Advanced Partial Differential Equations
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Lecture 16
Green's Function for Ball

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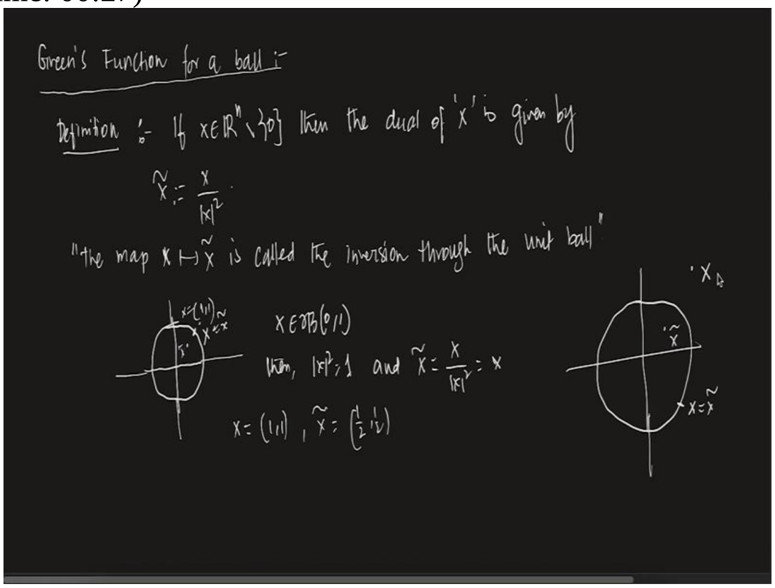

Welcome students and today this is the second part of Green's function. What we are going to do is basically construct Green's function for a ball. So, as I told you that Green's function can, you can of course, we saw that for a given domain, let us say omega is a sufficiently smooth domain, you can actually construct a Green's, you can actually get a representation formula for u of x. But the problem is you cannot, the problem is to construct the Green's function for such a domain. And for most domains it is not possible. But for some particular domains it is and the one such domain is a ball.

So, definition, let us say we start with the findings in this function so essentially we have to find a corrector function if you remember. How to find a corrector function and for that we are going to define some new concepts.

So, definition, so if x is in Rn minus 0, so x is in, x is non zero vector then the dual of x is given by, so basically we say x has a dual and the dual will look like this, x tilde is x by mod x-x square so if you, most of you understood what this means. It is called, the dual of x is given by x tilde which is by definition x by (())(2:14) so essentially what we are doing is we are basically taking the inversion of x with respect to the ball. So, this mapping, the map x going to x delta is called the inversion through the unit ball.

See, I will make all of this clear, why we suddenly started with something like this and what exactly does it mean. So, let us understand more in detail what it means. See, essentially let us say this is your art 2, let us do this, art 2 and that is your ball. Sorry, please forgive my art. It is not very good, yes. So, let us say this is your unit disc and now what happens is let us say I am starting out with the x here on the boundary. So, x is on the del b, let us say 0, 1. Then mod x-x square is what. It is 1 right and therefore hence x delta will be x by mod x-x square which is x.

So basically, if you reflect x with respect to this unit ball, the boundary, with respect to the boundary of the unit ball, that remains in the same place. It does not change direction or anything, nothing changes so x and x delta are same in this case. So, the inversion of x, so basically, we are inverting it, you understand.

So, think of this as a mirror, kind of a mirror and you are putting a point here. You will get another point on the other side of the mirror, something like this. So, but if you are taking a point on the mirror itself, so on the boundary, x is equals to x delta. For example, let us say x you are taking it to be something like, I do not know may be, let us take 1, 1. So, definitely that is not a point on the boundary and this is outside the ball that is the 1, 1 so as you can see x delta will be half, half inside the ball, half, half.

And see a point here, if you take, so this is your x and then x tilde will be inside, clear?

So, something like this, what it says is if x is on the boundary, x is on the boundary then x is equals to x delta. If x is here, then x delta is inside the ball in the disc, in the disc so x is there so basically the reflection you can think of or vice versa if x is here then x tilde is going to be outside the ball, something like this. So basically, we are inverting the point with respect to the boundary. I think this is clear now.+++

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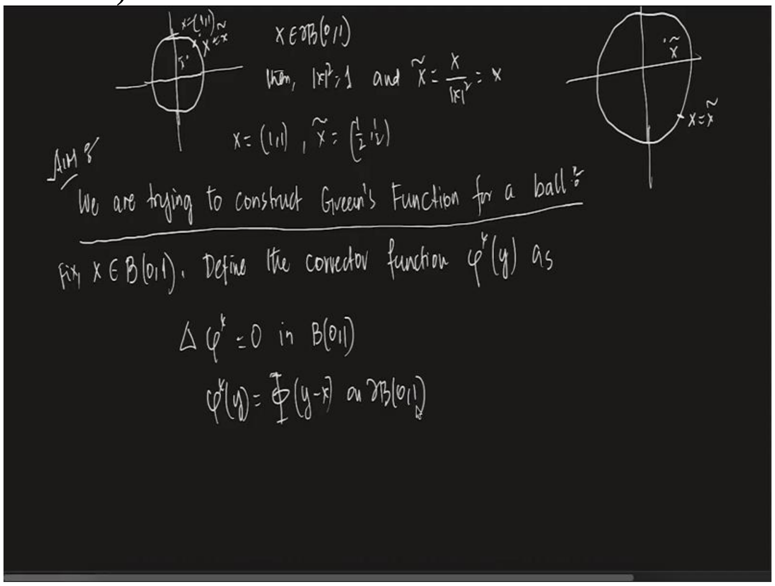

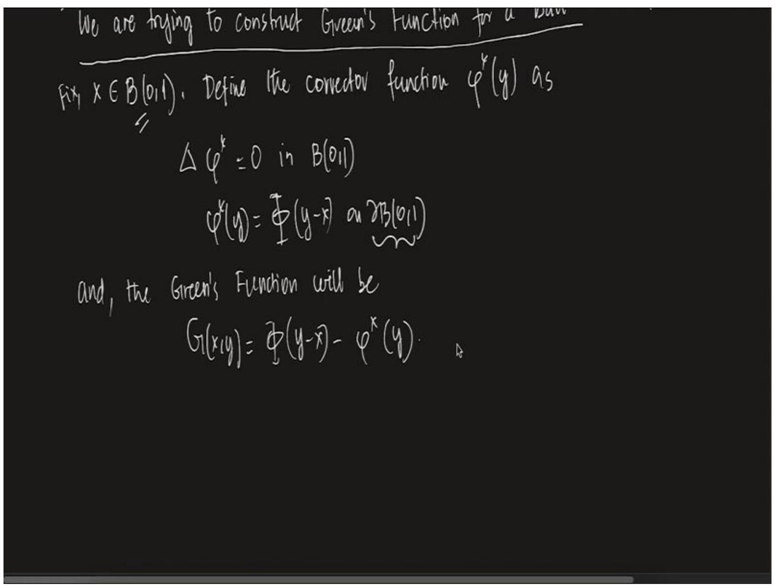

So, if it is not clear, what I suggest you to do is please take two three points and look at what exactly does it mean. Now, the thing is you remember what we are trying to do. We are trying to compute so basically we are trying to, let us write down our aim. We are trying to construct Green's function for a ball. We are trying to construct Green's function for a ball. This is our motive, I mean this is our aim, that is our aim and to do that what we are going to do? We are going to first of all construct a corrector function.

So, construct a corrector function. So, for that we are going to fix, you remember? You want to fix the x in the domain so in this case the domain is B 01. Once I fix this thing, the corrector function and define the corrector function for this fixed x, Phi x of y, f, Laplacian of phi x is 0 in B 01 and Phi x of y equals to, this is the fundamental solution, phi y minus x.

