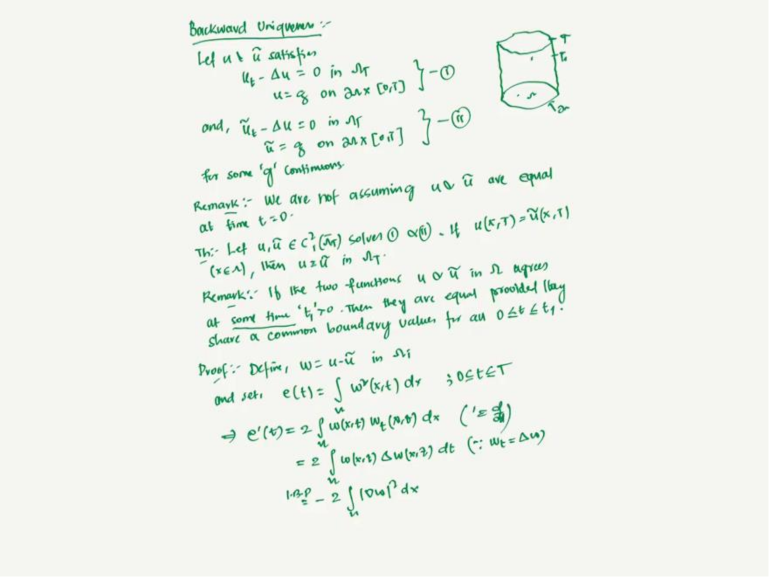

So, backward uniqueness. Now, you see in Laplacian what happens, if you just have a omega, and in that domain, you are just working out, if the solution is unique or not. But here there is a time variable which is involved. So, at natural question is this initially in unit is what we did is, we just assume that the u is g or some parabolic boundary and it satisfies the in homogeneous equation you have to show that the solution is unique or not and we have seen that the solution is unique.

But if the question is something like this, let us say if the question is, so let us say u, let u and u tilde this satisfies the ut minus Laplacian of u equals to 0 in omega t. And u equals to g on the only the dell omega across 0 t. So, dell omega cross 0 t. So, only the sides, if you will.

So, let us say, that is your domain. Let us say, this is our domain, boundary of omega is this, dell omega. So, let us say that is 0 omega, and this cross 0 t, so only this part, so only the top part, this part, only this for nothing inside, not inside, nothing inside, only this part, that part, this part, and this part.

So, a point here, does it belong to that set? No. So, the boundary we are only looking at the parabolic boundary minus the base is base. Of course, the boundary is included. The boundary of omega cross t equals to 0 is included, so that is here. So, let us say u satisfies this equation and, and u tilde also satisfies this equation. So, ut tilde minus Laplacian of u, this is 0, in omega t. And u tilde is g on the boundary cross 0 t sorry.

So, this is for some g continuous, clear. Now, one small remark here. So, what am I starting out with, I am starting out with two functions, u and u tilde. Such that u satisfies this equation and u tilde satisfies this equation. So, essentially this satisfies both the same equation here, but you note that we are not assuming u and u tilde are equal at time t equals to 0.

So, we are not assuming they are equal, you see it may be different and these points we do not know. We just know that on the boundary u is g and u tilde is there that is all. On the boundary, they are equal, of course on these boundary, not at these points. So, t equals to 0 it may or may not be equal. So, we are not assuming they are equal.

Now, what is the delta doing this says the theorem says. Let u and u tilde is in c 2 1 omega t. But of course, but also, we are always taking the boundary and solves 1 and 2. What is 1, this is 1 and that is your 2. So, let us say this solves 1 and 2. And if this is very important, u at x, t equals to u tilde at x and capital T, and x is in omega. So, what does it say? It says that u and u tilde they are same. For let us say this is some t.

I mean, see, here I am just putting this t here, there is nothing special about this t. It can be here also, it can be here also. So, let us say where t for now let us just assume that this is t. I mean, the cylinder may go on, it is not like you specified up t here only it can be. So, omega t can be here also. So, let us say, t1, so I can do it for omega t1 also.

So, if that happens then, so essentially if for any time t. For let us say, t equals t1 or t equals to t, the small t equals to capital T or small t equals to capital T, or whatever it is. If that is happening, if u is same for all x capital t, and u and u tilde are same, essentially at that those points then what happens to them say u is equivalent to u tilde in omega capital t.

