Advanced Partial Differential Equations

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Lecture 14

Solving Poisson Equation

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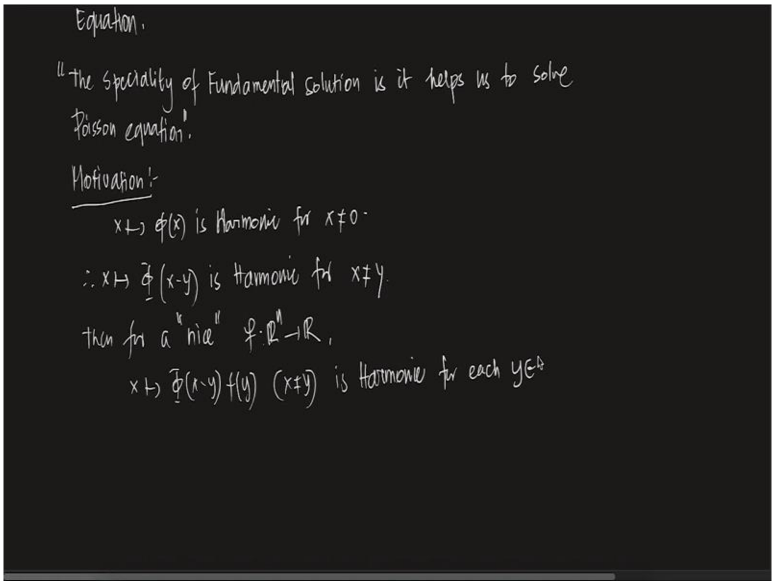

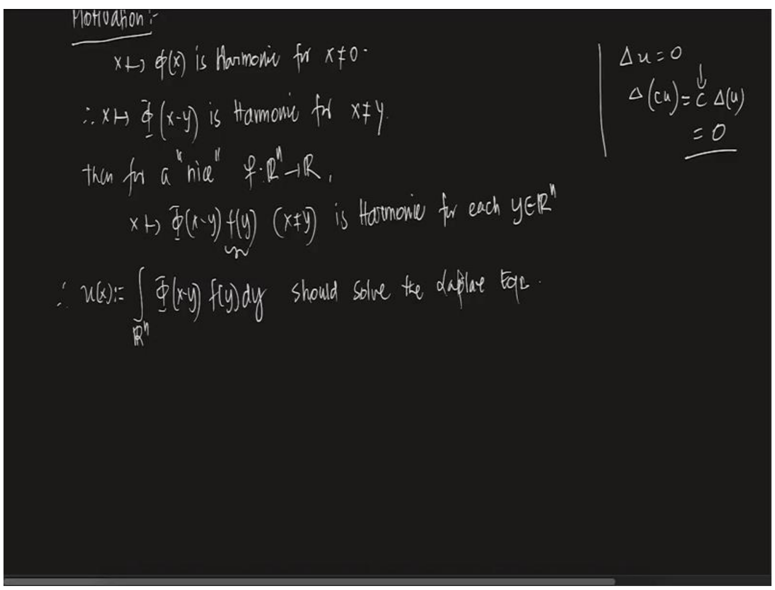

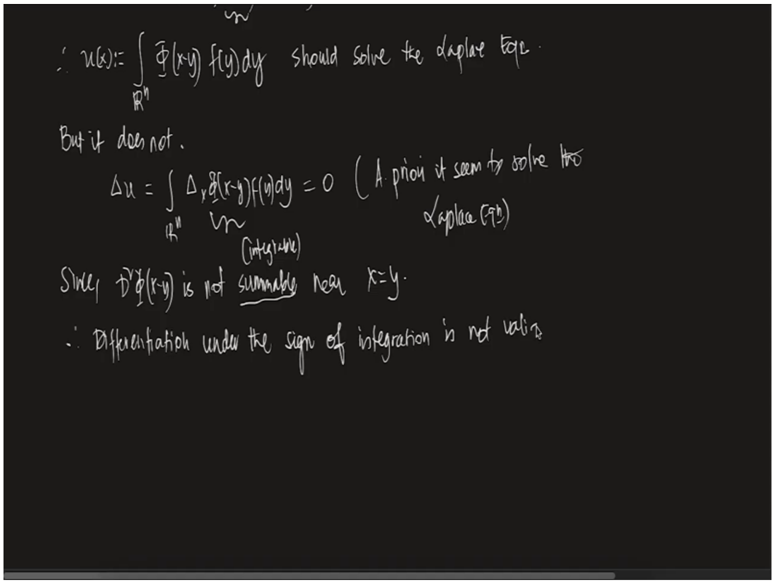

Welcome students in today’s class what we are going to do is we are going to use fundamental solution so what is so special. So, the speciality let me put it this way that speciality of fundamental solution is it helps us solve Poisson equation. So, non-homogeneous see fundamental solution essentially deals with homogeneous… Laplace equation Laplacian (())(01:03). What we are saying is we can use that particular function from the illusion of homogeneous solving Rn minus 0 and use it to solve something which is like in homogeneous problems. So, how to do that? Let us, see.