This phi and this Phi is different, yes. This is the fundamental solution. This is capital Phi, this is the small phi on the boundary of the domain. This is what we are taking so basically for a fixed x on the ball you take any x on the ball and you define a corrector function which satisfies Laplacian of phi x is 0 on the ball and Phi x of y is the fundamental solution at y minus x.

Now you see, this is well defined because y here is always on the boundary in this case. In this case y is always on the boundary, x is in the center of the, sorry x is on the ball, y is on the boundary. So, essentially y minus x is defined, so Phi of y minus x is defined here so y cannot be equals to x and hence it is defined. So, and the Green's function and once you get this corrector function, Green's function, what is it, what is the Green's function, will be G of x y is the fundamental solution at y minus x minus Phi x of y. So that is your Green's function.

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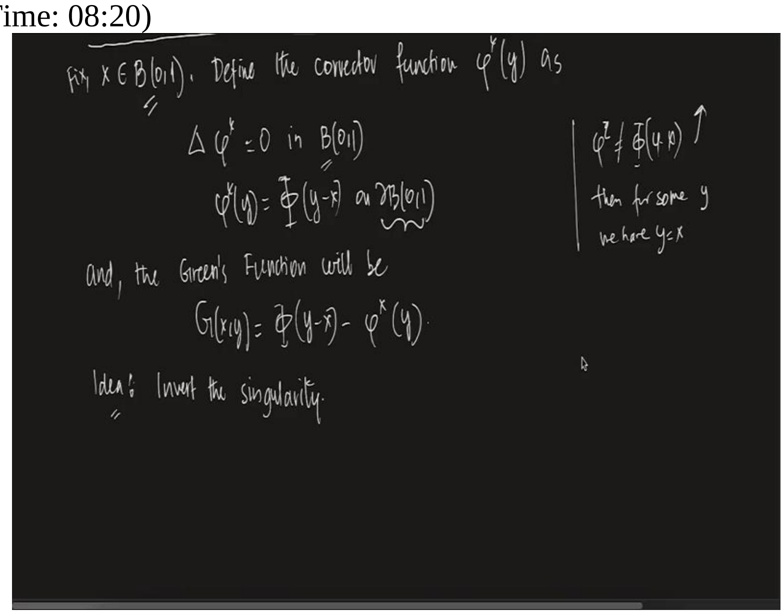

Now, how do I find this corrector function that is the question, yes? Okay, so what is the idea? The idea is to invert the singularity. Let me tell you what I mean by this phi essentially, I want a function which is, I want to find phi of x. See once I find phi of x, I already know what my fundamental solution, we just put it there and it is done. So, all I need to do is to find a correction function.

Now, I know that the correction function what it does it do? It is actually the fundamental solution, the value of the fundamental solution at y minus x. Please understand this thing. What is y, y, in this case y is on the boundary, x is a point on the ball, clear, inside the ball, inside the open set in B.

Now, here this is defined, no issues. If you define, let us say if you define phi of x to be the fundamental solution, of course it is 0. Laplacian of phi is 0 but for all x not equals to y. Is this clear? So, you see here there is not a problem because y is always on the boundary and x is in the interior of the ball.

So, y is never equals to x but here if you are putting phi of x to be, so let us say if I am taking phi of x to be our fundamental solution of y minus x, then for some y we have y equals to x. Of course, we have. X is any fixed point in B 01. There is some y where in the ball, there is some y where this has to be same and when this is same, this actually blows up.

So, this is not a good option in the ball. So basically, what to do? We want to find a function which is the fundamental solution. The value of the fundamental solution of y minus x on the boundary but phi of x is 0 in the interior of the ball. You do understand what I am saying?

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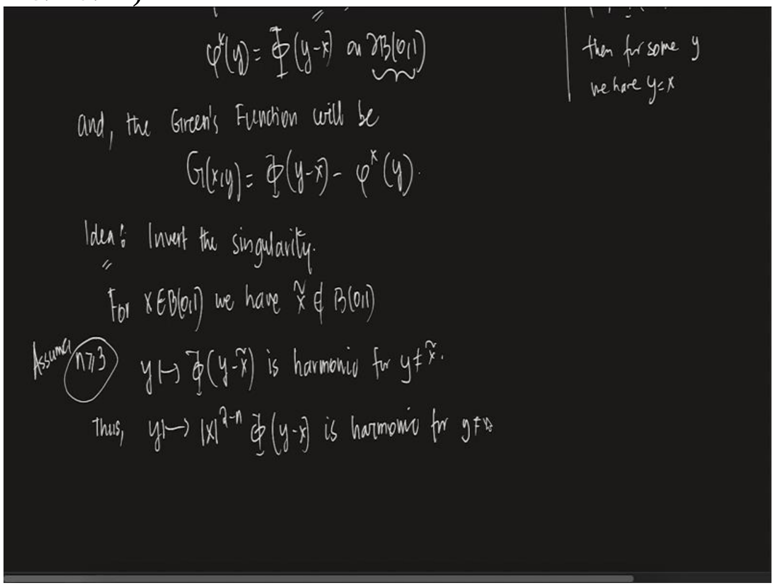

So, let us try to construct such a thing. So, you see that is why we want to invert the singularity because there is a singularity, definitely there is a singularity if I want to do something like this that x equals to y. So, let us just invert the singularity. So, what we are going to do is for x in B 01 we have x delta definitely not in B 01, so outside B 01. It is in the complement, so for now let us just assume n is greater than or equal 3, this is just an assumption. You can do it for n equals to 2 also, no problem, yes? So, assume. See, you have to do it in two different sets but for n equals to 3 I am doing. N equals to 2 is exactly the same thing, nothing changes.

So, assume this. Now, you see y going to, tell me something, phi of, think about it, y minus x tilde is harmonic for y not equals to x delta, this is true, why this is true? Because you see if I am taking y which is not equals to x delta so phi is harmonic everywhere in Rn except at y equals to x tilde so that is why it is harmonic. We have the function of y.

Now, if that happens, one can say that y going to mod x power 2 minus n, fundamental solution of y minus x is harmonic for y not equals to x delta, same thing. See here essentially this is a function of y so if I multiply it by mod x to the power 2 minus n does not matter, the same kind of idea. It is basically a constant as far as disc is concerned. See for a fixed x I am doing. I have fixed the x so basically this is a constant. I am just multiplying it by constant so harmonic function remains harmonic. It is not a problem.

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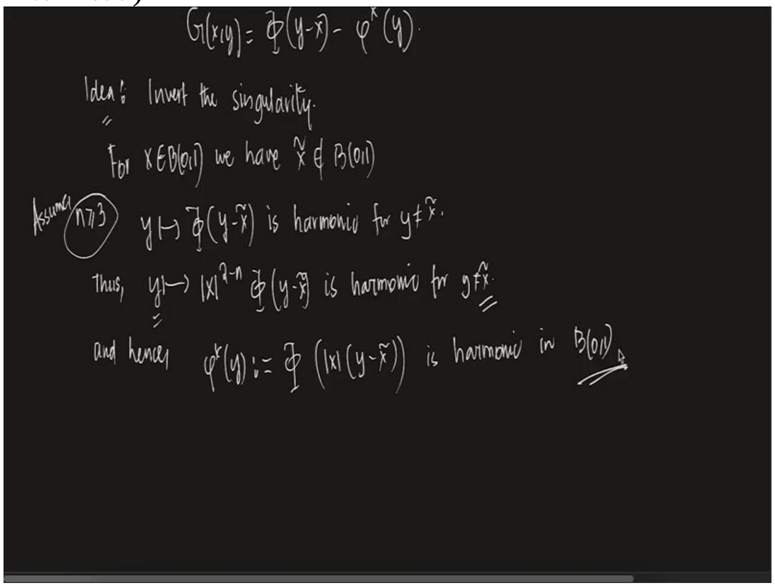