So, what this is saying is, let us put it another smaller remark, it is saying that if the two distributions on omega, if the two distributions with functions maybe, functions u and u tilde. Let us say they of course in omega it means at some time t. Of course, this is positive. So, let us say, t1 positive, then and of course the boundary values. So, you see the boundary value has to be same, and then they are equal provided they share a common boundary or common boundary values for all time 0t, t1.

So, essentially, I am taking t1 here you can just say t1 to be capital T, that is not a point, that is not the problem. So, here what it is saying is if it is same at one some point. So, this is important, at some point. If it is same at some point, then it is saying that they are equal of course the boundary condition has to be equal for all t. And then, so this is the capital, let us put it like this is this t, this is t1.

So, if this is holding, this hold for all t between 0 and t1 then the functions are identical equal. So, you have to have the function equal at for some t. And then, and on the boundary, it has to be equal for all those t, for all t below this capital T1 then the functions are same. So, let us look at the proof of this thing proof. So, what is the proof of this thing? First, of course, we are starting out with let I mean, you already have things. So, we define w, which is u minus u tilde. And of course, this is in omega t.

And once you define this thing, then you will and set, I mean this is our initial thing, whatever we did earlier, omega square xt, dx. So essentially, I am taking the I am fixing a t and for each fix t, I am defining a et, says that this is the square of w and this is hold for all t between 0 and capital T. If that happens, then I mean, of course, you can you see w is u minus u tilde, u minus u tilde is twice differentiable with respect to x and once that was differentiable with respect to t.

So, essentially, this is thrice differentiable that in this function e of t. So, I can take at least one different derivative here. If I take one derivative here that will give you two integral over omega. If you remember, this is omega of xt, omega t of xt. Why can I write it like this, because you see, I can take the derivative of with respect to t inside, because this is the integral with respect to x.

Now, if I just replace, you see Wt satisfies what? Wt satisfies 0. I mean, Wt equals to Laplacian w in omega. So, essentially this is equals to in t2 integral over omega w of xt and Laplacian of w, xt, dt. That is what you are going to get. And after that integration by parts, since Wt goes to the Laplacian of w. And see after the u integration by parts that will give you minus 2 times integral over omega gradient of w square dx. I hope this is fine. I mean, we did this earlier also. Of course, this prime, I mean the prime is DBT with respect to time.

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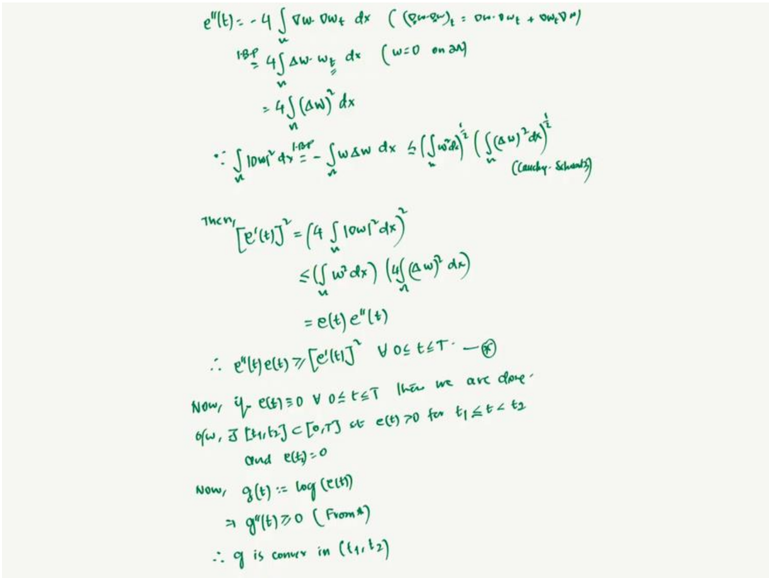

Fine now what we are going to do is we are going to take the another derivative. As I told you, ut is prime differentiable. So, I can talk about the e double prime of t. What is e double prime of t? If you can just I mean there is nothing to do here. So, let me just write it down. It is gradient of w, gradient of Wt, d of x. Why? Because see essentially you have a gradient of w dot gradient of w. I mean if you take the derivative of this thing, so it is basically gradient of w, gradient of Wt plus gradient of Wt gradient of w. And then you will get this thing.

