Advanced Partial Differential Equations

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Lecture- 15

Green's Function

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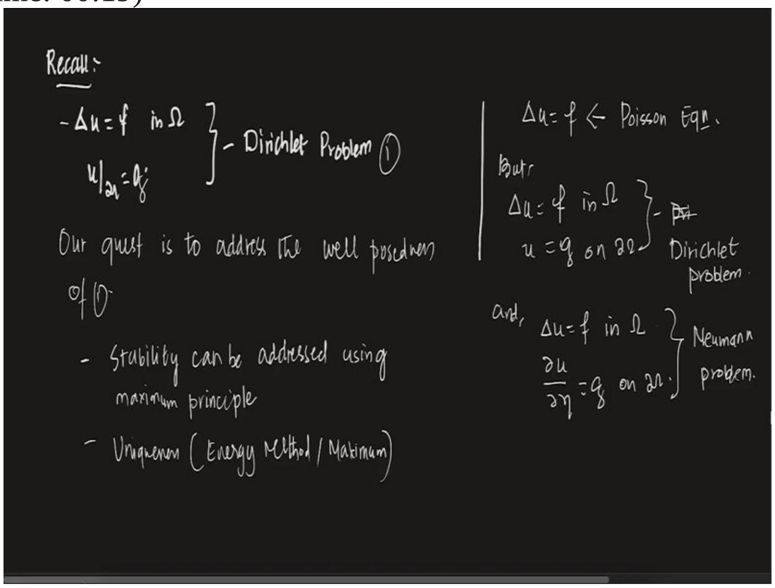

In today's class we are going to talk about existence. So essentially let us recall a little bit, quite small recall. Given a problem, let us just solve the homogenous problem and then we will dive into the non-homogenous, or maybe we talk about this. So, let us say you are given this problem, Laplacian of u equals to f in omega and u restricted it to the boundary is 0. This is called if you remember this is called the Dirichlet problem, the Dirichlet problem. I mean of course it is called a Poisson equation. See this equation is called Poisson equation, this equation.

So, let me give it like this, maybe Laplacian of u equals to f, this equation, only this equation is called the Poisson equation. But if you take this equation, but let us say this equation, Laplacian of u equals to f in omega and you are putting this initial condition, let us say u equals to g or 0, it is not a big problem. Essentially the data is given only on u. If that sort of thing is there, this problem will be called a Dirichlet problem. Sorry, it is Dirichlet problem. It is named after the mathematician, Dirichlet.

Now, let us say you can also have this kind of problem and the method which I am showing, you can actually modify the method to deal with this sort of problem also, del u del eta. So, this is outward normal on the boundary g on del omega. So basically, delta is the outward normal at every point of the boundary. So, del u del eta, that is equals to g on the boundary, so this is called a Dirichlet problem if the data is given on the partial derivative and not on u, then this problem is called a Neumann problem. It is called a Neumann problem.

So, today what we are going to do is basically look at the Dirichlet problem. So, let us call it a g, so let us call it, it does not matter, you can take minus not minus but let us just call it a minus. See the thing is this, this f is arbitrary. So, if you are not convinced, you just change f to minus f so Laplacian u equals to be (())(2:52) some minus f which is h. So, it does not matter. It is just a convention which we use minus Laplacian of u equals of f.

Now, the point is this, we want to solve this equation. So, you remember given (())(3:08) what are we trying to do. So, our quest is this, our quest is to address the well-posedness let us say the axial 1, well-posedness of 1. And in that respect what did we see. We saw that you can use maximum things equal to show that the stability is there. So, for small change in the initial data, the problem, the solution changes a little bit.

So, the stability, this is the recall. The stability questions or this change under the perturbation, stability can be addressed using maximum principle. But please remember this thing, maximum principle, most of the maximum principles which we did are valid only for bounded domains. Some are for unbounded domains, you can prove those, but the maximum principles which we proved are only for bounded domains.

So, stability can be addressed using maximum principles and there is another thing called as uniqueness. So, uniqueness we have seen that you can use uniqueness is there, so you can use energy methods for uniqueness energy methods and what is there, moreover you can also use maximum principles. So, those things are taken care of. Now, the only thing which is remaining and the most important thing is how do I find a solution for this problem.

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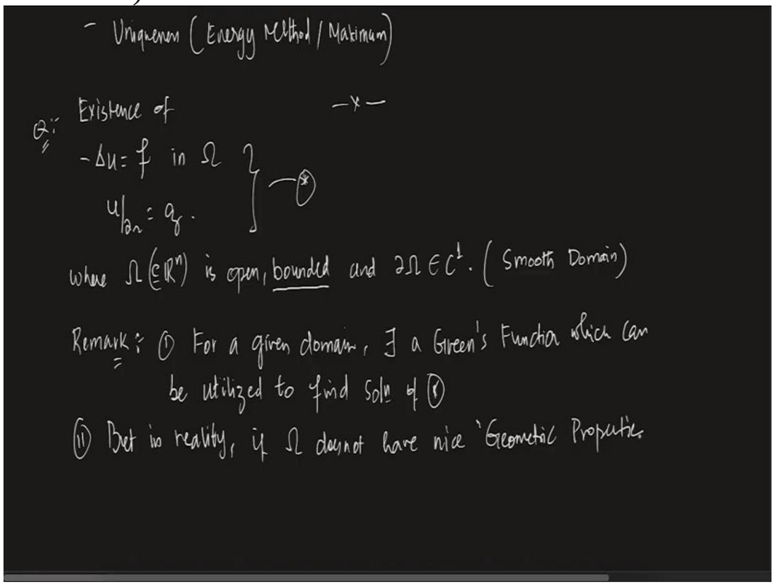

So, let me clarify this thing more. So, essentially what is the question now, the question is existence, existence of minus Laplacian now equals to f in omega and u restricted to the boundary is g. Now, for our case, now in this case we actually assume that f can be, f and g are smooth. So, do not worry about what is the regularity of f and g. They are smooth and what about omega. Omega, so where omega is of course is the subset of IRn, there is nothing to say so omega is open, bounded, very important.