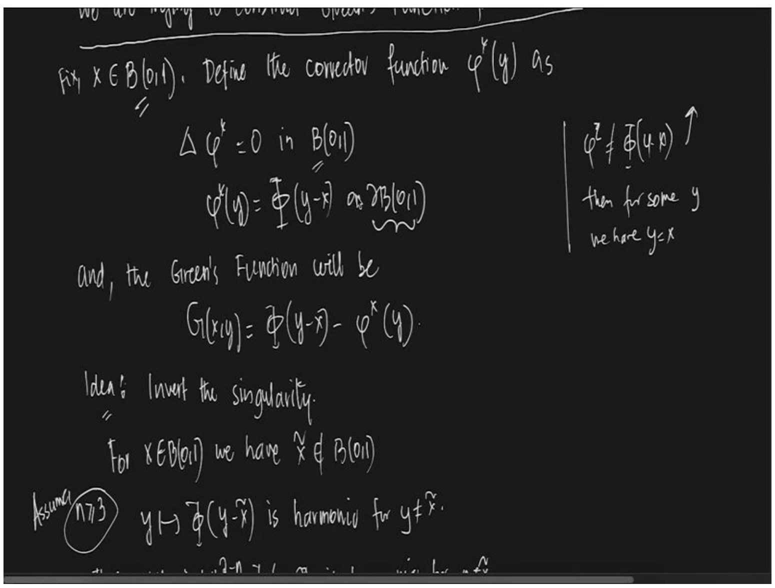

And hence if I am writing phi x of y, define phi x of y to be the fundamental solution of mod x times y minus, sorry this is x delta. So, this is harmonic in B 01. Is this clear why it is harmonic? See y going to this is harmonic for y not equals to x delta. Where is our y? In our case, we are choosing y from the ball. X is definitely never in the ball. It is outside the ball so basically this is always harmonic in this B 01. So, if I define phi x of y to be the fundamental solution of mod x times y minus x delta then that is harmonic in B 01.

Now, you may say that what is the point of putting this mod x to the power 2 minus 1. This is phi of y minus x, is actually harmonic in B 01. You are right there, but the problem is if you do that, you understand what I am saying? What is the point of this multiplying it by mod x to the power 2 minus n. If I do not do that, then what happens is, this is 2 that is capital Phi of y minus x delta will be harmonic in B 01 but I also have to have that phi x of y has to be phi of y minus x on the boundary.

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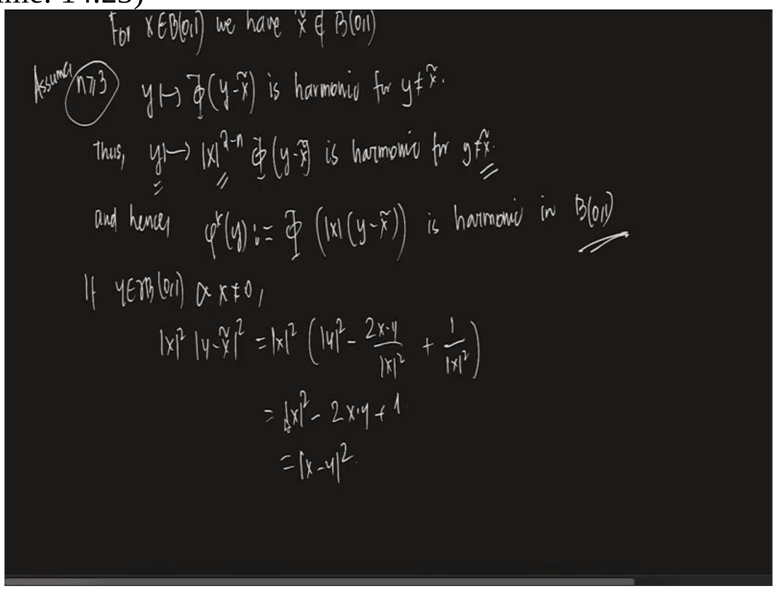

So, this I want to calculate. This I need to have. So, for that what I am going to do is this. See, if y is in del B 01 so that is why I am multiplying with this particular mod x to the power 2 minus 1. So, let us say if y is on the boundary of the ball and x not equals to 0, I am choosing x not equals to 0, then mod x-x square, mod y minus x delta square, let us calculate this thing. I will show you what is happening here, mod y square minus 2, x dot y by mod x-x square plus, sorry let us just break it up, plus 1 by mod x-x square. So, it essentially is, see mod y, y is on the boundary so mod y square is 1 so essentially this is mod x-x square minus 2 x dot y plus 1, correct? So, which is mod x minus y square. Clear?

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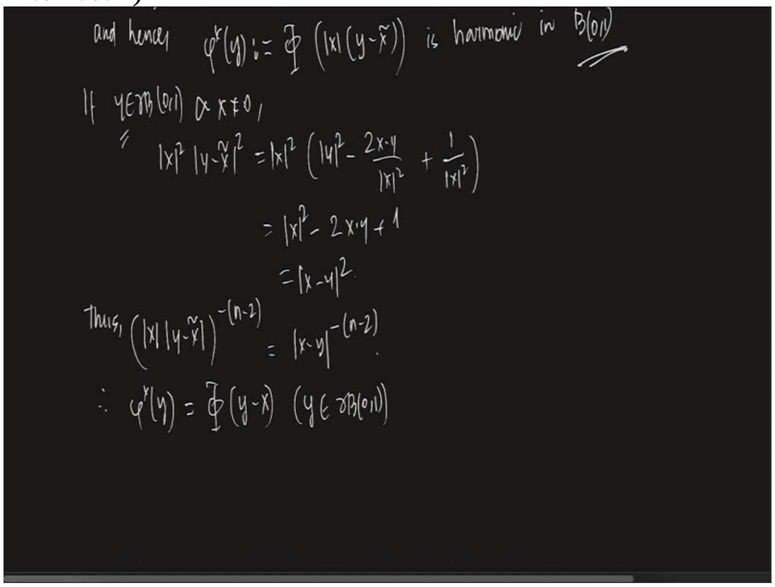

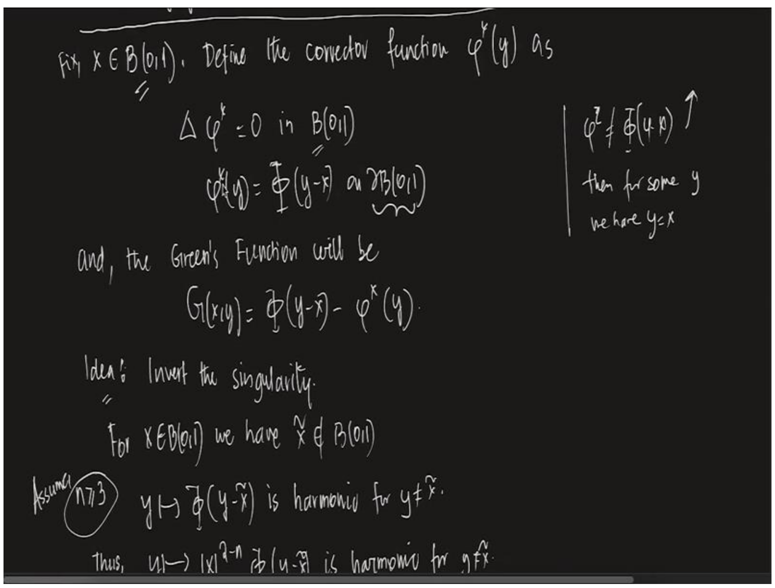

So essentially what is happening here is, thus, let me write it. Thus, mod x mod y minus x tilde power minus n minus 2, this is equals to mod x minus y to the power minus n minus 2. And therefore, what is happening is phi x of y, this is equals to the fundamental solution of y minus x for y on the boundary of 01. Is this clear?