See, here this gradient is with respect to x, that is why with respect to t this is coming going inside. So, you have this and again gradient of Wt is Laplacian of w. So, this will give you four integral over omega Laplacian of w, where is it, one sec, Laplacian sorry Wt, Wt we cannot change, we know what Laplacian of w is. What do we know, we know that so, no, I want to change it here. I will do an integration by parts here. We will do an integration by parts and I will put push this in derivative here.

So, it is integral of our w it will become I mean Laplacian of w dot Wt, d of x. Is this clear? And what about the boundary condition, see w is 0 on the boundary on dell omega. So, that is not here, clear. Because u, is u tilde on dell omega. So, this is true. So, now, what we want to do is this see. The Laplacian of w Wt is Laplacian of w. So, this essentially says this is whole integral Laplacian of w square d of x. This is what we are going to get E double prime of t. Now, therefore, what do we get see, therefore, integral over omega gradient of w squared d of x, this is equals to let is say integral. This is what we got last time. It is w Laplacian of w, d of x, gradient of w squared, if we just write it from integration by parts, we are going to get this. Because w is 0 on the boundary. So, we are going to get this.

Now, once you get this you can take the so the mod of this thing. The absolute value of this thing is less than, but I am not writing all that, it is usually less than equals to integral over omega w square whole power half and integral over omega Laplacian of w square dx, over half. This is true, see minus of this thing, if you take the modulus of this, that will be less than equal integral or omega mod of this thing. And then you use Cauchy Scouts inequality to get what you get so this is Cauchy Scouts inequality.

Now, once we do this thing, then what do have then e prime of t, let us take the square of this thing, e prime of t square, that is what? It is 4 integral over omega gradient of w square, dx, whole power whole square, clear. And that is equals to less than or equal to integral over omega w square dx and integral over omega 4, I will write it like this, Laplacian of w squared dx. And this is what? This is e of t, e double prime of t. I hope this is clear to you.

See gradient of w squared dx is this. So, I just wrote it here and after that put the 4 inside. If I am putting the 4 inside, that is what? This is essentially e double prime of t. So, that is e double prime of t. So, maybe I can write it like this, it will be more clear being put it this way. So, e double prime of t.

So, what do we have? Therefore, we have that e double prime of t times e prime e of t this is greater than equal e prime of t square for all 0 less than t, less than capital t. Now, this inequality, if you look at properly, see this is minus e prime t square is redundancy and that will give you some I mean, you know something like e of t by e prime of t that derivative of sorry, e prime of t by ut the derivative of that, something like that. I mean, if you just write it together. This is an od.

So, essentially, this gives us an idea. And what is that idea? So, let me write it. So, basically there is some log involved here. If I involve a log here, I will tell you what all of these means. So, for the next step, what we do is now, see, if e of t, I want to show uniqueness. So, if e of t is identically equals to 0 for all t between 0 and capital t, then we are done. Why we are done, because e of t is integral of w square. If ed of t identically equals to 0, the integral of w squared is 0, that will give you w is 0. And that will give you u equals to u tilde, so we are done.

Otherwise, let us say, there exists an interval t1 and t2, which is containing of course 0, capital t, such that e of t is greater than 0 for t1 less than t less than t2. And e of t2 is 0. This is always true. If it is not 0, if it is positive somewhat then this kind of thing is bound to happen. Why there is always a t2 says that e of t2 is 0 because e of t is always e of capital t, if you just look at, what is e of capital t.

So, see, e of see, this is for any t2 between 0 and t. And what is e of capital T, e of capital T is going to be 0. So, essentially what is happening is there is always such a t for which this all of this happens. So, now what it means, now we write g of t, you define g of t to be log of et. And once you define it, you can actually show that g prime of t this is I want you guys to do yourself, this is very easy. This is always greater than or equal to 0.

How? Let us say that is star from star. I hope this is quite fine. See, what is g prime t, it is e prime of t by et. And after the g prime t is essentially, g double prime t is essentially this thing, and this is greater than or equals to the both e double prime t or this is greater than equals to 0. So, what does that gives you, therefore, that says that g is convex in t1, t2. It says that it is convex I am sorry, this is open t1, t2, it has to be open t1, t2. That is what it is saying.

