Advanced Partial Differential Equations
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Lecture 17
Introduction and Well-posedness

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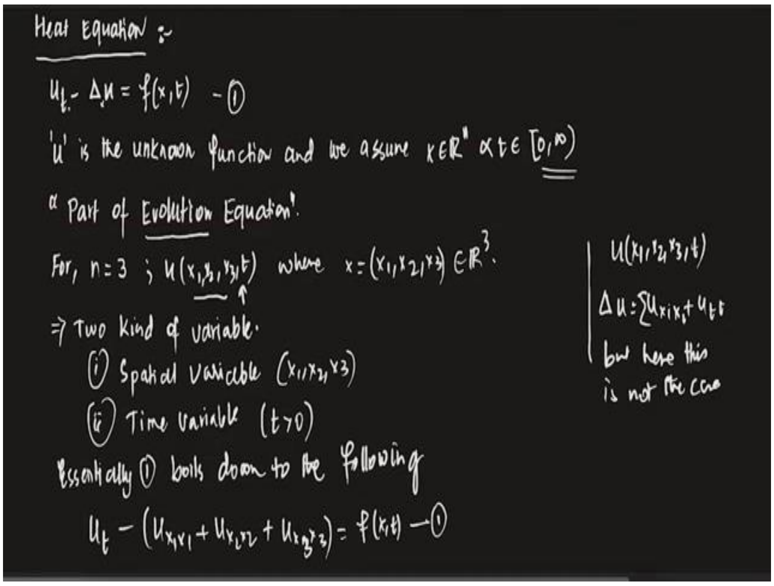

In today's class we are going to talk about heat equation. So, next few classes we are only going to concentrate on this, heat equation. Now, let us start by writing out what it means, what is a heat equation. So, essentially, generally speaking, we will write it like this ut minus Laplacian of u equals to let us say f, and this f will depend on x and t. Now, so these, let me write it properly and then we will explain what I mean by this. So, this will be in some domain. We will write down all this domain and everything, but first of all let us explore more what this means.

So, let us say that is your 1, this is defining some domain. Now, what is u? u here is the unknown function and you see here I write ut. So, basically u depends on a variable t, of course, because there is no point of writing ut and I also wrote a x here, yes. So, what all of this means? See, and here we assume x is in R n and t lies between 0 and infinity. So, essentially see t, here I mean this equation, this is a part of a, so let me put it like this, it is a part of something called evolution equation.

Now, what is the evolution equation? Evolution equations are those equations which evolve with time. So, over time, let us say you are modeling some phenomena which changes with time, those phenomena whatever the equation you could generate with those phenomena, they are called evolution equation because they evolve with time either degenerate with time or the other way around.

And here, what is happening is this one, as you can see, this equation has a ut term, so essentially this will depend on time and that is why it is a part of evolution, this is one of our evolution equations. The other one which we will study later is called a wave equation So, let us again come back to this equation, let us say x is in Rn, so let say for n equals to 3. How does this equation look like?

See first of all u, in this case, will be a function of x y, z, and t because you see u should contain, see there is one here you might get confused, let me put it this way it is x 1, x 2, and x3 where x is given by x 1, x 2, x 3 that is in R 3. So, you have a component from R 3, the first component and there is a component of time. This t, see t we are assuming from 0 to infinity, so t is the time variable.

So, there are two kinds of variables here. So, let me put it this way, too kinds of variables kind of variable, what are those? Number 1, one is called the spatial variable, so we will call it the spatial variable, what is the spatial variable here? The spatial variable is x, y, z here, not x, y, z, x 1, x 2, x 3, see I wrote x in R 3 here, so that is why I cannot use x, y, z. So, here x 1, x 2, x 3 that is the spatial variable.

And there is another variable which is the time variable. So, of course, whenever we are asked, you see working with time variable, you are assuming that this will be always positive. See, we always assume, whenever we are starting out the process, we are assuming that the time is 0 at that, and after that, we assume moving forward. So, time t will be always between 0 and infinity, so this is the time variable. So, this is given by t which is greater than 0, so spatial variable and time variable.

And now, let us look at this equation, see this equation ut is your partial derivative of u with respect to t minus Laplacian of u, these Laplacian, see u is a function of x 1, x 2, x 3 and t. So, generally speaking, in general case, you see if u is a function of x 1, x 2, x 3, and t, what is Laplacian of U? It is essentially, u x i, x i sum plus u of t t, that should be your Laplacian because if u will look like this, Laplacian of u, generally speaking, if it is written like Laplacian of u, it means this. But here, this is not the case.

So, here this Laplacian is with respect to the spatial variables. So, what I mean by this? So, essentially let me put it this way, essentially 1, boils down to this form, boils down to the following. This is ut there is nothing to do here, minus Laplacian of u is u x 1, x 1 plus u x 2, x 2. So, basically this Laplacian is with respect to the spatial variables with respect to x plus u x 3, x 3, this equals to f of x, t, that is your 1, so this is refreshed as 1.

So, of course, if x, i mean if you are taking x, to be in R n then we will go on to R x n and then u x 3, x 3 plus dot, dot, dot u x n, it will go like this. So, this is called the, see, in any of these cases, you see what is changing essentially is this x if x can be an R 1, R 2, R 3, R n, but this time variable is always there in between 0 and infinity. So, there is a small notation which we are going to use, so this is just a nomenclature you can say.