So, motivation now you see what happens is, let us say x going to phi of x, this is harmonic for x not equals to 0 yes or no. Because we have seen that C solve this problem. For all x in En except at 0. So, basically, this is a harmonic for if x not equals to 0. If it is happens, you just shift the origin to y. Here the origin is at 0, so you just shift it to y. What happens is, therefore x going to phi of y minus x or x minus y is harmonic for x not equals to y, I can do this thing, why? Because essentially, Harmonic function are translation invariance.

We talked about it earlier Laplace equations of r rotation and translation invariance. So, basically, if you translate the function, it is going to be harmonic it is not a problem. So, if that happens, then for a nice it not very mathematical but let me write it like this, f from Rn to r the math x going to phi of x minus y, f of y. So, this is not x not equals to y, of course is harmonic for each y in Rn yes it is of course harmonics. So, if you (())(03:07) y in Rn this is going to be harmonic. Why?

Because, you see, when you fix a y in Rn f of y f is some function at the point y just taking some real value, so basically you are essentially multiplying it by a constant yes or no, see, this function is a function of x, x is going to this thing, it is a function of x as a function of x, this is harmonic it is not a harmonic with respect to y, you do understand what I am saying, x going to phi x minus y is harmonic it is harmonic as a function of x.

Again, here, when I multiplied it by f of y for a fix y, f of y is just a constant essentially so I am multiplying by a constant it is a function of x and so it remains harmonic, it is not a problem. Because you see, if Laplacian of u equals to the 0 Laplacian of phi times u, let us say will it be 0 of course, because it is a linear function, so, this is 0.

So, this phi here is f of y, now y may change and that does not matter, I mean is going to be the same thing. So, essentially, what is happening is we are saying that this kind of function had to be harmonic. Now, if it is harmonic, then you see, then there is reasoning, so we can say that let us say if you just integrate it over Rn, so integrate over Rn, phi of x minus y, f of y dy this particular thing, this should solve, therefore, this should solve the Laplace equation.

You understand what I am saying, see, for every x it is harmonic as far every x this also Laplace equation. So, basically if I am taking the integration on Rn and let us say if I am defining u x to be that particular thing it is a new function u x, this is I am defining it like this then I can say that this also Laplace equation u. But the problem is it does not.

But it does not and now, what is the problem? The problem is see let us say if Laplacian of u if we just write it like this, I mean, if you take the Laplacian inside of this thing, it will be delta x of phi of x minus y f of y dy, it should be something like this and delta x of f of x minus y is 0. So, this (())(05:50) has to be 0. So, if I mean a priori it seems to solve the Laplace equation, but it does not why it does not because you see this particular function. Because it does not I should have wrote it here.

Since, you see the second derivative of phi at the point x minus y, this is not summable near x equal to y, is this clear? See, there is a singularity at x equals to y, if this phi blows up. So, D2 of phi that is this is not integrable summable whenever I say summable I mean integrable. It is not integrable (())(06:49) x equals to y and hence and therefore, differentiation under the sign of integration is not valid, then it is not valid, that is what.

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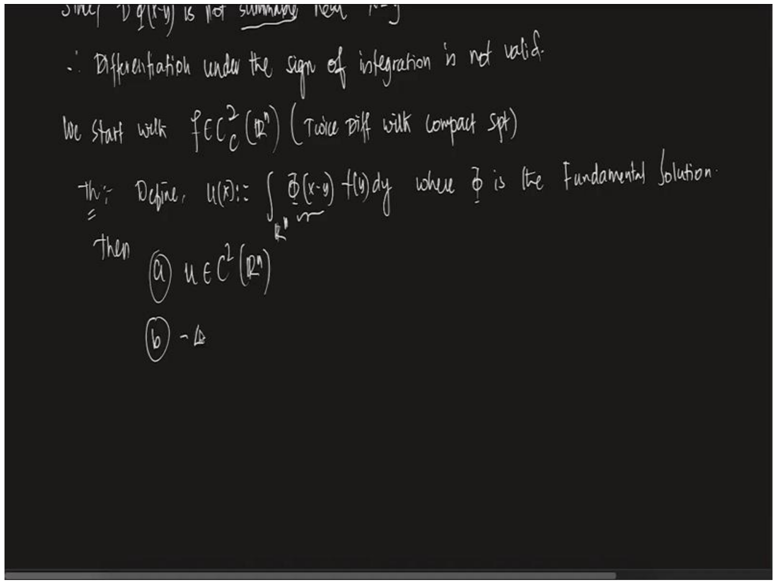
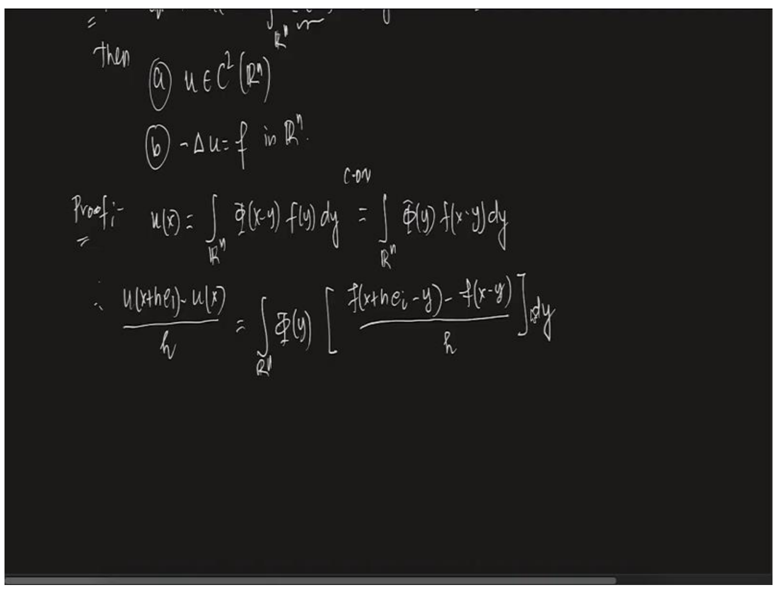