So, this what I am going to do now for that omega has to be open, bounded and the boundary of omega, del omega, is the C1. Or you can also say omega is smooth so basically the domain is smooth which means that the boundary is C1, or we also say the smooth domain. This is basically is saying that this is a smooth domain.

Now, what I want to do, I want to find an existence of this problem. Before I go on doing whatever I am trying to do right now, let me make some small remarks. So, please keep this thing in mind. Remark see what I am going to do is I am going to construct something called a Green's function. For a particular domain, for a given domain there exists a Green's function, Green's function which can be utilized to find solution of star. Is this clear? We will show that, we will do that, do not worry about it. So, we will write down the formula of what, you will see, look like, specific formula of (())(7:18) based on f and g.

So, if you are given a domain, for that particular please remember this thing, for that particular domain, you can find a Green’s function and this is a theorem right, that for a nice enough domain you can actually show that there exists a Green’s function. So, that is there, we will do that, we will show that.

But in reality so basically you may say that yes, then for any domain I can just find a Green’s function and write down the explicit formula for this thing any solution of that will look like this. You do understand one thing, see Green’s function, once for a given domain, take any domain. Let us just say that that domain is a ball with center origin and radius 1. So, for that given domain, let us say you found out a Green’s function and with the help of that you have just found out what the solution ux looks like, okay fine.

Now, and since you know that the solution is unique, you can say that is a solution which you are looking for. But the problem is this. See, for that particular domain, you need a ball. How do you find the Green’s function that is the question, but in reality what happens is, if omega does not have nice geometric properties. This is very important, geometric properties.

So, in some places you will see that they start constructing Green’s function out of nowhere, yes. Most of the places where you can see that do not have any motivation of how those Green’s functions are coming and most of the times they are like, it is not based on any logic.

You just try your hands on something and kept the Green’s function. And most of the times you would not find that is the point. See Green’s function exists for C1 function that is for sure. It exists but finding an explicit Green’s function is almost always impossible, you understand, except for a few nice domains. If omega does not have a nice geometric property and I cannot specify what sort of geometric properties I am talking about right now but we will have some idea about what exists (())(9:34).

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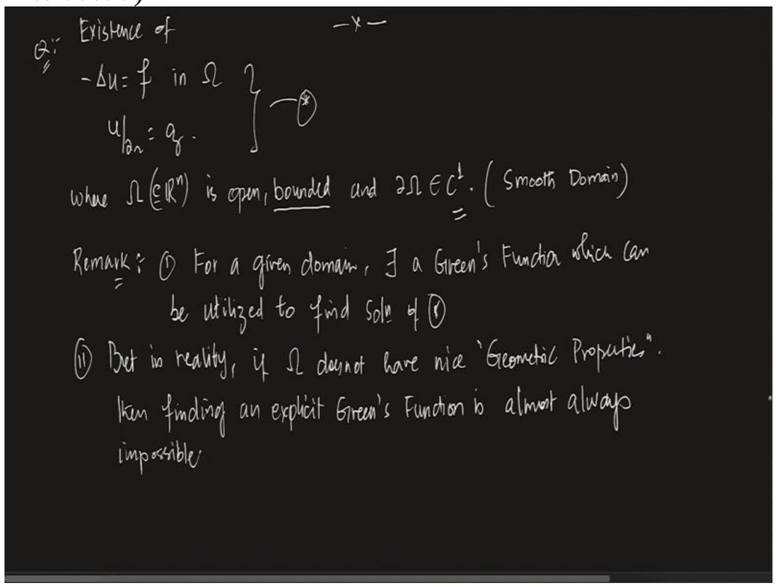

Then finding an explicit Green's function is almost always impossible. Now, you may by some hook and crook for some domains you can get it but generally speaking, it would not be easy. I mean not easy it is kind of an impossible task to do. So, with that in mind, please, let us start with finding at least for some problems, we can find. Let us do that.

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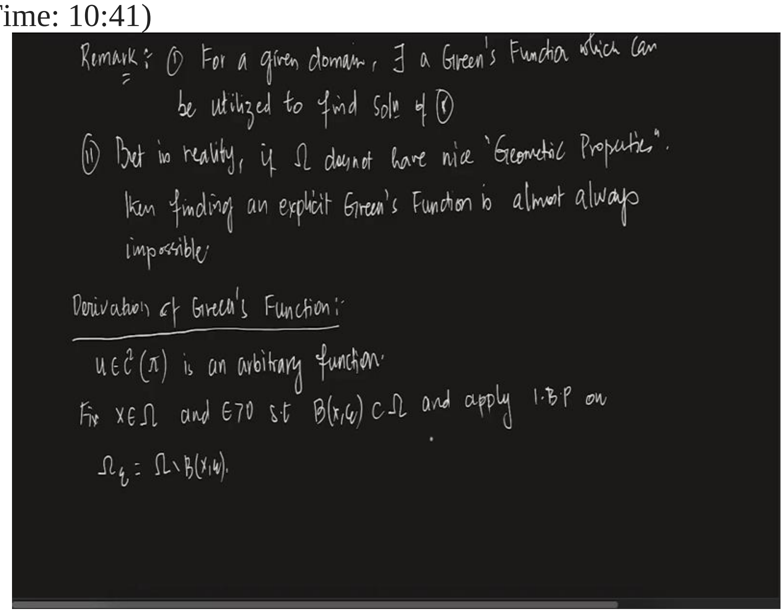

So, first of all let us show that there is a Green's function which solves our problem. Let us just call that problem star. So, this is star. We want to solve this problem. We want to find ux. How does ux looks like. So proof, basically the derivation, I am doing the derivation of Green’s function. So, let us say, I am starting out with a u which is C2 omega bar. There is nothing special about this u. I am not saying that this is a Laplacian (())(11:10) or all that, nothing. It is just an arbitrary function, is an arbitrary function. So, you start with any arbitrary function which looks like this, clear. Once you do this then what I am going to do is I am going to imply a similar tactic which we did in the last class of the convolution thing which we did last class, that sort of thing.