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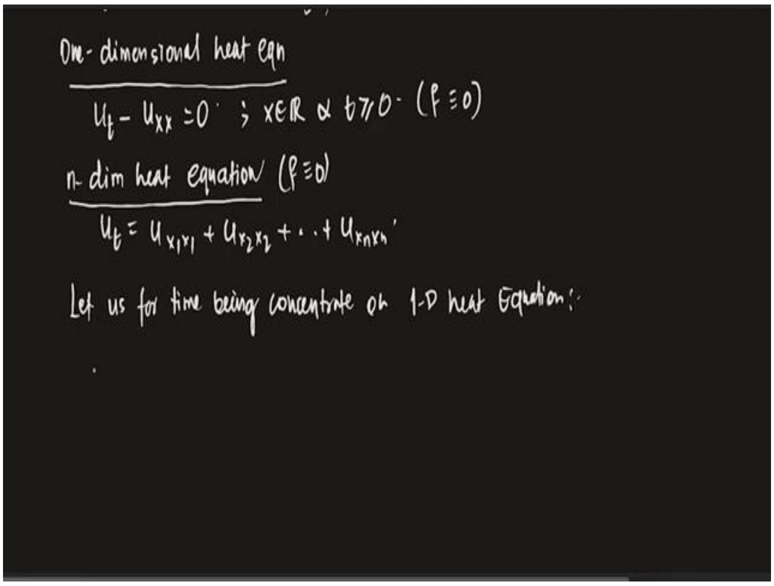

So, one-dimensional we say, one-dimensional heat equation, heat equation means that the dimension x varies in R, so x is one-dimensional. So, that will look like u t minus u x x equals to 0 that is your one-dimensional heat equation, see t is always between 0 and infinity, so there is nothing, I mean, t does not do anything, all i mean, whatever if there is a change in dimension that has to be in x, so that is why whenever we say one-dimensional heat equation it means u t minus u x x, only x variable is there, so x is in R and t greater than equals 0.

Now, if I say let us say n-dimensional heat equation, you can understand n-dimensional heat equation, of course, it means that you write it like this, ut minus u, so let us say ut equals two. So, you see here, I am just assuming f is 0, here I am just assuming the homogeneous problem. I mean, of course, if you do not have a homogeneous problem, of course, there is a f, that is there. So, I am just writing it to f equals to 0 for now, I am just assuming f equal to 0 here also f equal to 0 I am assuming, I mean, it may not be 0. In that case, you just add a f.

So, this is n-dimensional heat equation u x 1, x 1 plus u x 2, x 2 and it will go on, it will be u x n, x n, clear? So, that is your n-dimensional heat equation, so it is very easy. Now what we do is we look at some boundary conditions. So, what sort of equations are we talking about here? So, let us look at some boundary conditions which we encountered here.

And for now, let us just look at this problem only one-dimensional I mean, you see in n-dimensional same sort of boundary condition will work but for now, let us just concentrate on one-dimensional. So, let us for time being concentrate on 1-D heat equation. And what we are going to do is we are going to write down the different boundary conditions which you can encounter here.

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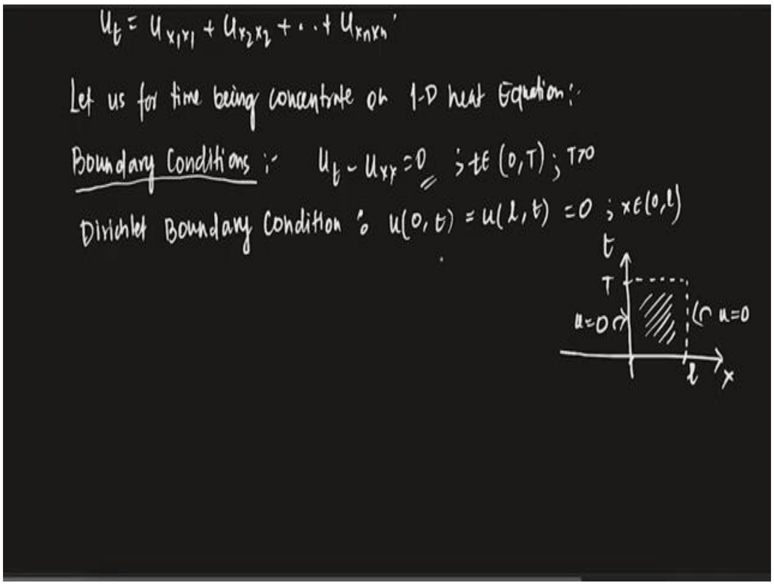

Boundary conditions, clear? What are the boundary conditions? So, essentially you have this equation ut minus u x x equals to 0, but what are the boundary conditions? So, before we look at the boundary condition you can guess that we do have to define what the domain is, but you see, I do not want to jump into a more complicated domain. First of all, let us do it for a one-dimensional thing. And we will look at a boundary condition and then we will go from there to a higher dimension, that will make things easier because to be honest with you, heat equations are a little tricky, it is not very, it is not easy, like Laplace equation.

So, first thing, first, let us write out the Dirichlet boundary condition. So, essentially, I am looking for this equation u t minus u x x equals to 0 the one-dimensional heat equation and with this, I am writing down the Dirichlet boundary condition, what is the Dirichlet boundary condition. Dirichlet boundary condition is essentially you see, the x I am choosing it from let say, some 0 and T, of course, T can infinity that is not the point here, but just for now, let us just say T positive and 0, T, let say I am choosing from 0, T.