See, I want my phi of x, I want my phi of a, where is it, yes, I want my phi x of y to be the fundamental solution at y minus x. It should be equals to the value of the fundamental solution at y minus x. If I am choosing only this thing, not with mod x to the power 2 minus, not with this. If am choosing only phi of y minus x, that is never going to be phi of y minus x delta on the boundary for x in B 01, it cannot happen. So, for that I am modifying this thing. I multiplied it by mod x to the power 2 minus 1.

Once I do this, then for any y on the boundary you can show that mod x times mod y minus x tilde whole power this is equals to mod x minus y whole to the power n minus 2, for n greater than 2, this is the fundamental solution. So, phi x of y this is your phi x, so phi x of y is basically capital Phi fundamental solution at y minus x. So, this is what, this is our phi x y is defined by the fundamental solution of mod x y minus x tilde which is essentially this, so that is why phi x of y I wrote it like this. Clear?

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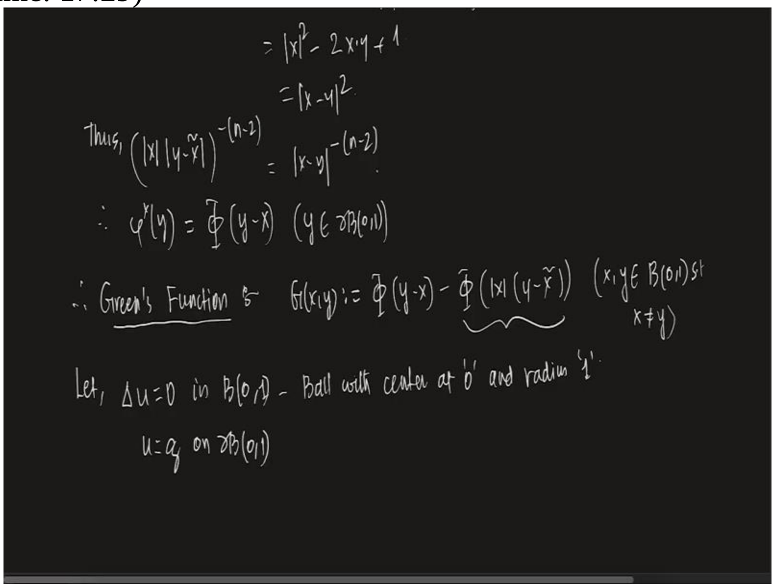

So, consequently this is happening. So, let me write down, therefore the Green's function how do you write Green's function, G of x y will be defined as the fundamental solution of y minus x minus the fundamental solution of mod x y minus x tilde. This is for x y in B 01 such that x not equals to y. Is this clear?

See as fundamental solution minus phi y that is our Green's function and phi x of y is essentially our, this is the fundamental solution with respect to fundamental solution it is written like this, so the corrector function in terms of fundamental solution so G of x y is this.

Now, assume I am just doing a particular thing. See once you find this Green's function you know how to write u of x. So that is always there so let us solve a small problem using this thing. So, let Laplacian of u is equals to 0 in the ball with center at 0 and radius 1. So basically, this is ball with center at 0 and radius 1. So please remember if I am not saying, generally in this course whenever I am saying B 01 you just think of it as a ball with center at 0 and radius 1 until I say otherwise. And u is G on the boundary of the ball.

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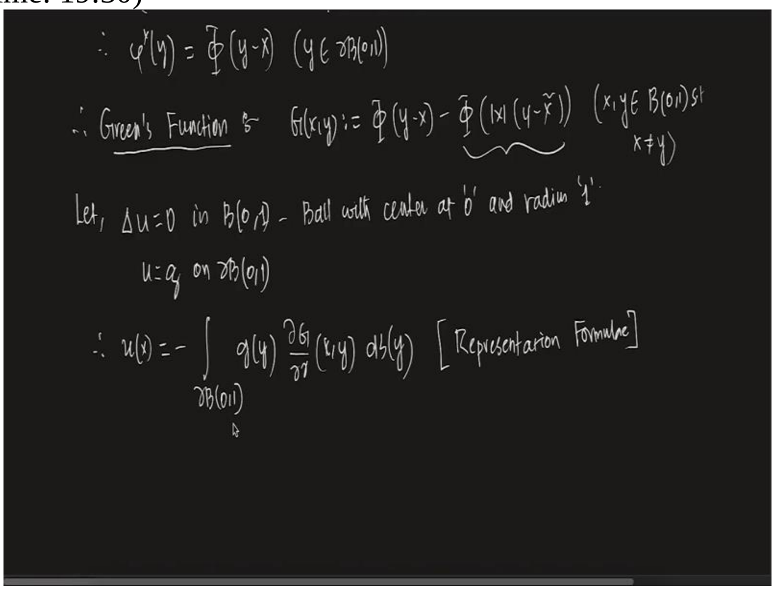

So, let us say this is given. So, I am not taking it, let us just start with Laplacian equals to 0, harmonic. So therefore, you use the representation formula. If you use the representation formula you can see that u of x will look like the minus integral over del B 01, G of y del G del gamma of x, y, ds of y, clear? So, u of x will look like this. This is, where is it coming from you remember in the last lecture, we talked about the representation formula, clear? In the representation formula there is another part, but that part is 0 because of Laplacian e equals to 0, that part is not there. Essentially it blows down to the boundary.

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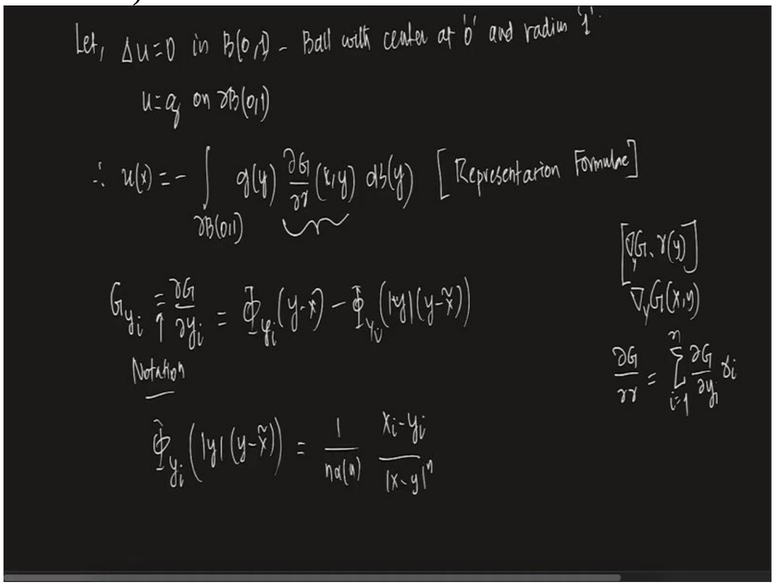