So basically, we fix the u, we fix the x in omega and epsilon greater than 0 such that B x epsilon is containing omega and we apply integration by parts on, let us say we call it omega epsilon which is omega minus B epsilon.

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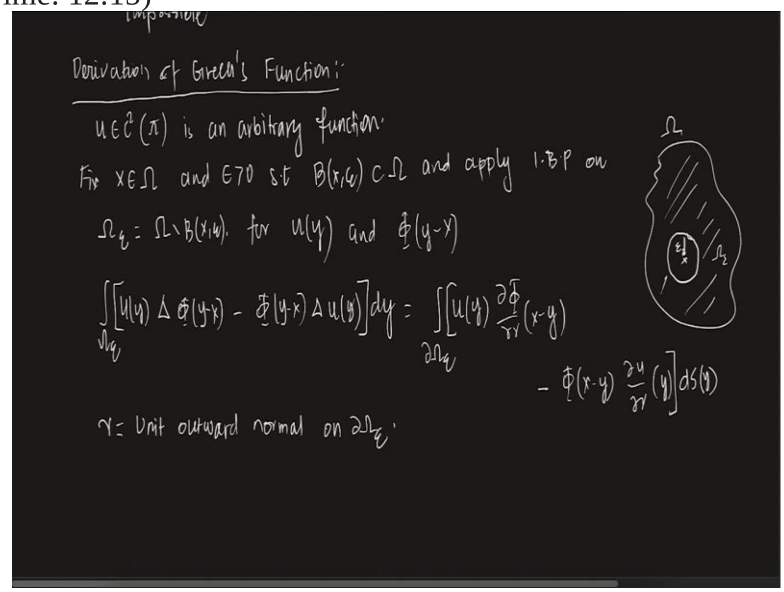

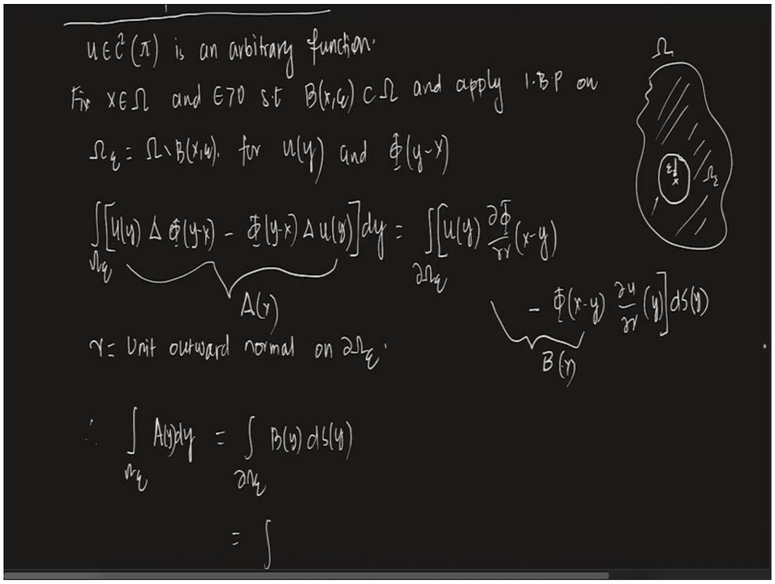

Why we are doing it, it would be clear right now. So basically, what we are doing is this. Let us say that is your omega, this is your omega and that is the point x. Take a small neighborhood of x, let us say this is x, that is epsilon neighborhood. I am just throwing this part out and this is your omega epsilon, this is your omega epsilon.

So, I want to work on omega epsilon right now. What I am going to do here is this. See, I am going to use integration by parts in this domain for which functions, u of y, u is a function of y. See this x is fixed. I am fixing an x in omega and for that I am using integration by parts on this u which is a function of y right now and the fundamental solution if you remember, phi of y minus x. And now I am sure you have some idea of why we are taking this deleted neighborhood, because y, when y is equals to x, phi blows up right.

So, we do not want that thing to happen so in this omega epsilon you see y, if I am starting now to the y in omega epsilon, that can never be, y can never be equal to x so phi is well defined here right and of course u is always well defined anywhere in the domain so it is not a problem.

So, I can use the integration by parts and once you use that what happens, let us say omega epsilon, u of y, Laplacian of phi of y minus x minus phi of y minus x Laplacian of u of y, dy. So, this is equals to integral over the boundary of the omega epsilon, u of y and del phi del gamma of x minus y and minus, let me write it like this and here also let me put it like this minus phi of x minus y del u del gamma of y, dfy.

So, this is just integration by parts**,** I just took two parts and put it together, I mean nothing much. So, once I have something like this. Gamma I did not specify, gamma is the unit outward normal on del omega epsilon. That is your gamma.

Now, let us look at, see hence let us call this thing as, it is a big expression. Let us call this for now, let us call this an A and this expression as B. So, what do we have, I have omega epsilon A, dy, A of y of course, A of Y dy equals to, so this is A of Y and this is B of Y and that is equals to del omega epsilon of B of Y, d s of y. So, now let us break this part. D epsilon Y so this is, see what is the boundary of epsilon Y, epsilon Y is this interior point. It consists of the interior points so the boundary is of course the boundary of omega plus the boundary of this ball.