So, Dirichlet boundary condition is the u at 0, t that equals to u l, t this equals to 0. So, x I am choosing from 0 to l, clear? So, here, I think, I write x, this is t, I mean, of course, t can be capital T can be infinite. I mean, you can just choose it for now just choose t between 0 and T but T is positive, it can be infinity also. And x, let us say, for now, I am just choosing x between 0 and l, and this is the Dirichlet boundary condition, so it is saying u at the point 0, t equals to u at the point l, t equals to 0, so what it is saying is something like this, let us draw some pictures here.

Let us say this is your x variable, and that is your t, t-axis, and 0 to l, that is your x and 0 to capital T. So, this is the domain of where you are working with, and in this domain, u satisfy this equation and what is the boundary condition u for x equals to 0 and t equals to t, u 0, t and u l, t is 0. So, essentially u 0, t and u l, t, so on this line, u is 0 and, on this line, u is 0, clear? u 0, t equals u l, t equals to 0. So, if this something like this happens then you, of course, have, it is a Dirichlet boundary condition, this condition is called a Dirichlet boundary condition.

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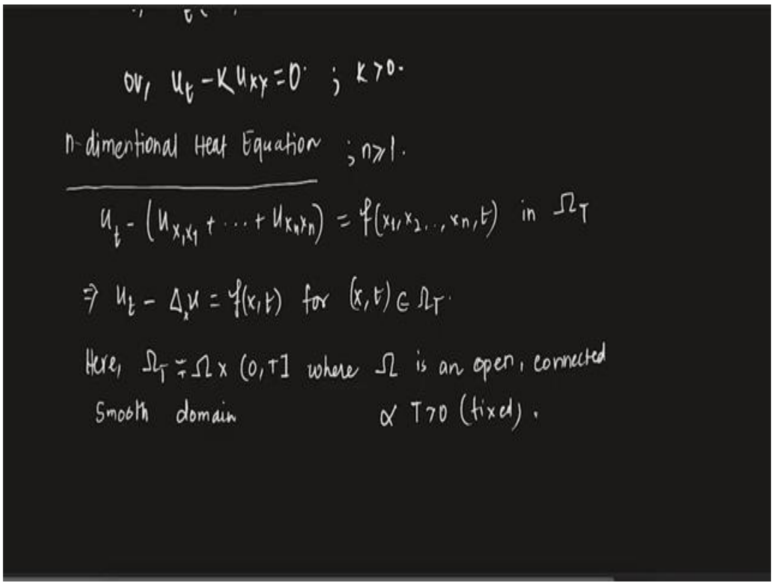

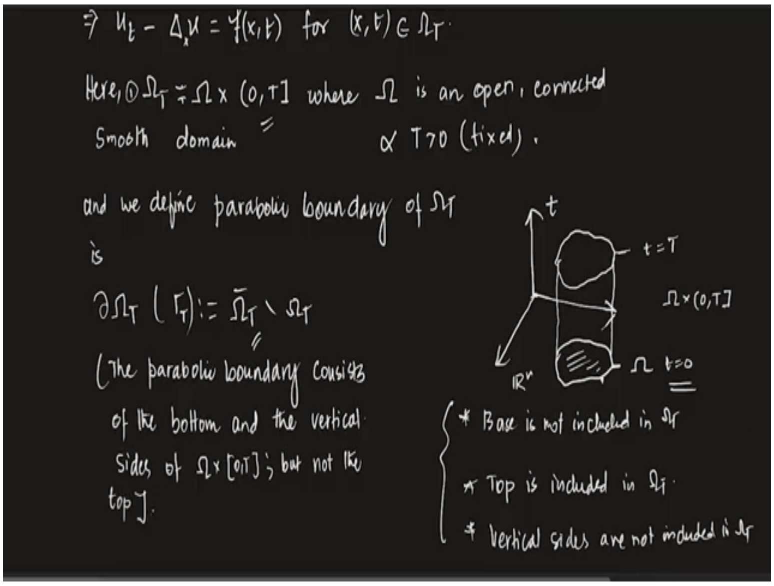

Some idea of what did equation looks like, let us look at the n-dimensional heat equation, yes, and how to work with that equation. So, first thing, first n-dimensional heat equation. As you can understand, whenever I am talking about n-dimensional heat equation, I of course mean n is greater than equal 1. So, essentially whatever I am going to do, I mean of course, one-dimensional heat equation is involved here.

So, again let me write down what is the n-dimensional heat equation. So, in general terms you can just write it like this, I mean without any specific notation you ut minus u x 1, x 1 minus u x 2, x 2 minus u x n, x n this equals to let us say a function which of course depends on x 1, x 2, x n and you have this t variable and these is defined in omega T, I will define what omega T is, these are complicated objects here, I have to define what omega T is.

Or essentially if you want to write it properly, yes, we write it like this, ut, generally speaking, we write it like the Laplacian of U. So, please remember this Laplacian is with respect to x only the x variable equals to f of x, t for x, t in omega T So, that is your n-dimensional heat equation we have explained again and what is omega T? Now, here omega T is very interesting, what is omega T?

See, whenever we say omega, what do we mean by omega is, of course, an open set, it is an open smooth, smooth open set we are always considering. So, here omega T which is defined by omega cross 0 open T closed, this is omega T equals to this, where omega is an open connected smooth and bounded domain and T greater than 0 fixed, is this clear? So, what is omega? For now, you see whenever I write omega T like this, I mean, see this boundedness I will explain to you when we need boundedness when you do not.