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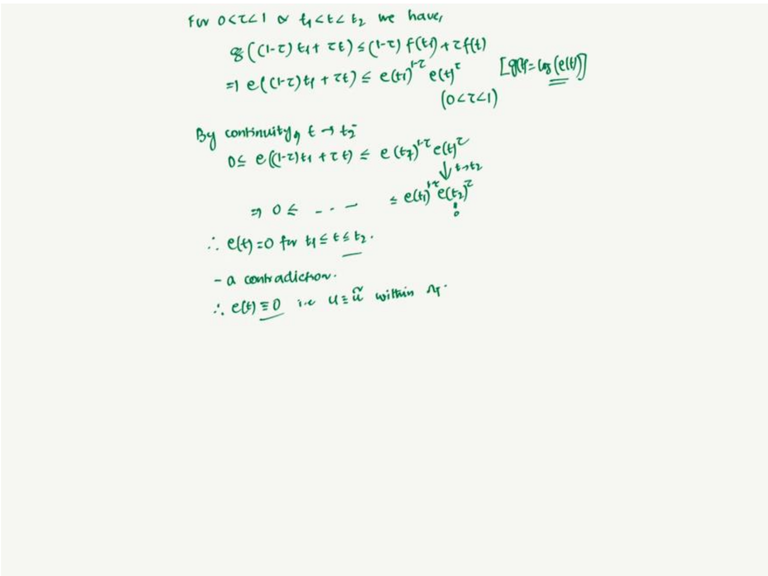

So, now once it is convinced in open t1, t2, want we can do is we can just write it like you see for 0 less than tau less than 1, and t1 less than t less than t2, we have g of 1 minus tau, t1 plus tau t. This is less than equal 1 minus tau f of t1 plus tau f of t. I can write it like this. So, for any t between t1 and t2, I can just write it and the right the expression like this.

And see, if what is g, g is log of exponential t. So, if I just put it together what is going to happen, log of not explicit sorry, sorry, g of t is log of et. G of t it is log of et. If you put it together, what is going to happen, this will just be exponential sorry, it is not exponential here, it is 1 minus tau times t1 plus tau t, this is less than equals to log is there.

So, this will go inside e of t1, 1 minus tau e of t to the power tau, this is for 0 tau 1. And if this is happening by continuity what can you say, once eu is continuity and take t towards t2 minus. By continuity, when t2 goes t2 minus what happens. This will convert to see, e is a continuous function. So, this will convert. So, e power tau is also continuity for tau is equal to 0 or 1.

And hence this will converts, to e power t2, this whole for all t between. So, this will converts t power t2. So again, exponent e I am sorry, the function e is always positive function, because e of t for all t, so this is positive. So, essentially what is happening is 0 less than equal, so non-negative, exponential is always sorry, not exponential, what am I saying. See, e of t is integral of w square. So, it is a, integral of a square function. So, it is never negative, so it is always non-negative.

So, this is 0 less than equals to exponential 1 minus tau t1 plus tau t less than equal exponential t2, 1 minus t1, 1 minus tau exponential t to the both tau. Now, when this thing t goes to t2, this will go to exponential t2 to your tau, exponential t1 1 minus tau, this is less than equals to 0. So, what does that mean? It means that. Therefore, it means that exponential t is 0 because exponential sorry, no, so, what am I saying, why am I saying exponential? This is e, I am really sorry about this here.

So, this is e, not explanation. So, e power t, this is 0 for t1, t, t2. See, this is given to be 0. This is what our assumption was, this is 0. And so, whatever is inside you know e for all these t which is looks like this. So, the learning t and t1, t1 and t2. For all those t, e of t is 0, because this is 0, this is 0. So, this is finally between 0. So, this e power, this power e of this all these points are 0. So, that is going to be 0.

This is a contradiction. That is a contradiction to the fact that e power t is positive, that is what we assume. And hence it is proved that therefore, e power t has to be identically goes, to 0 that is u should always be u delta within omega t.

One small thing I want to again explain. I am really very sorry, if these are not exponential, see, I do not know why I am just calling it exponential, because we know whenever we see e, we just write it at exponential, but this is not an exponential. This is just the function e. So, remember, I am really sorry about that. So, with this we are going to end this particular topic, and next topic we are going to start-up with fundamental selection.