$$\therefore \frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i-y)-f(x-y)}{h} \right] dy \quad (e_i = (0,0,..,1,0..)) \text{ } i^{th}$$

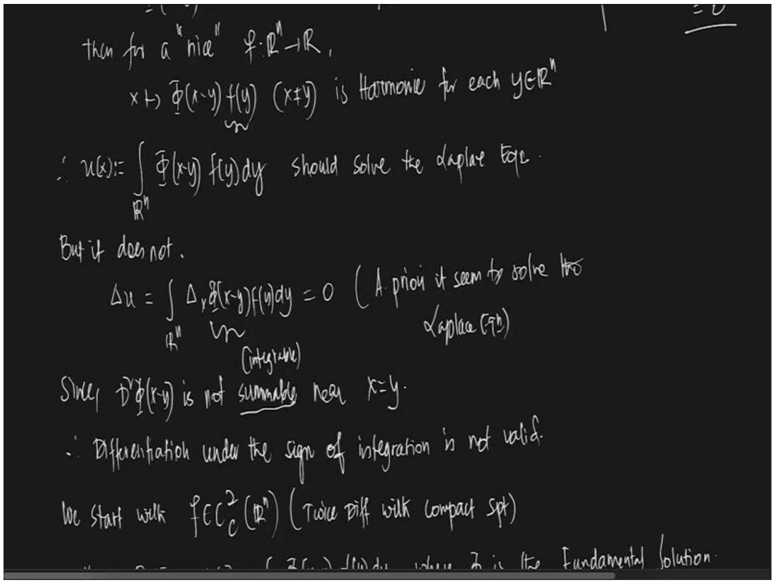
Now, $\dfrac{f(x+he_i-y)-f(x-y)}{h} \xrightarrow{u} f_{x_i}(x-y)$ as $h \to 0$. $(f \in C_c^2(\mathbb{R}^n))$

∇f exists & it continuous, $\nabla f = (f_{x_1}, .., f_{x_n})$

and, $$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y)\, dy \quad (i = 1,2,..,n)$$

Ily, $$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\, dy \quad (i,j = 1,2,..,n)$$

$\therefore u \in C^2(\mathbb{R}^n)$



So, what happens now, so what we are going to do is the see, so, this is a theorem we start with this we start by assumption on f. So, essentially, I said nice f, what is nice? Let us see. So, we start with f which is C infinity or you can say C2 I mean it is not a problem let us take C2 and with the compact support in Rn.

So, twice differentiable with compact support. So, once I start with the f then there is the theorem which I want to solve. So, the theorem says that define u of x, you define it like integral over like this integral over Rn phi of x minus y f of y dy. It define it like this. Where phi is the fundamental solution then number a, u which you get u, it is C2 of Rn.

See if you are defining like this, phi is not twice derivative this has a singularity at x equals to y but still it is saying that if you define it u of x like this, then you have to be C2 of Rn. Moreover, you solve see we were trying to make u to solve this equation Laplacian is equals to x. Actually, what happens is you solve this equation, the Poisson equation in Rn, is this clear?

So, basically, we were thinking that it has it should be solve the Laplace equation, but in reality solve the Poisson equation. And we will see later on that I mean, this actually can be useful to provide a formula for the solution. So, we will look at it later. I mean, we revisit this particular thing later.

So, first of all, what we are going to do is we are going to start with a proof of this thing. So, before I start with the proof of this, let me give you a small it is not a remark, but let me give you suggestion the calculations, which I am going to do. So, this is you should be very familiar with the sort of the estimate. So, we are basically going to do some estimates, please understand this thing in PD research problems.

If you do theoretical PD these are the type of things which you need to do and it should quite this should be bread and butter for you. So, essentially, these are, this is going to be a little dry at first, but I mean, you will get the hang up it if you give enough time. So, first of all, let us say we start with u of fx,

So, again, whatever I am doing, please do it yourself check one's. I mean, without that you are never going to understand I mean, I can just do it here it does not matter, please, I mean, you can just look at the calculation that is not point but do it yourself that is the important thing. So, first of all, ux is given by a phi of f minus y f of y dy over Rn.

Now, I am just doing a change of variable this I do not think I have to show you can do it yourself just change of variable phi of f minus i f of y dy characters getting change to phi of y f of x minus y dy, just change of variables so change of variable. Now, therefore, I want to show u in C2.