But generally speaking, this boundedness, so remark generally speaking boundedness is not required whenever it is required, I will just let you know. So, maybe I can just remove boundedness from here, it does not I mean right now, it does not matter I will explain to you when we need. So, let us just write on this domain and T positive, yes. Boundedness is not required. See boundedness we will assume later, for now, it is only the. So, let us take up an example and see what this omega T means.

So, essentially what is omega T? Omega T is omega which is an open set, that is the important opens set. So, of course, this connected smooth all this is included, but the important thing is omega is open and T lie between open 0 closed T. Please remember this thing, open 0 closed T, so what does that mean? See, let us say, that is your omega let us draw a nice omega, yes, that is a nice omega and let us just do it like this.

Let say this is your three-dimensional (())(18:02), so that is why, how do I put it maybe I can just, let me draw this thing first, so let us say this is your R n space, not this direction, the whole this, this is your R n plane and that is T relation here, this is your R n. So, I mean, for intuition just there is this thing, that if your R 2, this plane is R 2 and that is the third dimension which is the time dimension and we start from 0 to infinity, clear?

And in this thing, let us say that is your domain and where is this domain omega? This domain is in R n, I mean for intuition think of this as a R 2. So, this omega is in R 2 and then you have 0, t. So, 0, t something like this. The same thing will happen over here, so, it is up can think of a can of a soda or something a can of coke or something. So, omega T, let us understand what omega T is.

So, omega T contains omega, so what is omega? See, here I drew it like it but omega is whatever is inside, it does not contain the boundary. So, omega cross 0 t, so t, where is t 0? In Rn, this is on the base t, 0 here t, 0. So, this is t equals to 0, and what is t, here? t equals two T, here t equals two T. So, essentially 0 to 2 is this, what it says is what is omega T? It contains, I mean of course, does it contain this, so think of this can.

Now, does the base of the can, is it included in it? Of course, not why Because t equals two base corresponds to omega cross t equals to 0. Is t equal to 0 there? No, of course not. So, the base is not there. So, important thing is base is not included in omega T. Of course, it is true, base is not included in omega T. Let us just see that, so omega T is omega cross 0, T closed.

Now, what about the top part? The top, the lid of can, of the cylinder whatever you want to call it, is a lid included.

So, the lid means you see, forget about this boundary, that boundary is never included but whatever the inside lid is that included t equals to t is included. So, the upper lid, how do I put it, the upper portion, and then just maybe the top portion. The top portion or the top is included in omega T. Now, let us see what else is included? Is this boundary included here, clear? For t equals to 0, is this boundary included here? Of course, not, why?

Because omega does not contain a boundary, omega does not have any on the boundary it contains points in the interior it does not have any point in the boundary. So, boundary you take any point on the boundary and t equals to 0 in this here. So, that point is not included in omega T, so the boundary term, so this particular line, I mean this thing cos t equals to 0 and the sides this thing, does it i mean, is it there in omega T. Of course, it is not including omega T. So, the vertical side let me put it anyway.

So, this is important, the vertical sides, what I mean by vertical sides, is the side of the can which you can see. So, think of a coke, those 330 ml coke can, the side of the cans which you can see from your eyes that is not included is omega T, what is included? Whatever is inside, of course, the base, is the base included. So, think of the base to be smooth base it is not smooth in the can, but just think of that smooth, the base is included? No, the base is not included, whatever the boundary, boundary is also not included.

What about the side profile which you see from the side that side vertical side they are not included? Why it is not included in omega T? Because you see, any side if you take any point on the side, what is it? It corresponds to some t equals to t 1 and x is on the boundary of del omega That is the sides, the vertical side, and the omega does not contain the boundary. So, the vertical sides are also not included in this.

Now, so, the top is included, the top part is included, vertical sides are not included, the bottom is not included and whatever is inside of course it is included, that is your omega T. Now, we will define another important property. So, this is omega T, let us look at this here this is 1. And we define, so here this is the things, which makes parabolic equation a little complicated and we define a new boundary, we define a parabolic boundary. What is parabolic boundary? We will write it like this of omega T, which is given by is del omega T, clear?

That is your parabolic boundary, I mean generally, in some books you will see it is written as del omega T or in some books it is also written as gamma of T, clear? And that is again defined by, so I mean, this is not the definition it is just a notation it can be written as gamma of T, which is defined by omega T bar minus omega T. Just think about it, what does it mean? It means that you take the whole omega T, take the closure of that thing the omega T is, whatever is there you take the whole closure.

So, now if you are taking the closure, just think of omega T and if you are taking the closure of that T, what will happen? Think of that the coke can, it includes everything. So, whatever is in the top, whatever is in the bottom, whatever is on the side, whatever is inside everything is included in omega T bar. So, the liquid along with whatever the surface you can see from outside everything is included there. Now, what is omega T?

As I have told you omega T whatever is inside and the top but excluding these vertical sides and the bottom. So, what is del omega T? If you just think of the parabolic boundary, the parabolic boundary consists of the bottom part of course, and the vertical sides of omega 0, T but not the top, not this part, not the top part. So, let me put it this way, the parabolic boundary consists of the bottom and the vertical sides of omega cross close 0, t. Please remember this is closed 0, t, yes, of course, but not the top that I explained.

Now, before we move on, I want you to understand that what omega T specifically is, so take some points on the omega T, omega T bar and just think about and take some time where omega T bar or of where omega T I mean, which points are in the omega T bar, which points are in the parabolic boundary. That is important and why is this important? Because later on you will see that while we talk about maximum principles and all, it is very important to understand where the maximum exactly.