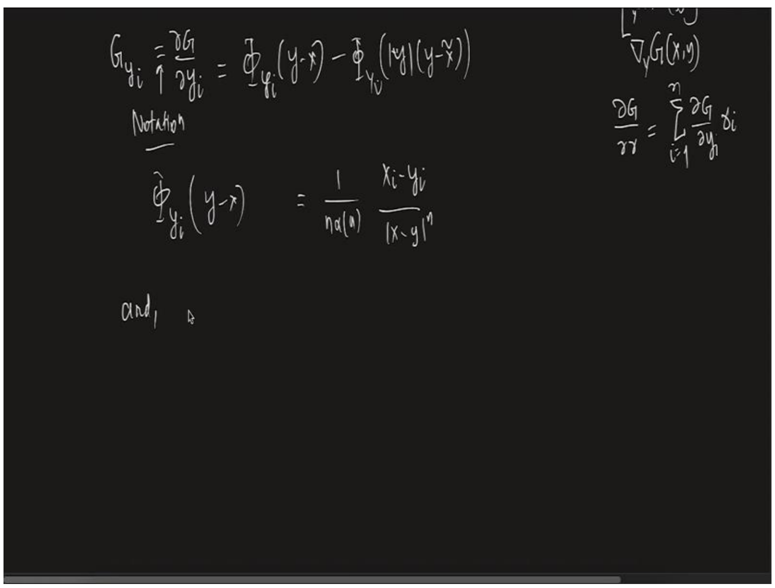

Now for this, let us just solve this problem and see if we can do this. So basically, you see if I want to solve this problem, if I just calculate this thing on the boundary, then I am done, right. I just put it in the g and I am done. So, let us just do that. G, here, I will just show you how to do this thing.

So, G of y i, you remember this is basically what is it? It is G, y, so the derivative is with respect to y, whatever, I do not know what, let us just call it like this. This is just a notation, whatever notation you want to write it, it is not a problem. So, essentially this is del g del gamma is this one. So, what is del y with respect to see, basically I am taking g of x is a function of 2 variables x and y less than y. I am just taking the derivative with respect to y. Is this clear? And that y is y1, y2, yn so basically, I just have to take the partial derivatives and after that put it together.

So, let us just put it, so basically if I just write it down, then del g del gamma is summation, it will look like this, del g del yi, gamma i, i equals to 1 to n. It will look like this, clear? So, I just have to calculate del g, del yi first so that is what, g y i. This is, I hope all of you guys know these notations, g y i is del gamma, this is just a notation. This one is just a notation that is what I am saying. So, this is equals to the fundamental solution with respect to y minus x, minus capital Phi of small y, y minus x delta and this is with respect to yi, maybe you can write it here, yi.

Now, once I do this thing, let us calculate this one, capital Phi yi, mod y, y minus x delta. Do I really need to calculate, you guys can do it yourself. Let us just write it down, 1 by n alpha in phi. You know what phi is right? You just calculate the derivative with respect to yi that is all. This is xi minus yi by mod x minus y whole power n. You guys already know how to do this. It is just, you write phi and take the derivative with respect to yi that is all. And also, sorry, sorry, what am I doing. It should be phi i of y minus x, the first expression.

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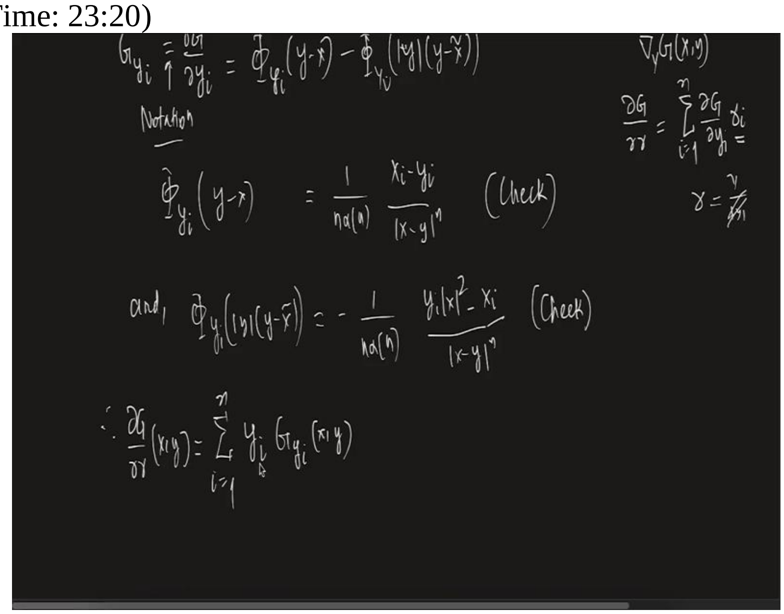

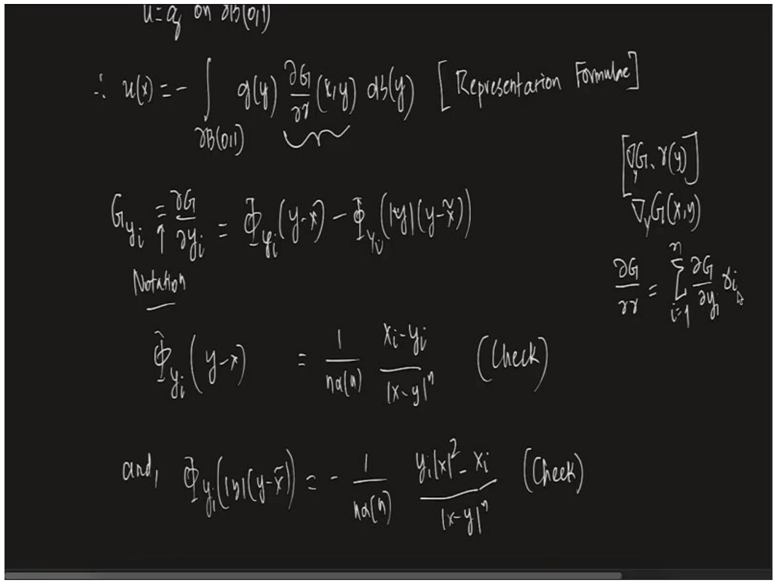

And you also can do the same thing here, mod y, y minus x delta. What exactly is this thing? Again, if you calculate, again I am not doing this thing, I am just writing it down what is exactly happening. So, you have to check this part, check this part, minus 1 by alpha in yi mod x-x square minus xi by mod x minus y whole power n.

So, please let me be very clear about it. You already know what phi is, just put what phi is on this variable and after that take the derivative with respect to yi. I am absolutely certain that all of you guys can check this part so please do that. So, once you do that, what do I need to do, once I do that, so u of x will look like this. So, I just have to put it there. You see del g, del gamma is essentially this thing and gamma 1 i of y I already know. I will just put it there and I will get what my del g del gamma is.

So, let us say therefore del g del gamma at the point xy, what does it look like? It will look like summation as I wrote it here i equals to 1 to n, yi, gyi at the point xy. Now, I do not think you are wondering why yi is there. What is gamma i, gamma is the unit outward normal and that is a ball so basically this is y by mod y and mod y is 1 here so essentially that is not there so gamma is y so gamma i is y i so that is why it is yy. So, this is the unit outward normal at the point y in the ball that is the unit outward normal, unit ball, unit ball.