So, I first of all, let us say so u of x y (())(11:30) the first derivative of u, partial derivative of u respect to any aspect. So, what is it? It is u of x plus h ei minus u of x by h. So, what is it? This integral over Rn phi of y, f of x plus h ei minus y minus f of x minus y by h dy, I can do it like this. What is the idea behind this change of variable, if I am not doing is change of variable I have to do all of this in phi. I do not want to do that in phi. Why I do not want to do that in phi? Because phi has a singularity at x equals to y, I want to avoid that.

So, I just replaced it with C of y. So, there in singularity is getting replaced at y equals to 0. And after that, this is all happening in with respect to h I am just putting everything on here. And I am working with it because f is C2 we know that f is C2 we are assume if you remember we are assume f is C C 2 C2 with the compact support and what is ei sorry ei is 0 0 this is just a 1, 0. So, ith coordinate this one. So basically, this are the highest unit factor.

Now, you see, f is C2. So, f of x plus h ei minus y minus f of x minus y by h what happens to that? This goes to f xi at the point x minus y, I think you agree with me here, this is uniformly. So, whenever I write uniformly, I will just write it like this because uniformly at h tends to 0, I think we will agree with me here why because f is C2, C2 of Rn. It is means that it is twice differentiable.

So, basically, it is not differentiable twice differentiable defiantly one differentiable if it is one differentiable, it means that gradient of f exists and it is continuous exist and is continuous. And, what is gradient of f? Gradient of f is f x1 f xn. So, this particular every each component is exists and is continuous if f of xi so let us say f of xi exist and is continues. What does that mean? It means that this particular thing converges, this is just the definition of f of x, y.

So, this converges uniformly and hence and thus what happens u of xi at the point x this we will given by integral over Rn phi of y del f del xi let the point x minus y dy this is for I equals to 1 to n, I can write it. Why? Because I am just you see u xi if I am just taking limit x tends to 0 on both the sides of this thing I can take the limit inside x tends to 0 because I mean the integration with respect to y there is nothing into worry about I can take this inside, if I take this inside this particular thing converges to u.

So, u xi is basically, integral over Rn phi of y the left side x minus y dy. And similarly, you can also take another derivative here u xi xi. Now, you think of u xi are some phi and do the exact same thing. So, what will happen, it will be phi of y the fundamental solution Del square f Del xi let us do like (())(15:50) Del xj I mean does not matter. For now, but so this i, j is between 1 to n.

So, this expression, I am taking again with respect to xj and doing exactly the same thing, which I did see f is C2 so the same kind of argument works on the gradient of the partial derivative of f to the respective xi. And that will give d2 f dxi dxj. So, that is, you know, you are u xi xj. So, what does that say? See this expression on the right hand side. This is continuous with respect to y. Yes or No, you do understand what I am (())(16:34) this is a continuous this expression, this is continuous with respect to x, please remember this is continuous see here there is a singularity with respect to y. But I do not care because this is with respect this is continues function with respect to x. And so, what we can say is u is a function which is a function of x. So, essentially, therefore, I can say that u is in C2 of Rn.

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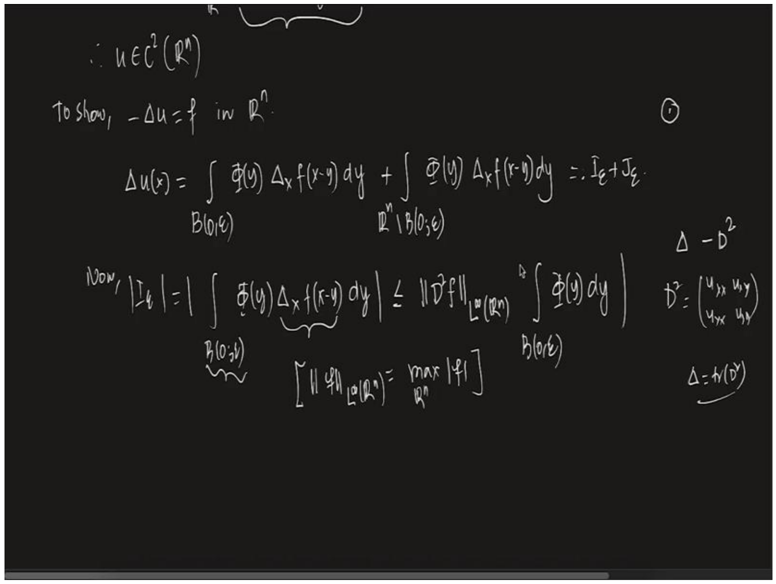
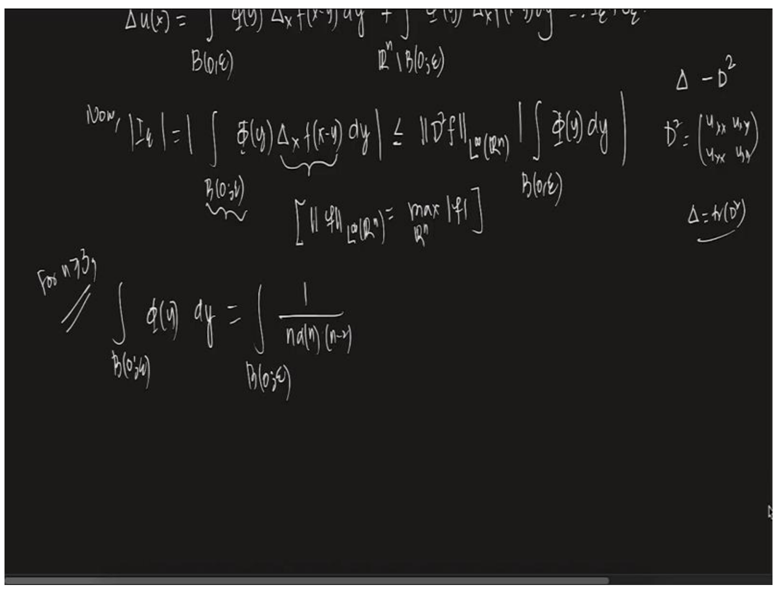