So, it is basically the boundary of ball with center at x epsilon B of y, let us call it B delta because ball I am writing as b so let us say this, call it a B delta. B delta is explicit expression with a big one. So, B delta of y, ds of y plus the integral over the boundary, del omega. If you remember the boundary is this boundary, the boundary of omega epsilon is this one plus this one right, so I am just breaking it up like that, B delta of y, ds of y.

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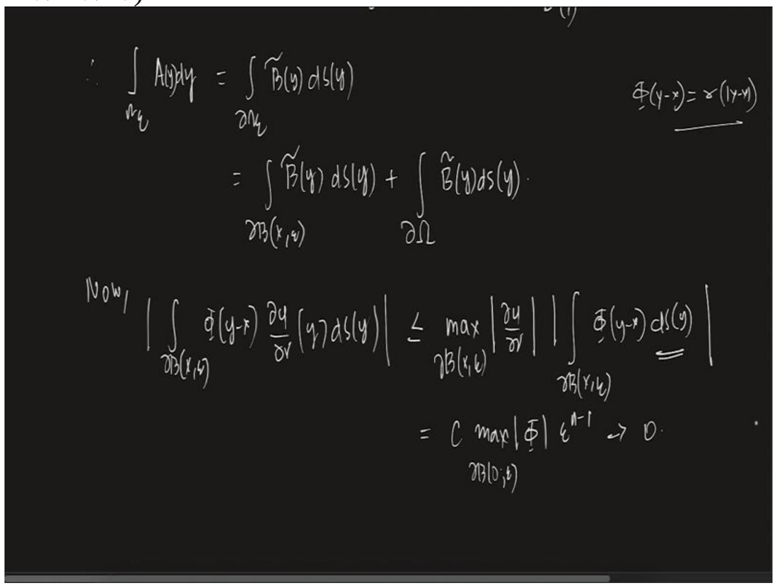

Now, let us look at this expression, omega epsilon, del B x epsilon, phi of y minus x, del u, del gamma of y, dsy. This expression I want to calculate what happens to this. See we did the hash two expressions right. B delta is this one right, so I am just starting off with this one. I am not touching this one, I am just starting out with this, this is phi times del u del gamma and integral over the boundary, integral over just this one. I have also this part to take care of but let us just stick with this.

For that what do you have, see del u del gamma, u is a C2 omega bar function, u is a C2 omega bar function, so this is continuously, see u is a C2 omega bar function and this ball is contend in omega. So, u is continuously differentiable. So, on the boundary of the ball there is a maximum (())(17:59) and that maximum I can just take it outside so basically this is less than equals to the maximum of del u del gamma on B x epsilon, sorry del B x epsilon. That I can just push it outside and once I do that, I am left out with is epsilon phi of y minus x, dsy, clear, that is what I have left out with.

Now, see essentially if I am taking y from this, see I am doing this integration such that y is on the boundary of this. Where does y lie, y lies on the boundary of the ball centered at x and radius epsilon and I want to integrate phi of y minus x there. So, as you guys know, phi of basically y minus x is some function R of Mod y minus x because phi is the radial function.

So, if it is a radial function what happens is this is basically a constant on the ball, y minus this is a constant there so essentially you can just take the maximum of this phi outside. If you take that particular thing, so essentially what I am trying to say is see, if you are taking y from the boundary, y minus x is something epsilon, y minus x is epsilon. So, essentially what is happening, you are basically integrating one by or I mean if you want you can just, you see y is never equals to x.

You can also think of, see there are many ways of doing this thing. You can also think of it like this. X is not equals, to y because x is in the center and y I am taking from on the ball, from the boundary of the ball so x is never equals to y. There is always an epsilon distance between them. So, this is always defined, well defined and it is a particular function so you can also take the maximum outside so basically maximum of this and maximum of this let us say the axial C, let us say this is your C and I am again taking maximum of phi. I can take that outside also.

So again, this is one and the same thing and you have the integral of del B x epsilon dsy. So, if you take the integral that is basically the surface area. So, that will look like epsilon power n minus 1. Surface area that is epsilon power n minus 1, of course some constant is there and that is getting absorbed here. I am not writing all that. So, this goes to 0, that epsilon goes to 0. That is quite evident, is this clear.

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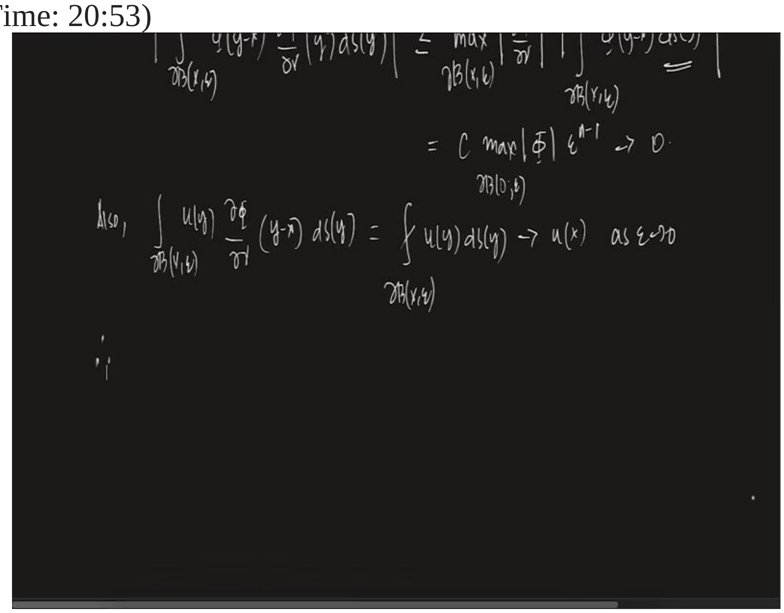

Now, what we have is this. Also integral over del B x epsilon, u of y, the other part, this is there, the other part is u of y, del del gamma of phi. So, u of y, del phi del gamma of y minus x ds of y. Now, what we can see is this thing. If you remember from the last class you remember, we did this kind, exactly this sort of calculation.