So, parabolic boundaries are very important to study. So, let us say for example, what I mean by these, let us say you take a point here on the top of the can, t equals to T and x equals to some x naught using the Omega. Do you think that point is there in the parabolic boundary gamma t? This is gamma t, do you think it is in gamma t? Of course, not, it is not there. What about this point, a point here? Do you think this point is on the parabolic boundary? Of course, it is. That point is on the boundary.

So, let us say, for example, let us say this point, this point is x, let us say this is x 1 and capital T. So, x 1 and capital T, do you think that is on gamma t? Yeah, of course, it is here this is in gamma T, but not in omega T. Of course, it is not by (())(27:59) definition of gamma T. But of course, it is gamma T, I mean, that is (())(28:04). Let us say there is a point here which is like a x naught capital T. Do you think that point is there? So, it is not capital T, it is not in gamma T but it is in omega T, clear?

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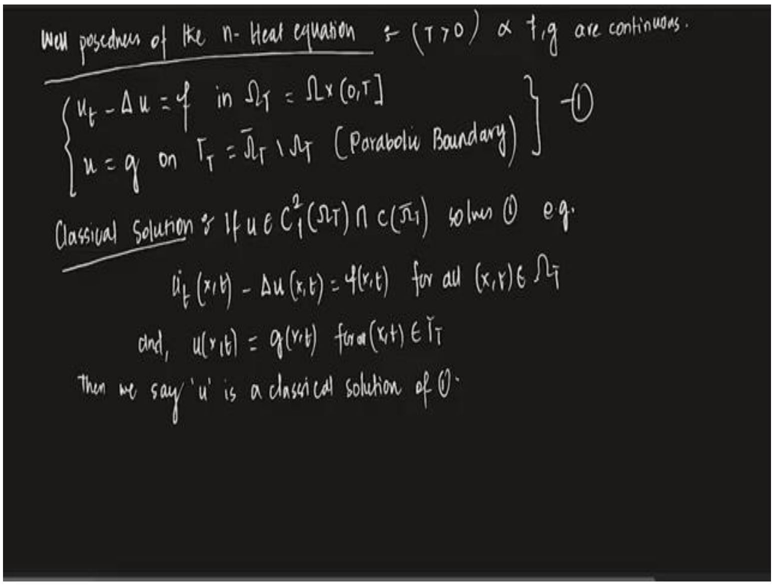

As you know, that we want to study well-posedness of the n, so I will write it like this, heat equation I will write like this n-dimensional heat equation. So, please remember these things whenever I write it like n-heat equation, I mean, the spatial dimension is in R n, n-dimensional heat equation. So, my question is to, our idea we want to achieve is to talk about the, I mean know about well-posedness of n-dimensional heat equation.

Now, if you remember, if we want to study something like this, what are the things which we need to do. So, first thing first, of course, yeah, but before we talk about, well-posedness. I mean, of course, well-posedness of which problem let us put that first. So, essentially, we are looking for this for ut minus Laplacian of u equals to f. As I told you, f (())(29:31) on x and t. And this Laplacian is with respect to x and this is omega T, omega T is just a parabolic boundary.

So, you remember what is omega T, it is omega cross 0, T closed, clear? T is greater than 0 for some T positive, T positive we are all in. So, I am not writing all that every time you do realize that T is always positive, capital T is always positive and these particular things specify time variable. So, ut minus Laplacian is equal to 0 and I will be talking about the well-posedness of which problem, the judiciary problem.

As you remember, I talked about different boundary conditions of side. In one-dimensional heat equation we talked about, I mean Dirichlet problem, (())(30:15) problems here I am just generalizing this thing and I want to talk about the I will poses how Dirichlet problem, how does that look like? It looks like u equals to g on gamma T, which is omega T bar minus omega T, so the parabolic boundary. So, is this clear what am I saying? I am saying that I want my u to be easy for, equals to some function g.

Of course, we want all of this to be some function g on the parabolic boundary and such that u satisfies this equation in omega T, clear? So, whenever we say something is a solution what do I mean by this? So, let us write down the definition of solution, classical solution, this is called a classical solution. So, u in, this is a new notation C 2 1 omega T, yes? C to 1 omega T.

So, essentially if you remember what is, I mean whenever we are writing something like the C 1, C 2 you remember what I mean by this thing? What do I mean? I mean that you are looking at Laplace equation if you remember. We are talking about five differentiable functions, Laplace equation if you remember but here what we are going to do here is whenever we are saying that, classical solution we mean a u which is in C 2 1 omega T intersection C omega T bar. So, let me explain what I mean by this? C 2 1 omega T is to these apart the subscript or superscript is 2 that will represent the x variable.

See, in earlier case, in the Laplacian case there are only spatial variables because there was no time variable, it was not evolution equation for Laplace case. So, there were no, two parties. So, there is no spatial and time variable and hence you do not have to simplify, C 2 gives you everything. Here, you do not need two variables for the time You do not need twice regularity. So, see you do not exceed two variables in the time, you do not need twice regularity.

So, you do not see 2 with respect to time because the only ut is there. So, you just need one derivative with respect to t. So, that is why this is one derivative with respect to, so this will specify what the derivative with respect to t this will specify what the derivative is with respect to x spatial variable, this is spatial and that is the time variable, omega T, of course, it has to be C 2 1 in the whatever the domain is omega cross 0, t because in this omega T only.