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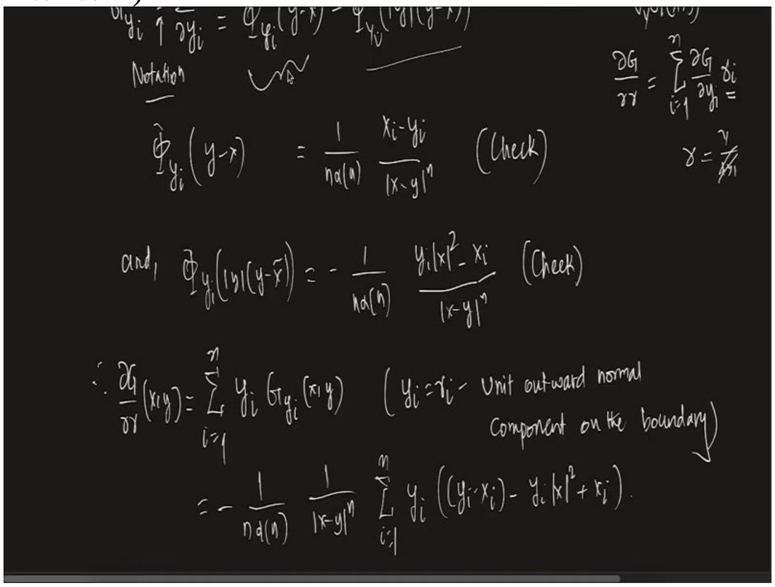

So yi is equals to gamma y, the unit outward normal and component on the boundary, clear, so that is this. Now if this happens you see, you just put it there, minus 1 by n alpha n 1 by x minus y whole power n summation i equals to 1 to n, yi minus xn minus yi mod x square plus xi. I am not doing anything; we are just calculating because I am just putting it together. Del yi of this thing I am just writing it and then minus of this particular thing because phi yi of y minus x is this so minus of this particular thing if you just put it together with the summation it will look like this.

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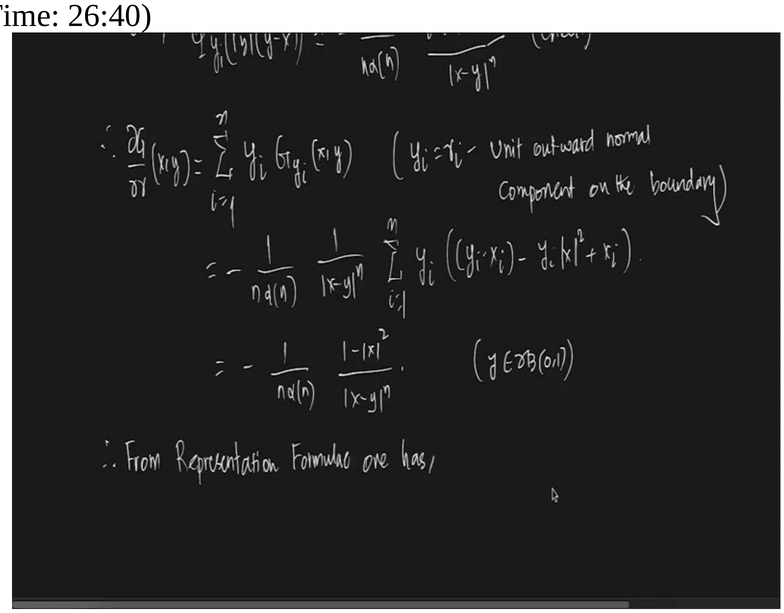

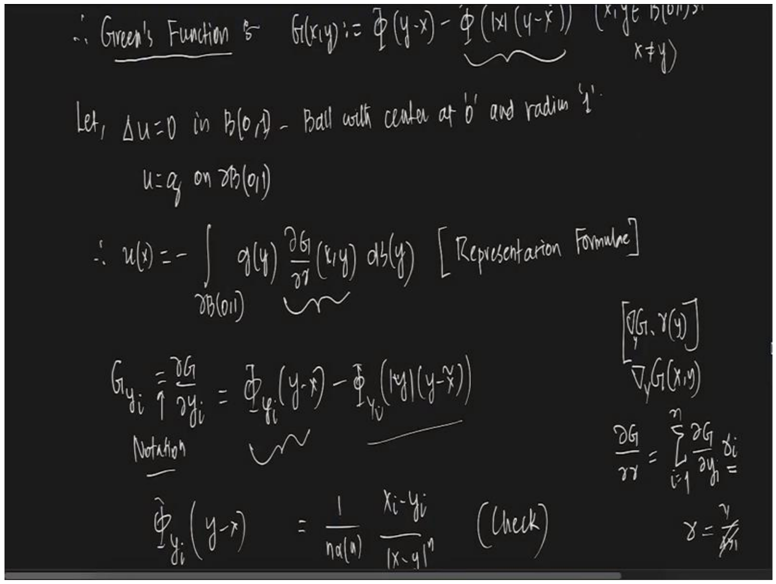

So, once I have this thing, calculate this thing. Once you calculate this what is going to come out is, alpha n 1 minus mod x-x square by mod x minus y whole power n. So, please remember, guys y, where is in this. You cannot choose y from anywhere and you can, so because you see here, y is on the boundary of the ball. We want y from the boundary of the ball. I do not care what y does outside. So basically, all of this calculation is called y on the boundary of the ball.

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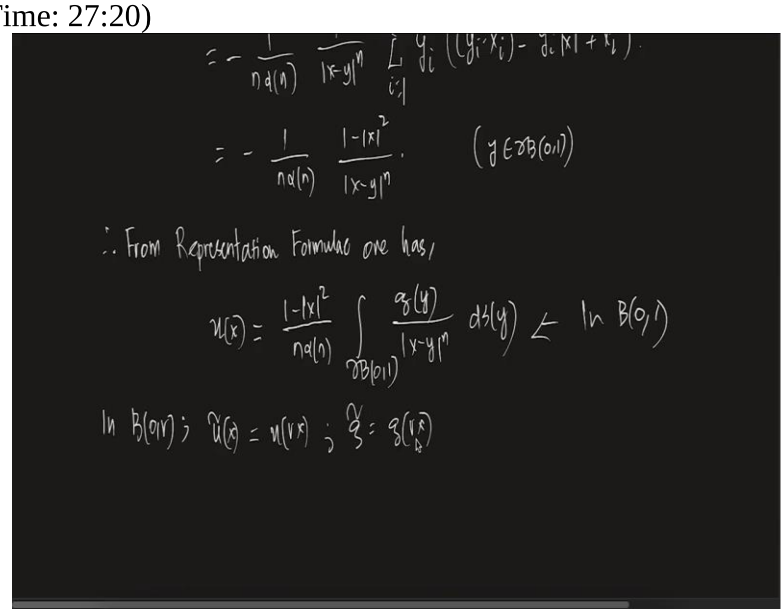

Once I do this thing, what is our vector therefore from representation formula, from representation formula one has u of x will look like 1 minus mod x-x square by n alpha n integral del B 01, g of y by x minus y whole power n ds y, clear? So, now this is on a ball which is centered 0 and radius 1. You can of course do it for a ball with center 0 and radius on B 0r, you can do that, that is not a problem. So, in B 0r, in B 0r you can write it like this. See, this is on B 01, this is on B 01. So, if I ask you to do it on the ball with radius r, of course you can do it. We just change u to u tilde which is, so basically you just replace u to u delta which is u of r x.

See, in a ball with center 0 and radius 1, this is the formula. So, in a ball with center 0 and radius r, you just have to scale u accordingly. That is x is getting scaled to r of x that is all. Similarly, g will be, you have to do it like this, g of rx.