So, what did we get, we get something like this. I am not doing this part again, s y which goes to u of x. If you remember while proving that the convolution, convolution with phi and s that is also Poisson equation in Rn, when doing the last calculation, this is the last part of that calculation, if you remember. Please go through that if you do not remember that. So, this is also true as a (())(22:00) of course as (())(22:02).

We did the exact same calculation, I am just using that, I am not doing anything special here. Once you have something like this, then therefore the formula which you have is this, you see this is u of x. If you just put it together, let us just put it together, this is u of x, this is going to 0. So, essentially you see here this term will be there, this term will be there, from here you are only getting u of x because one is this part, B delta has two parts, one is going to, this is going to 0 and this is going to u of x.

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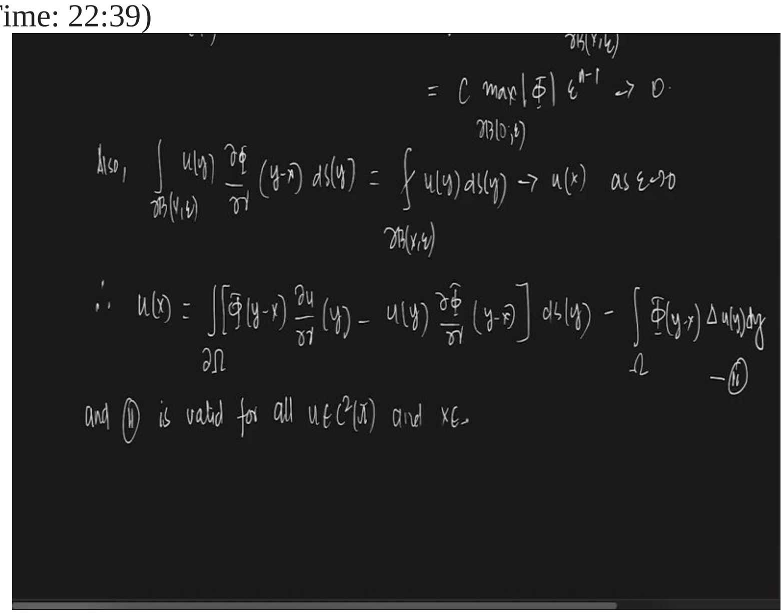

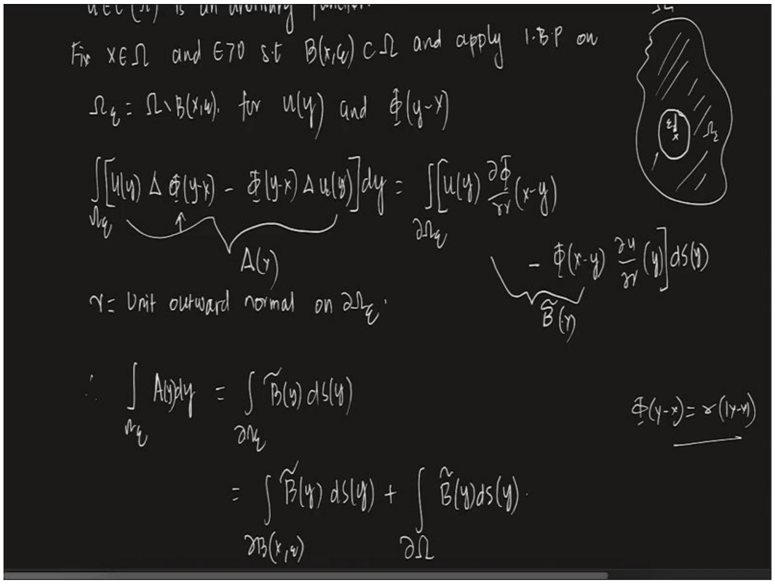

So essentially, I can write it as u of x is equals to integral over the boundary phi of y minus x, del u by del gamma of y minus u of y del phi by del gamma of y minus x and dsy minus integral over omega phi of y minus x, Laplacian of u y, d y. So, this is what we are getting, correct. If you see where is it, yes, we are getting this. Why, because what happens to this term. This particular term is 0, this particular term is 0, y is never equals to x so Laplacian of phi, this is a fundamental solution. This is always 0 so essentially that is gone.

I am always left out with the minus integral over omega epsilon phi of y minus x Laplacian of u of y, v y so omega epsilon as explained (())(23:56) omega so integral over omega only this particular thing. So, that is what I wrote, where is it, you see that is what I wrote, clear.