and for $r>0$;

$$v(r) = \begin{cases} a \log r + b & (n=2) \\ \dfrac{a}{r^{n-2}} + b & (n \geq 3). \end{cases}$$ where a, b are constant

"Check".

Definition:- The function $$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log|x| & ; n=2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}} & ; n \geq 3. \end{cases}$$

defined for $x \in \mathbb{R}^n \setminus \{0\}$ is called the Fundamental Solution of the Laplace Equation. ($\alpha(n)$ = Volume of the unit ball in $\mathbb{R}^n$)

"The speciality of Fundamental solution is it helps us to solve Poisson equation."

For $n \geq 3$,

$$\int_{B(0;\epsilon)} \phi(y)\, dy = \int_{B(0;\epsilon)} \frac{1}{n\alpha(n)(n-2)} \frac{1}{|y|^{n-2}}\, dy = C(n) \int_{B(0;\epsilon)} \frac{dy}{|y|^{n-2}}$$

(By Integration of Radial Function)

$$\int_{B(0;\epsilon)} \frac{dy}{|y|^{n-2}} = \int_0^{\epsilon} \int_{\partial B(0;r)} \frac{dr}{r^{n-2}}.$$

$$\int_{B(0;\epsilon)} \Phi(y)\, dy = \int_{B(0;\epsilon)} \frac{1}{n\alpha(n)(n-2)} \frac{1}{|y|^{n-2}}\, dy = C(n) \int_{B(0;\epsilon)} \frac{dy}{|y|^{n-2}}.$$

(by Integration of Radial Function)

$$\int_{B(0;\epsilon)} \frac{dy}{|y|^{n-2}} \overset{UC}{=} \int_0^\epsilon \frac{1}{r^{n-2}} \cdot r^{n-1}\, dr \overset{\text{upto a constant}}{=} \int_0^\epsilon r\, dr = \frac{\epsilon^2}{2}.$$

$\therefore |I_\epsilon| \le \|D^2 f\|_{L^\infty(\mathbb{R}^n)}$

Now, $|I_\epsilon| = \left| \int_{B(0;\epsilon)} \Phi(y) \Delta_x f(x-y)\, dy \right| \le \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \left| \int_{B(0;\epsilon)} \Phi(y)\, dy \right|$

$\left[\|f\|_{L^\infty(\mathbb{R}^n)} = \max_{\mathbb{R}^n} |f| \right]$

$D^2 = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}$

$\Delta = tr(D^2)$

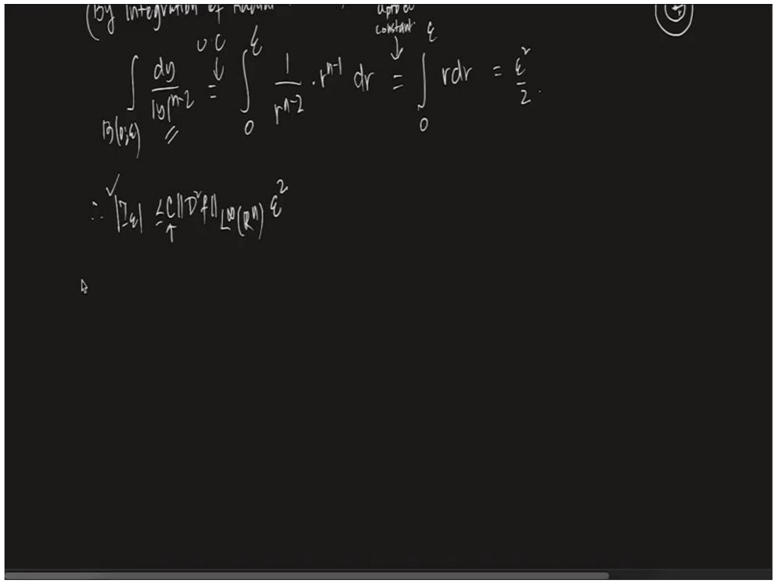
$|\int f| \le \int |f|$

For $n \geq 3$,

$$\int_{B(0;\epsilon)} \Phi(y)\, dy = \int_{B(0;\epsilon)} \frac{1}{n\alpha(n)(n-2)} \frac{1}{|y|^{n-2}}\, dy = C(n) \int_{B(0;\epsilon)} \frac{dy}{|y|^{n-2}}.$$

(by Integration of Radial Function)

$$\int_{B(0;\epsilon)} \frac{dy}{|y|^{n-2}} \overset{UC}{=} \int_0^\epsilon \frac{1}{r^{n-2}} \cdot r^{n-1}\, dr \overset{\text{upto a constant}}{=} \int_0^\epsilon r\, dr = \frac{\epsilon^2}{2}.$$



So now, the second part and this is the difficult part actually to prove. So, please, stay with me. You have to show miners Laplacian u is equal to f in Rn. So, how to show this part 2. So, to show minus Laplacian of u equals to f in Rn, how do we show that? Please remember that phi blow up near y is equal to 0 phi blows up near y is equal to 0.