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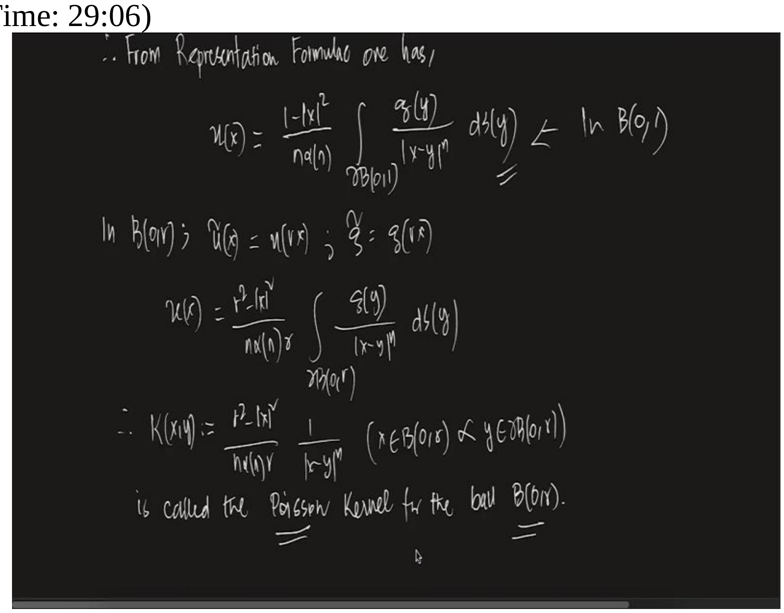

Once you do this thing, what do you get, you get u of x will look like r square minus mod x-x square, r by x, yes, by n alpha n, r integral over del B 0r. I mean there is nothing special, it is just I am writing like this, x minus y whole power n ds y. See why I suddenly delete for B 0r because I wanted to write this K of xy. So, this K of xy given by r square minus mod x-x square by n alpha nr and 1 by mod x minus y whole power n. So, x is in B 01, B 0r and y is on the boundary. This particular thing is called the Poisson Kernel for the ball B 0r, clear?

So, what I meant is this, see this is a representation formula on for B 01. If you want to do it for B0r, you just scale it. You just scale u and g and you have what is u of x and this is u delta of x. For the ball, the formula will look like this so it is not u tilde, basically u of x. So, the ball, if the solution is u of x then that is given by u of x to be this.

And see here the main idea of writing this is the following. I just want to write this r square minus mod x-x square by n alpha n in r times 1 by 1 minus x whole power n. This is a Kernel. This is called a Kernel. This is a special kind of function, it is a singular function actually. This is the Kernel, and this Kernel is called the Poisson Kernel for the ball B 01.

This is very important. We will look at why this is important later. In an assignment I will show you, you can see that what is so important about this Poisson Kernel. There are a lot of things which you can do with this thing so that is why.

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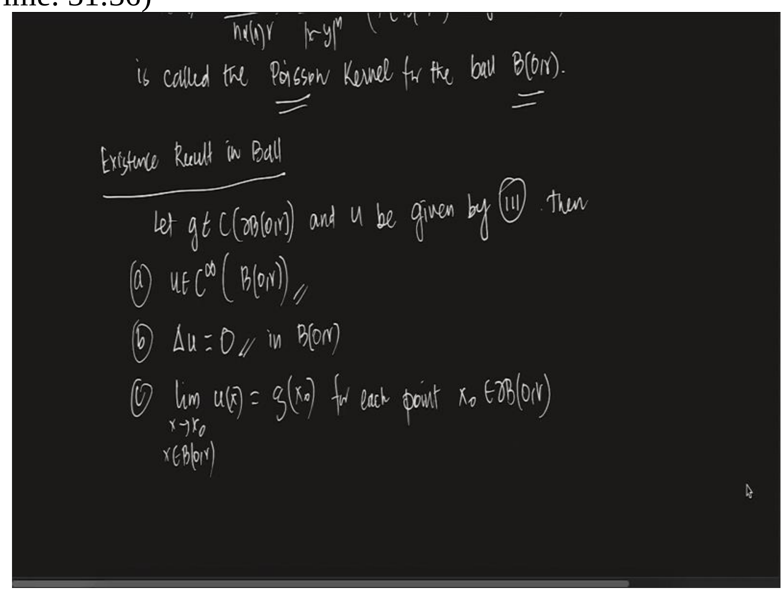

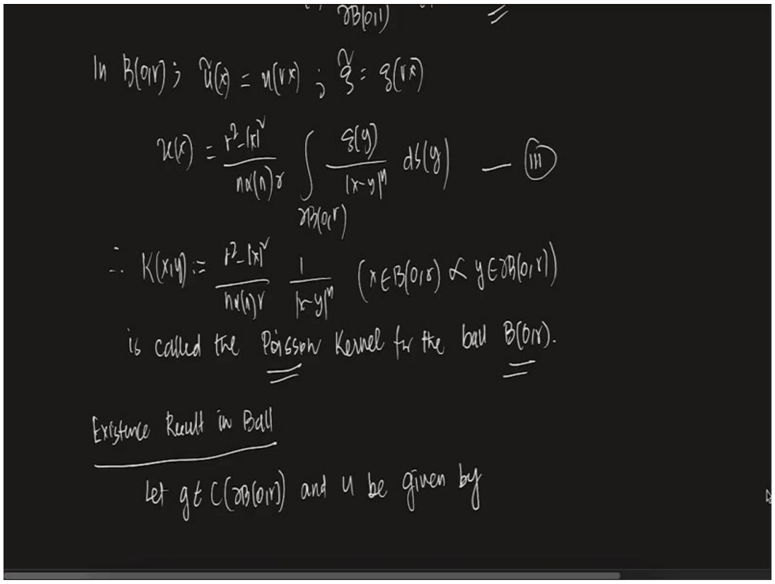

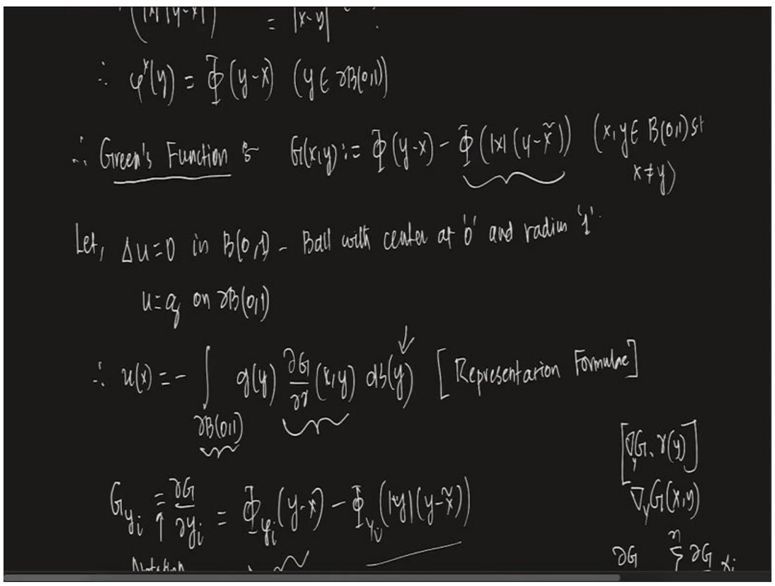

So, I am just ending up with this small theorem which I am not going to prove because this proof is exactly the proof of what we did earlier for the convolution thing. So, the existence result in ball. So let g is in continuous function del B 0r and u be given by let us say, let me call it 3 may be, 3, it is given by 3. If that is the case then a u is in phi infinity. You can show that this is infinitely differentiable on B 0r. Is this clear? See initially what we did if you remember, we did the convolution of the fundamental solution with f and showed that this is C2, given f is C2.

So, here what we are saying is if you are given g which is continuous on del B0r and u is given by phi, then u is infinitely differentiable, b Laplacian of u is equals to 0 because this started with Laplacian of u equals to 0. See we started by solving this problem, Laplacian of u equals to 0, u equals to g. So, what the theorem says is you can show that the u is infinitely differentiable, Laplacian of u is 0 in B 0r and c you can show that limit, how is the boundary condition satisfied, this is what, limit x tends to x naught, and x is in B 0r, u of x is equals to g of x naught.