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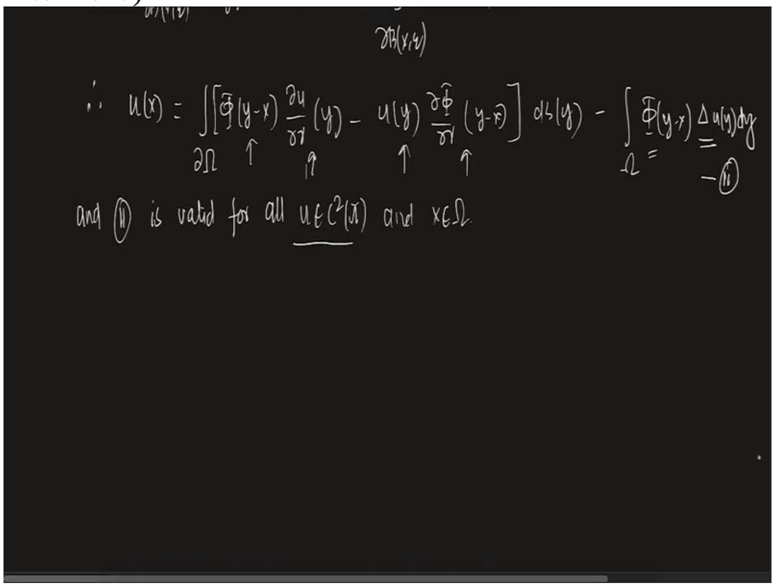

So, of course and, so let us call that as 2 and 2 is valid for all u in phi 2 omega bar and x in omega. You can do it for any x in omega it does not matter because for any x in omega you can always find a ball doing that because omega is open. So, here you see what is happening. If you know, let us just assume that if you know what Laplacian of u is in omega so let us say that is your f, phi is already known to you, phi you already know because phi is the fundamental solution, there are no changes.

So, essentially you can just calculate what is integral of omega, integral of this particular over omega, once you know what Laplacian of phi is and that is given to you so basically it is given to us where Laplacian of u is (())(25:05) and u equals to 0, g equals to boundary.

If you look at the star, if you look at the star, where is the star, Laplacian of u equals to f is given to you, so just explain that there and u restricted the boundary g which is given to you. This function, this holds for any u in C2. So, what I am going to do is I am just going to write down here. If I put it there you see this Laplacian of u, this is f, minus laplacian of u equals to f. We have replaced it by f, this already we know so we can calculate this.

Let us look here, u on the boundary, we know if this is g and del phi del gamma on the boundary we can calculate because we know what phi is, we can just calculate what del phi del gamma is. What about this one, phi we know, but del u del gamma. This is a problem, del u del gamma, see we know what u does on the boundary. We do not know what del u del gamma does on the boundary. So, we have to somehow eliminate this particular expression and then we can go through what we are trying to do so this is where we are just trying to construct a Green's function.

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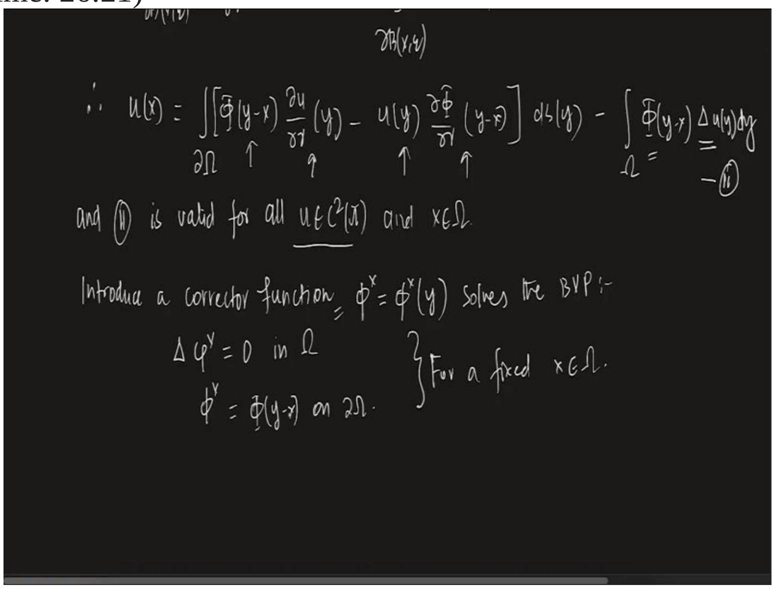

So, to do that what we are going to do is we are going to introduce a corrector function. So, that is given by phi of x which is phi x of y and it solves the boundary value problem. It solves the boundary value problem. How does it solve, it does this, Laplacian of phi x equals to 0 in omega, phi of x is equals to phi of y minus x on the boundary. So, what we are going to do is for a fixed x, so this is for a fixed x in omega. What we are going to do is this, see we are going to fix the x in omega and for that we are going to introduce a corrector function, which is phi x given by phi x of y.

So, for every, please understand this what we are trying to do here. We want to eliminate this del u del gamma term, to do that we are fixing x in omega. So, once we fix that x we are introducing a new function, based on that fixed x which is phi x of y and such that Laplacian of phi x is 0 in omega and phi x is the fundamental solution valuated at y minus x on the boundary. Is this clear?

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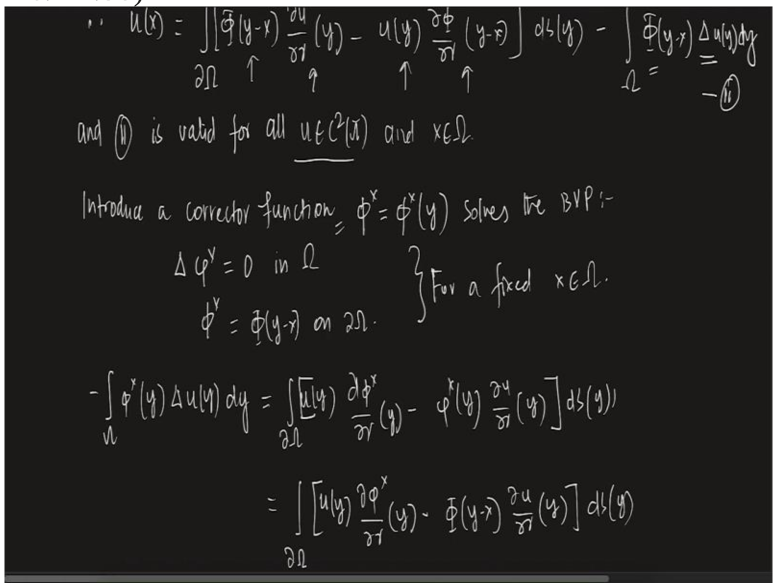

Now, let us apply the integration by parts again. Once we apply integration by parts, we of course get this, let me write it down. It is phi x of y Laplacian of u y d y equals to integral over the boundary del omega u of y, sorry, del phi x by del gamma of y minus phi x of y del u del gamma of y dsy. So, basically, I am just using the integration by parts to write this thing. I hope you understand where this is coming from. So, this is just whatever we did earlier.