This has to satisfy equation and for this equation to be satisfying you have to be at least twice differentiable with respect to x and once differentiable with respect to t, that is what I wrote here. Intersection C omega T bar, see here whenever I am writing all that of course, I am assuming that and if g is continuous, wherever they are defined they are continuous that is always you assume. So, if g is assumed, you are saying that u equals to a continuous function on the parabolic boundary.

So, I am assuming u is continuous on the boundary also, up to the boundary omega T bar, clear? So, that is what it is, so let us say if let us call it 1, if u in C 2 1 omega T intersection C omega T bar, solves 1. So, that is ut let me write it properly, ut at the point x, t minus Laplacian of u at the point x, t equals to f of x, t for all x, t in omega T and u of x, t equals to g of x, t for x, t in gamma t, which is the parabolic boundary. So, this should hold for all x, t for all, and this also should hold for all omega T. Then, we say u is a classical solution of 1, clear? Then, u is a classical solution of 1.

Now, once this is taken care of now, we want to talk about the well-posedness. Now, I can talk about well-posedness. So, first thing first, what you have to do? You have to find existence, yes, and then you have to do the uniqueness part and then after you do uniqueness, then the continuous dependence on boundary data. For now, we are just interested in finding the uniqueness, clear? We are just going to do uniqueness.

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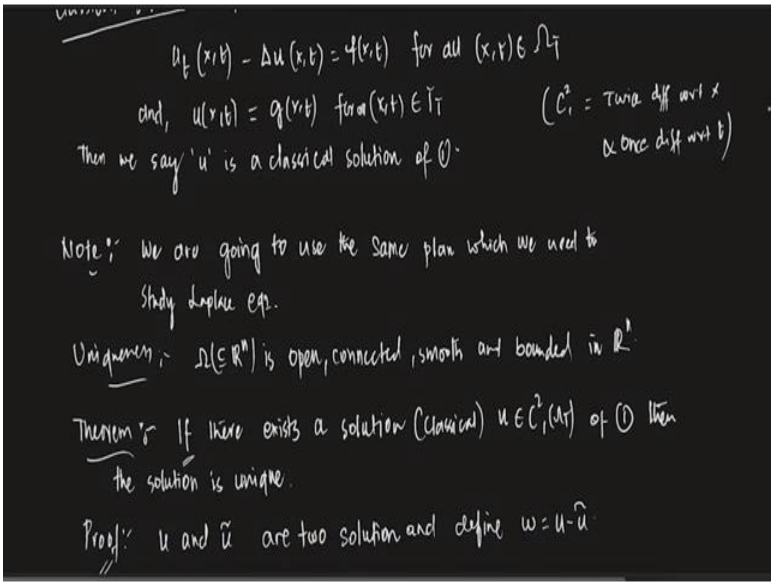

So, essentially, so we are going to ley me put it small note kind of, this is not a remark it is a note. We are going to use the same plan, how do I put it? Plan may be, which we used to study heat equation, where we have Laplace equation. So, first of all the Laplace equation if we remember we started out with the uniqueness, and then we went from there. So, and here let me again emphasize what is C 2 1, it means twice differentiable with respect to x and once differentiable with respect to t, clear?

Now, so uniqueness, let us talk about uniqueness, and then we will go from there uniqueness. So, we assume for uniqueness, we assume that omega subset of R n is open. Of course, all of that is truly open, connected, as you guys know that this is always as it is, smooth but this is the extra term which has been for this and bounded in R n, clear? Now, what we want to do is we want to talk about. So, if you are taking omega like this, yes, we want to see if there is a unique solution.

Let us see if we can solve the problem does there exist a unique solution for that and what sort of problem are, we talking about? Let me write down that problem. And then it will be clearer about what the problem is. So, essentially the problem will look like. So, consider the problem or maybe I can write down the theorem that will be better, let me write down that the theorem first. The theorem should be, if there exists a solution, classical solution whenever I mean solution, I mean classical solution, so basically classical solution, u in C 2 1 omega T.

If there exists a solution, because we do not know if there is a solution, I am just assuming that there is a solution. If there is a solution, of 1 then the solution is unique. Is this clear? So, it says that if there is a solution, it has to be unique. Why I am doing it like this because we did not prove that there is the solution. So, I initially was going to write that there is the existing solution, but in that case, see initially, so let me tell you this particular thing. So, you should not do this mistake in the future.

I was about to write the theorem like this, that there exists a unique solution of 1. If I write it like this, there is no harm in that. But the thing is, in that case, you have to show that there is a solution. And then there, the solution is unique. I am not saying that, please remember this. Here. I am assuming that I am saying that if there exists a solution. So, I am saying that you give me a solution.

Once you give me a solution, I can tell you that whether it is unique or not, and in this case, it is unique. But the thing is, initially, I do not know whether there is a solution. So, proof, let us look at the proof, we will prove all this, that how to find the solution or everything. But for now, let us just assume that if there is a solution, it has to be unique. So, let us say u and u tilde, this is our usual thing u and u tilde are two solutions, you guys know that whenever we want to find uniqueness, we started two solutions. And we define w to be u minus u tilde, this is what you define. Now, once you do that, then you see the linearity kicks in. Since u and u delta both satisfies this equation.