So, what we are going to do is we are going to break Rn in two parts one part is I mean, we will take a ball of radius centre at original radius epsilon and outside the ball, we will take this integral in two parts. So, from here what do you get Laplacian of u of x is given by integral Rn or just (())(18:07) up B0 epsilon phi of y delta x of f of x minus y dy, why I am writing delta x there are two variables x and y.

So, if it makes sense to write to what is this Laplacian is with respect which variable so here it is with respect to x variable. So, delta x of f x minus y if I am taking the sum it will be sum. So, if you take the sum of this thing over i it was (())(18:41) j from 1 to n then it becomes this side. And again, it is in Rn I am taking up into two parts one is ball with the 0 epsilon and one is Rn minus B 0, epsilon.

And you have phi of y delta x of f of x minus y dy. Now, once I have this thing, I will just write it like this I epsilon plus J epsilon two part this is I epsilon this is J epsilon this is what I am writing. So, now, let us do what I epsilon. So, mod I epsilon this is what this is integral B 0, epsilon phi of y delta x of x minus y dy, this is there see here this particular expression this expression f is in C2. So, delta x delta, what is the relation between D2 and delta. If you are confused here, D2 in two variable just think it is uxx, uxy, uyx, uyy and what is delta? Delta is just the sum of these two.

So, basically, delta is face of D2. So, essentially here also, this delta is just some coefficients of the D2, I your some of some coefficient since f is C2 this particular thing has a bound on the so this a ball so it is bounded set. So, now f is C2 is means the double derivative exists and discontinuous.

So, all the parts are double derivatives means derivatives exist and they are continuous. So, delta x of f is continuous function on these balls, I can take the maximum and I can put it outside. So, let us take that and do I mean, get it outlet. So, this is less than equal D2 f, I am taking the maximum of this thing L infinity of Rn.

The for some of you who know some major (())(21:08), you guys understand what I am trying to write here. But for others, let me give you this. So, let us say f is a continuous function on a bounded set let us say L infinity of Rn, if I am just writing it like this, this is basically the maximum of mod f over Rn, Rn can be change to some omega so L infinity Rn is maximum of f. Whenever I am writing the norm of L infinity norm, it means I am just taking the supremum maximum norm.

Here the maximum exist (())(21;39) f is in Rn, it does not matter f is in Rn but the maximum at this because f is in phi phi there is a compact support that is why the maximum is attain hence I can write otherwise, we just write it as a supremum, just think of it like this. So, this is I take the maximum of B2 f and put it outside. Now, I am only left out with phi of y dy over B 0, epsilon mod of this thing. Now, phi let us do this part, B 0, epsilon phi of y dy. So, this is B 0, epsilon. Let us, do for n greater than 3 so for n greater than equal 3 n equals to 2 I want you guys to set yourself.

So, this is what it is 1 by n alpha n. n minus 2 is there I forgot to write it n minus 2. If you remember please check what is phi of y let us just check this part where is phi of y 1 by n n minus 2 alpha n 1 by mod x whole power n minus 2. If you remember this alpha n I did not write this alpha n is the volume of the unit ball in Rn.

So, for all of you guys who understand measure theory, this is just a measure of the unit ball. So, n minus 2 1 by mod x to the power mod y to the power n minus 2 dy this should be (())(23:30).

So, let us you see this is basically a constant dependent on n throw it outside. So, this is some Cn I will write it like sub Cn I am not really interested in that thing B 0, epsilon dy by mod y to the power n minus 2.

Now, you remember your integration on radial function. So, by integration of radial function formula. What do we get? We get that this particular integral dy but mod y whole power n minus 2 over B 0, epsilon can written like this 0 to epsilon del B 0, r dr by r power n minus 2 sorry I should write it like this 1 by r power n minus 2 and d xi let us say.

Now, it should dr, so essentially, what I am doing is to see, first of all, I am integrating this thing on the unit ball, integrating this thing on a unit ball on the unit ball, this is a radial function. So, basically this is constant. So, essentially, what am I going to get let me write it properly. So essentially, what I am going to get from here, 1 by r to the n minus 2 and then the volume of the del be 0 epsilon, that is r to the power n minus 1 dr from their r to the power n minus 1 dr of course have constant is there. But I am not writing all that (())(25:43).

So, if I calculate this thing, it is basically 0 to epsilon yes is this is up to a constant this mark equals to up to a constant. So, 0 to epsilon 1 by r power n minus 2 to the power r minus n minus 1 dr this is just the integration with respect radial function. See what am I doing is here is this I want to integrate it over this particular ball. So, I integrate it over this I mean, just the boundary of the ball on the boundary this function.

So, let us say this boundary is of radius r this is a boundary of a ball with radius r. So, if that happens on that boundary, this particular thing is 1 by r power n minus 2. And after that, what happens is you are just integrating it between to epsilon you are just taking r to r r from 0 to epsilon.