This is the same integration by parts so this is just this formula, this formula, Laplacian of phi x is 0 so I am not writing that part and except that everything is same. I am just writing that particular formula you see.

So, Of course not for that function. I mean I am using that formula but for this function, phi of x and u. Once I do this thing so let us say this over del omega this will look like u of y del phi x del gamma of y minus phi x at twice Y minus X, del u by del gamma y dsy. You have this.

Now, you see what is our point, our point was to eliminate this particular term del u del gamma at the point y on the boundary and this is the term right, phi y minus del u del gamma on the boundary. So, if I add this term and this term then the two will get canceled out. I want to eliminate that only.

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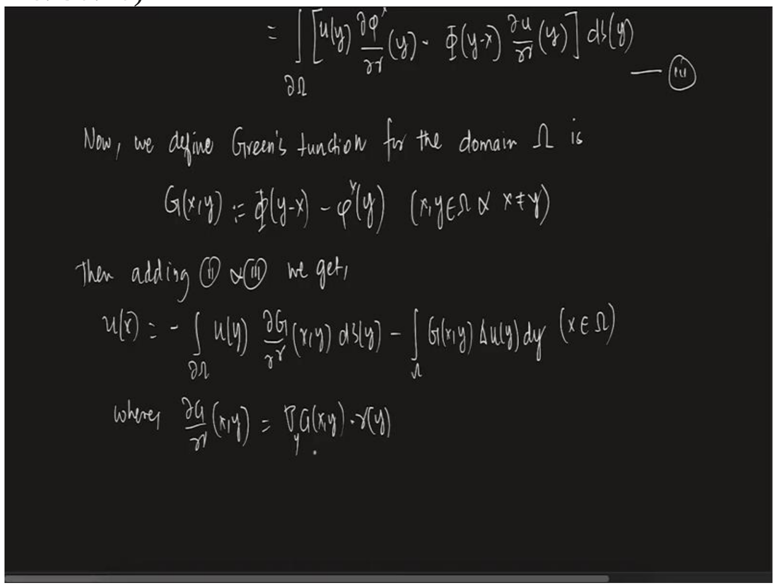

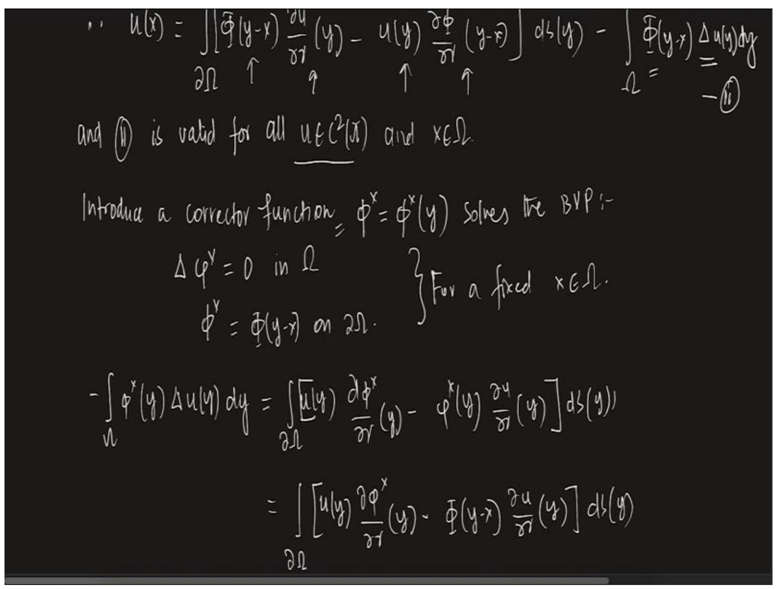

So, what we are going to do is now we want to add those two terms and to do that we want to make it a little simple that is only. So now we define, you understand what I am saying see if I add those two terms this term is gone right. If I add this particular term and this term, I want to eliminate this term. So basically, if I add this expression and this expression together, that term is gone and I want to do that and I want to write it in a more compact form and that is why we introduce a new function, so we define Green's function.

This is what we call the Green's function for the domain. You see Green's function is not for anything. It is only for the domain omega, is g of x y. We always write Green's function as g of x y so this is equals to, this is by definition, the fundamental solution at y minus x. Please remember the fundamental solution is defining whole of Rn and that is why you can just write it like this, phi x of y. So, this is your, x and y is in omega and x of course not equals to y because otherwise this is not defined. So, this is a Green's function which we defined.

Now, once we define this thing, then adding, let us call it so that is your 2. This is your 2 and let us call it 3. So, adding 2 and 3 we get, what do we get? We get u of x is equals to minus integral over the boundary u of y del g, I am just writing it down, please calculate this thing. It is not very difficult to do, this is very easy to do.

Please check that part, that is all, dsy minus integral over omega g of x y Laplacian u of y. So, this is for x in omega. That is u of x, of course here where del g del gamma of x, y, this is given by gradient of g at the point x y now when I am writing gradient of g I mean it is with respect to y, so essentially g with respect to y that is all times gamma y. You understand this is g with respect to y, dot gamma y and of course the gamma you can eliminate out what normal, I am not writing all that, gamma, unit output normal, so whatever the domain is.