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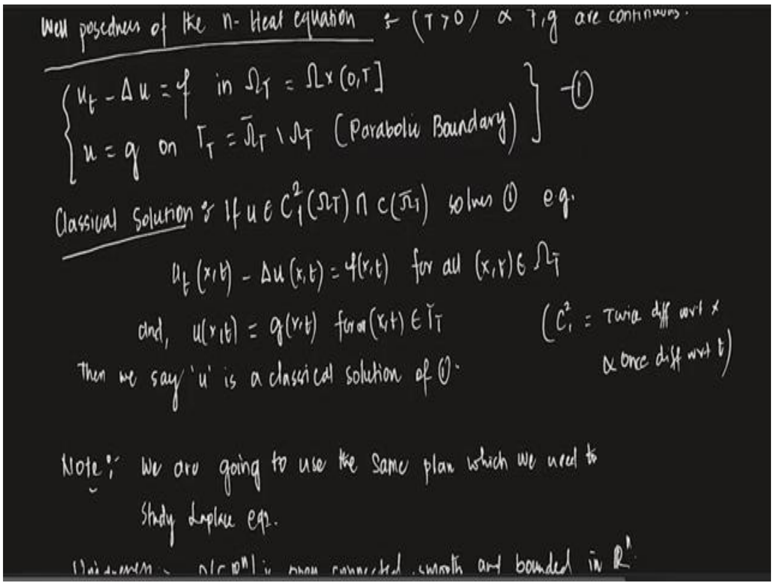

What will happen that u minus u delta will satisfy the homogeneous problem if it is getting canceled out here.

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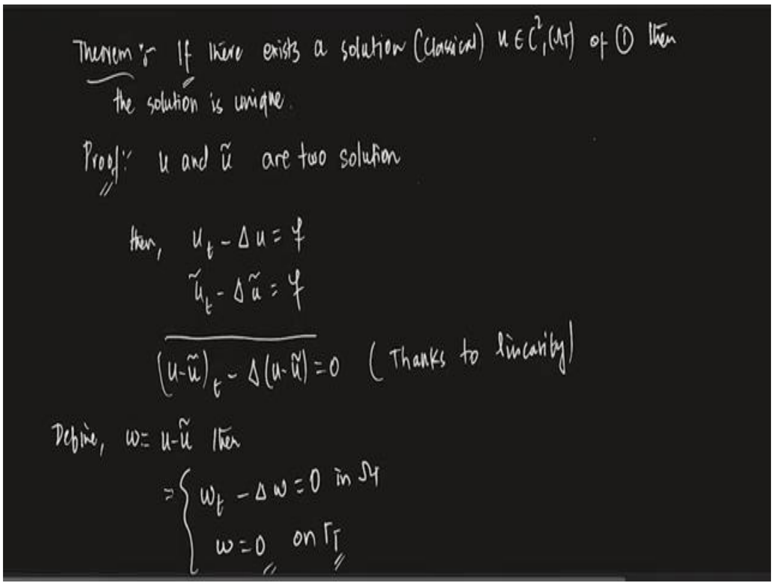

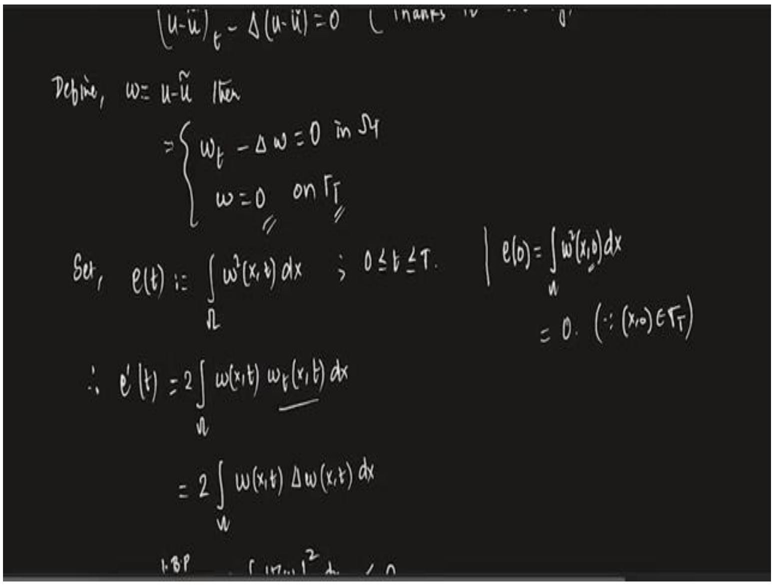

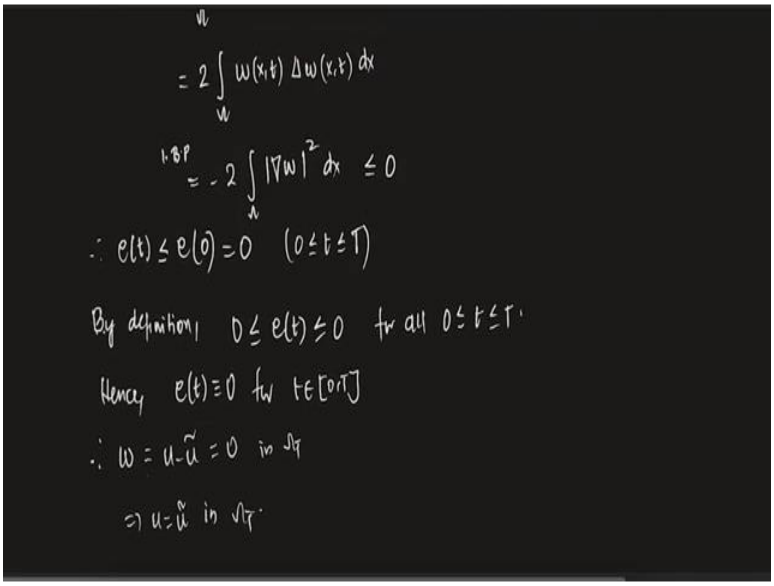

So, let me write ut minus Laplacian of u delta equals to f and ut, u tilde t minus Laplacian of u delta was to f. If you subtract it, what will happen is u minus u tilde with respect to t minus Laplacian of u minus u tilde equals to 0. So, this is thanks to linearity. See all of this is possible because these equations are linear.