So, this is why it is coming. So, 1 by r power n minus 2, r to the power n minus 1 this is the volume of the ball I mean, this particular sorry, the surface area of this ball inside. So, that is why r power n minus 1 now it gets cancelled out and these becomes r dr. So, this is r square it becomes r dr. So, this is up to a constant, please remember this is up to a constant I am not writing all that up to a constant.

Here also same thing up to a constant. So, this is essentially epsilon square by 2. So, see what is happening here is this, if I take the bound on this thing, if I write it so, I epsilon mod I epsilon therefore mod I epsilon this is less than equal D2 f the L infinity norm of this the maximum of D2 f L infinity of Rn again why this happens, because f is have a compact support.

So, we can write it like this and you see the mod of the integral and mod of the integral is less than or equal to the integral of mod. So, mod of the integral is less than equal to the integral of mod, mod of integral f is less than equal to integral mod x. So, if you write it like this, then essentially that is, what is mod of (())(28:17) this is epsilon by 2 up to a constant.

So, essentially, there is a constant which depends on n times epsilon square by 2 again it is getting absorbed by the constant see here, we are not really interested in these constants all we are interested in is getting some estimate the constants are always there. Why? Because you see now if you take epsilon toward 0 essentially this is I epsilon is 0, this constant does not contribute anything I mean, this half and there are other constants like this, this constants all of these are getting absorbed in this thing C.

So, this C is there and this particular expression D2 f the infinity norm times x square epsilon square, now this is fixed the maximum f ix fixed. So, essentially, I epsilon is bounded by epsilon square. Now, epsilon is small if you take epsilon small enough, what is going to happen if is you can actually say that the mod of I epsilon this is small so this goes to 0.

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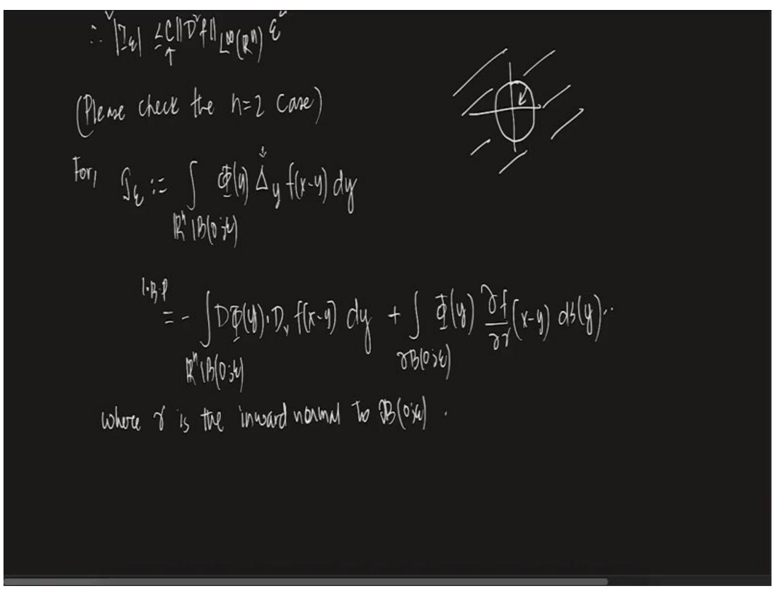


$$\mathbb{R}^n \setminus B(0;\varepsilon)$$

$$\overset{\text{I.B.P}}{=} -\int_{\mathbb{R}^n \setminus B(0;\varepsilon)} D\Phi(y)\cdot D_y f(x-y)\,dy + \int_{\partial B(0;\varepsilon)} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)\,dS(y) := K_\varepsilon + L_\varepsilon$$

where ν is the inward normal to $\partial B(0;\varepsilon)$.

$$\therefore |L_\varepsilon| = \left| \int_{\partial B(0;\varepsilon)} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)\,dS(y) \right| \le \int_{\partial B(0;\varepsilon)} |\Phi(y)| \left|\frac{\partial f}{\partial \nu}(x-y)\right| dS(y)$$

$$\le \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0;\varepsilon)} |\Phi(y)|\,dS(y)$$

$B(0;\varepsilon)$ $B(0;\varepsilon)$ $B(0;\varepsilon)$

(By Integration of Radial Function)

$$\int_{B(0;\varepsilon)} \frac{dy}{|y|^{n-2}} \overset{C}{=} C\int_0^\varepsilon \frac{1}{r^{n-2}}\cdot r^{n-1}\,dr \overset{\text{upto a constant}}{=} C\int_0^\varepsilon r\,dr = \frac{\varepsilon^2}{2}.$$

$$I_\varepsilon \le \begin{cases} C\varepsilon^2 & \text{if } n \ge 3 \\ C\varepsilon^2 |\ln\varepsilon| & \text{if } n=2 \end{cases}$$