So, in this course please just keep this in mind whenever we write the (())(33:37) gamma kind of things, you will always understand, it is always understood that this is the unit outward normal unless otherwise written. So, now you see once, so basically what I am trying to say is once you find this g, you can just replace this g here and you can you find what u of x is and we are done.

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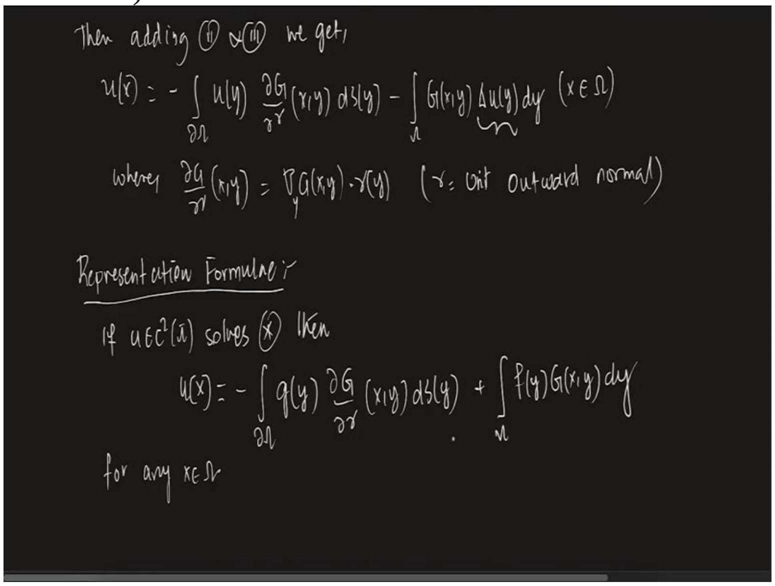

So, let us write down the representation formula. This is called the Green's representation formula. What it says, if u is C2 omega bar solves star then u of x is written as minus integral over the boundary, g of y, del g, del gamma of x y, dsy, plus integral over omega f of y, g of x y, dy clear. I just wrote this and see Laplacian of u is f, minus Laplacian of u is f so that is why this is plus f and g minus integral over the boundary u of y times this. So, this is remaining del g del gamma, I am just writing and u on the boundary is g so I am just writing it like this and that gives you a representation formula for any x in omega.

Essentially what we did is, let me again just recall. We wanted to find the solution of the Poisson equation with the Dirichlet boundary data and to do that what we did is, we found out a Green's function. So, the main important thing here is to find this integrator function, to find this, sorry, the corrector function.

Once you find the corrector function like this so the important thing is see, if u, if omega is C1, this is easy to find, this is always true, it is not a problem, for any given C2 if omega is a smooth domain this is always true. You do not have to worry about it. The only worry is once you find a corrector function you have to find a corrector function for that omega. It may look like not a very difficult thing to do but it actually is the most important thing which you can, I mean this is the difficult part, to find a corrector function. Here I defined it like this but in real life situations if you have to solve this problem explicitly you have to find a corrector function explicitly, that is going to be difficult.

And once you find the corrector function, you just define a Green's function like this. I mean there is nothing special actually, you cannot, g I just defined it just to make it short that is all. So, once we define this thing you do realize that what happens to g on the boundary, it is going to be 0, g on the boundary of the domain is going to be 0. So, you just have to find that g with this and then you just write down the representation formula in terms of this function g.

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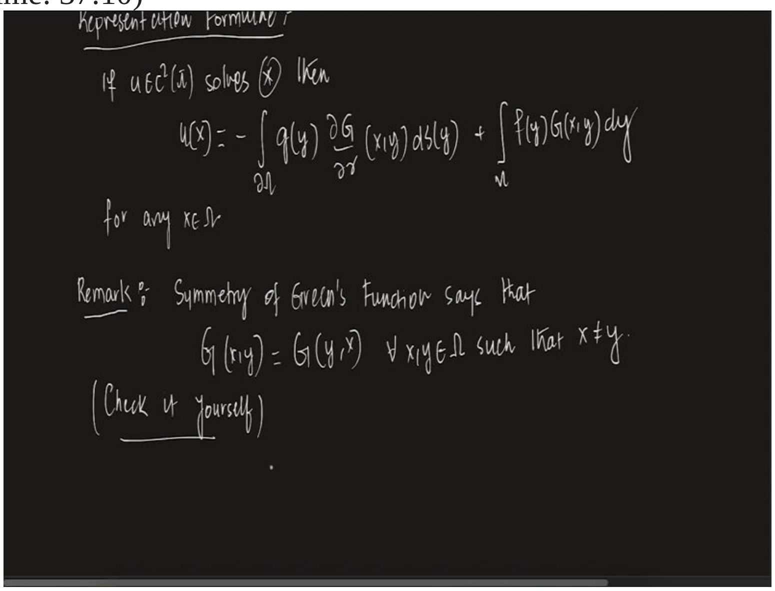

So, one small thing, so our remark and this I want you guys to do it yourself. It is just some calculation, but I hope you can do this. This is called symmetry of Green's function. So, what does it say, it says that g of x, y this is equals to g y, x. This holds for all x y in omega such that x is not equals to y. Is this clear?

This is called a symmetry. So, essentially it is saying that if we change x to y or y to x, does not matter, it does not change anything, it will be accepted. So, you guys have to check it yourself. Please check it yourself. So, please do that and this is just a property of Green's function. So, once you do that, you are done. Now, in the next part what we are going to do is we are going to see how to find Green's function for a particular domain. So, with this we are going to end the lecture.