Now, define w which is u minus u delta. Once you define something like this then wt, we did define wt to be u minus u delta. So, I defined it twice. I mean, let us just not define it here. I will just define it here, it does not matter, you do realize that it does not matter. So, then wt minus Laplacian of w, this equals to 0 in omega T. And what about in the parabolic boundary w is equals to 0 on gamma t.

Again, the same thing happens g and g get canceled out. So, in the parabolic boundary w is mean to be 0. Now, set the new, this is the new thing, we have set an e of t, this is kind of energy of the system at any time t. So, this is defined by integral over omega, please remember, this is not integral over omega T, it is only integral over omega w square x, t dx because this is with respect to omega, x is in omega, t is in 0, t. et I am defining this is defined for all t between 0 and capital T.

So, essentially what am I doing? I am taking any t between 0 and capital T, and I am defining energy at that particular time given by integral of the square x, t. So, if I do that, then what is e prime of t, prime is with respect to t, the derivative, so that equals to integral over omega. If you take derivative with respect to t, the t gets inside because this integral is with respect to x.

So, I can take it inside it becomes 2 omega of x, t omega T of x, t dx, I hope all of you will agree with me here. So, this is the prime of t and you see this will give you integral over omega, omega of x, t. What is omega T? It is Laplacian of omega x, t dx, yes or no? Why? Because you see omega satisfies this equation. So, omega T is Laplacian of W, I am thinking for omega. So, this is the, I mean you can call it omega, you can call it w, I am really very sorry for the confusion.

So, let us just call it w and this is omega and this is w. So, I am taking the integral over omega of w square, here 2 integrate over omega w wt and 2, since w satisfies wt minus Laplacian w is also 0. So, it equals to two times integral w Laplacian of (())(44:40), clear? Because wt is equals to Laplacian of w. So, that is why I just wrote it like this, for any given t in omega.

So, omega cross t equals to t let us say, wt is Laplacian of w because it satisfies, w satisfies our problem, so this is 2. Now, integration by parts, you know our usual friend integration by parts, so that will give you integral over omega gradient of w square dx, that will give you because you see this is w and that is, I mean the integration will give you a gradient of w equals to 1 derivative will go here, it will gain 1 derivative which will lose 1 derivative that is becoming the and there is a boundary term.

What about the boundary term? The boundary term w is equals to 0 on the boundary, so there is no boundary term. This is particularly always non-positive because there is a negative sign involved here, this is non-positive. What does that say? Therefore, it says that, what about e of 0? Let us look at what is e of 0. So, e is defined in such a way that t equals to 0, it is w square x 0, for t equals to 0, so you are looking at w on the base.

On the base, this is 0 because the base is containing gamma T. So, see let me put it in this way, e at the point 0 that is integral over w, w square x, 0 dx, x, 0, t equals to 0 is the base that base is included in gamma T, yes or no? So, w is 0 on the gamma T. So, we are taking the integral of 0, since x, 0 belongs to gamma T. I hope this is fine because this is the bottom part of the lid of the can. So, what do you have?

You have e prime of t is non-positive. So, that will give you e of t is less than equal e at the point 0, which is 0, for 0 less than t less than equal to capital T, this is true. So, since e is, I mean, e prime is less than equal to 0. So, therefore, what does that say? That says that and you see what is the definition? So, by definition again, you see what is e? e is w square integral of w square, so w square whatever it is, w square is always greater than equal to 0, it is a positive function, I am taking the integral of positive function.

So, it is e t is always positive. So, 0 less than or equal e of t less than equals to 0 for all t between this, that will give you e t identical equals to 0. So, hence, e of t is identical equals to 0 for, I am writing thing to many times, t it is 0, T, capital 0, T, so it is always 0 and hence, if e, t is 0, what does that mean? Let us see what is the e, t? e, t is integral over omega w squared, this function this is equals to 0. So, w is a continuous function because w solves the problem, so is a continuous function.

So, you have a, I mean w square the integral the identically equals to 0, that will give you w squared is identical equals to 0, that will give you the w is identically equals to 0, so it has to be because if it is positive, if w squared is positive in some positive, in an open set containing omega and then the total contribution of e t will be positive, that is not happening here. So, hence w which is given by, u minus u delta, which is going to be 0 in mega T, which will imply u is u delta in omega T. I hope this is fine. So, there is a uniqueness here. Now, what we are going to do, is it is that time-dependent variable.

So, essentially, I am just taking one fixed time and I am doing all of this thing, and I am saying uniqueness So, what did we prove? We prove that if you have an equation like this, what is the equation? The equation which looks like this, ut minus Laplacian equals to f, u equals to g on the parabolic boundary, then if omega is bounded, please remember this is for a bounded domain. If omega is bounded, then we can say that we can have a unique solution, if the solution exists at all. So, we did we are going to end this lecture.