See u is such that on the boundary is g, what does that mean? It means that u of x, see inside I know what u does, so I am saying that as u tends to x naught, g, so basically what will happen is ux will tends to g of x naught. That is how it is satisfied, for each point x naught on the boundary of the ball with radius r.

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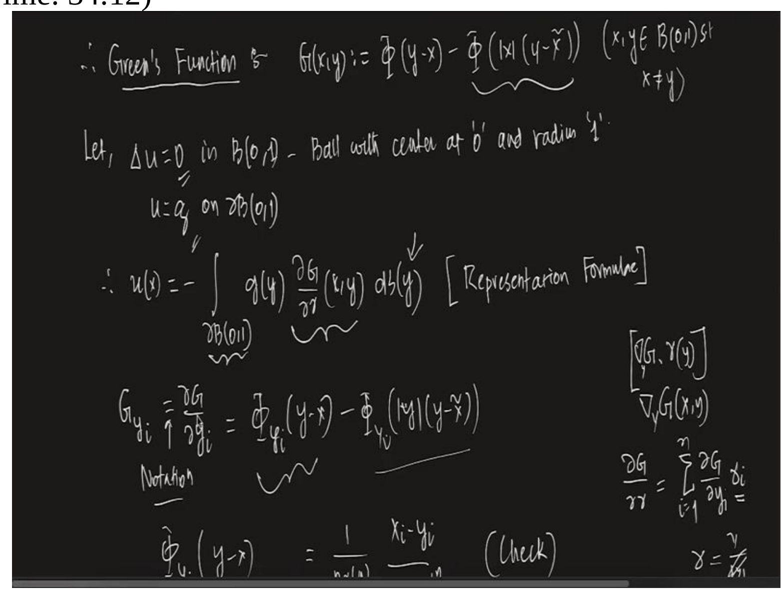

Fix $x \in B(0,1)$. Define the corrector function $\varphi^x(y)$ ...

$$\Delta \varphi^x = 0 \text{ in } B(0,1)$$
$$\varphi^x(y) = \Phi(y-x) \text{ on } \partial B(0,1)$$

| $\varphi^x \neq \Phi(y-x)$ ↑ then for some y we have $y = x$

and, the Green's Function will be

$$G(x,y) = \Phi(y-x) - \varphi^x(y).$$

Idea: Invert the singularity.

For $x \in B(0,1)$ we have $\tilde{x} \notin B(0,1)$

Assume $n \geq 3$: $y \mapsto \Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$.

Thus, $y \mapsto |x|^{2-n}\,\Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$.

$$\varphi^x(y) = \Phi(y-x) \text{ on } \partial B(0,1)$$

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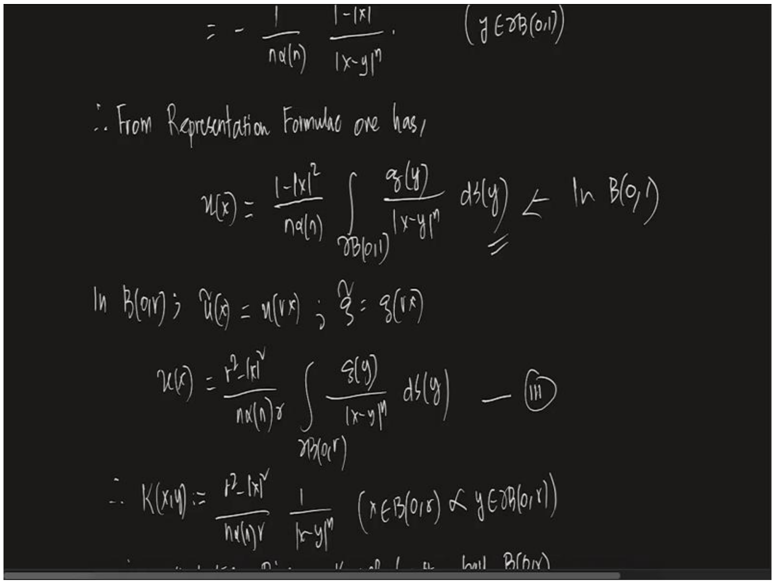
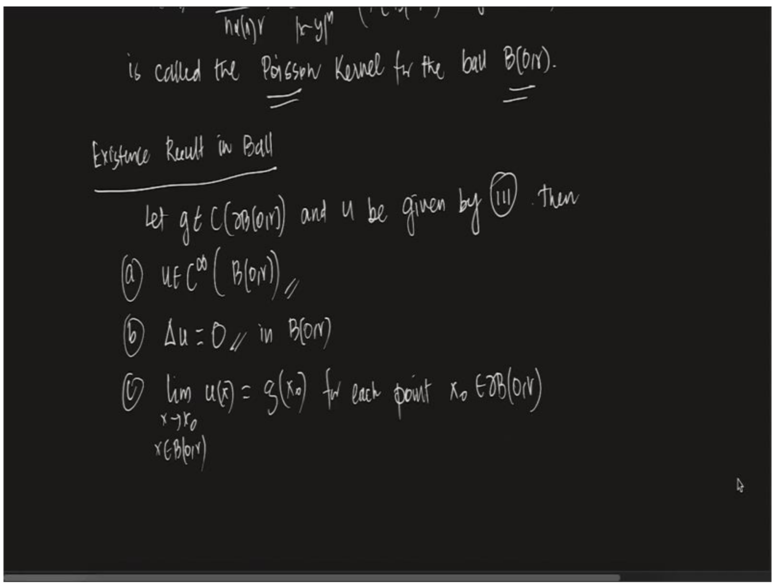
Thus, $y \mapsto |x|^{2-n}\,\Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$.

[We multiplied by $|x|^{2-n}$ because φ^x has to match $\Phi(y-x)$ on $\partial B(0,1)$]

and hence $\varphi^x(y) := \Phi\big(|x|(y-\tilde{x})\big)$ is harmonic in $B(0,1)$

If $y \in \partial B(0,1)$ & $x \neq 0$,





So, the proof is exactly the same what we did, so we are not going to do that thing but I think it is more or less clear. So, let us summarize what we did here. Once you find a Green's function, so basically you have to find a corrector function. To find a corrector function in this case of ball what we did is, we basically look for the inversion, we look for the inversion. Once we, here, where is it, yes, we look for the inversion of a point x delta. Why do we look for inversion because there is a singularity at x equals to y. I just do not want that singularity so I will just prefer an inversion and made the function harmonic.

We did we multiply mod x to the power 2 minus n? Because we had to match our corrector function on the boundary that is why we multiplied it by mod x to the power 2 minus n. Let me write it, we multiplied by mod x to the power 2 minus n here because phi x has to match capital Phi of y minus x on the boundary. That is why we did this.

Now, once we did this, we know that they are harmonic and we showed that, this is the reason why it matches on the boundary and then you write down the Green's function. You already know what phi of x is. There is nothing else, you use the representation formula, write it down and calculate what g is, put it together and you get this representation formula. This is u of x so the value of u at any point x, so the representation formula.

Now, do you think there is another formula for u, no. Because you see the point is u is unique. So, whatever the value of u at that point this is a unique value because of uniqueness theorem for this problem. So, and on B 0r what you can do is you can just tell the domain and the functions appropriately and you get the solution like this. Please remember this is important. This is called a Poisson Kernel. This Kernel is called Poisson. See g, nothing special is there, g is given to you and we just incorporated it with g. This is important and this is called a Poisson Kernel.

And what did you get? The u which you get from here that u is phi infinity, Laplacian of u is satisfied with the 0, and boundary condition gets satisfied in this way so basically you take any point on the boundary, that is getting approached by u so basically like this, limit x tends to x naught, u of x goes to g of x naught. So, with this we are going to end this lecture.