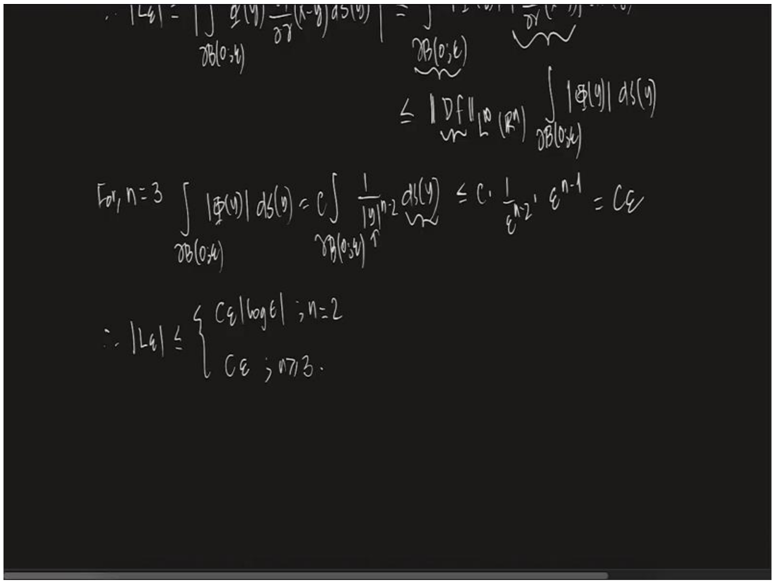
$\uparrow$ C.

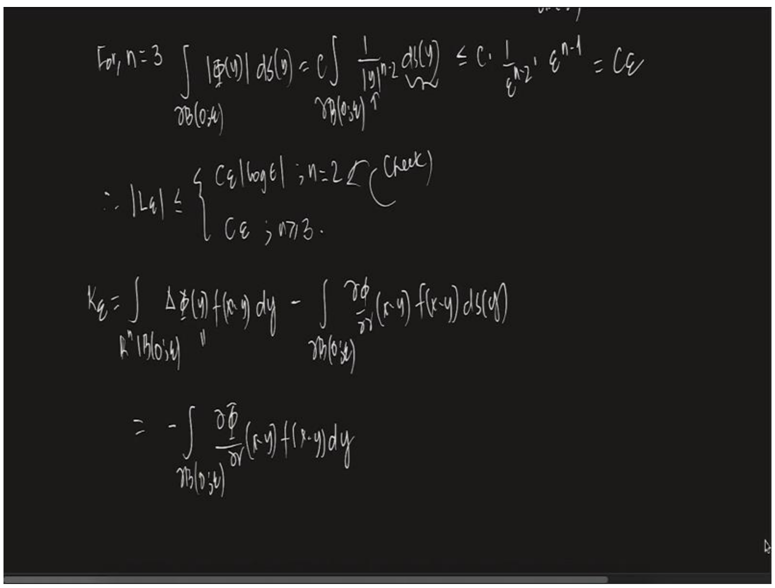
$$\therefore |I_\varepsilon| \le C\|D^2 f\|_{L^\infty(\mathbb{R}^n)}\,\varepsilon^2$$

(Please check the $n=2$ case)

For, $J_\varepsilon := \int_{\mathbb{R}^n \setminus B(0;\varepsilon)} \Phi(y)\,\Delta_y f(x-y)\,dy$

I.B.P





Now, please check the n equals to 2 case. Now, for j epsilon now let us j epsilon part this is given by integral over Rn minus B 0, epsilon phi of y delta y of f of x minus y dy, is that f. So, now, this is equals to minus integral Rn minus B 0, epsilon I am doing integration by parts. What is it? It is phi of y sorry it is D phi of y D y with respect to f of x minus y dy plus on the boundary del B 0, epsilon phi of y del f del comma of x minus y ds y.

So, let us explain what we why we write it like this see integration by part as we know that when we do integration by parts one derivative comes to this part with a negative sign so negative sign one derivative comes to this part. So, this loses one derivative this gains one derivative so, that happens plus on the boundary see what is the boundary of this domain this domain is essentially this will B 0, epsilon let us say the domain is essentially this thing, what is the boundary? The boundary is the shared common boundary with the ball.

So, that is why this boundary is, what is gamma? Where gamma is the generally we write unit outwards normal to the domain the domain is here outward normally this side. So, basically this is an inward normal to the ball you understand what I am saying see generally this gamma is outward normal to the domain this the domain is outside inside out actually.

So, the outward normal is basically inside the ball. So, gamma is that inward normal to the ball B 0 epsilon this is a invert normal. I hope this is clear the normal to the boundary of the ball B 0, epsilon inward normal. Now, let us I mean let us write this particular thing as again this is a K epsilon and the next one is L epsilon this is K epsilon this is L epsilon .

Now, therefore, mod L epsilon, what is the this let us check, mod L epsilon I am doing this part this is mod of del B 0, epsilon phi of y del f del gamma of x minus y ds y this is less than equals to integral del B 0, epsilon mod phi of y and the derivative sorry the norm of del f by del gamma at a point x minus y ds y.

Now, you see all what happens to this thing f is C2 with a compact support. So, definitely the one the first derivative is also continuous and this is bounded on this ball. So, essentially this is a compact set I can take the maximum of these outside. So, essentially this is I can just write it as D f L infinity of if you want you can just write it like Rn only no problem. The maximum f Del f on Rn I can just say.

See, why this work because you see if you remember f has a compact support f is twice continuous differentiable. So, it is one continuous differentiable and the first derivative is also continuous it is one differentiable and the first derivative is also continuous because the second derivative is continuous. So, the first derivative have to exist and it is continuous.

So, if the first derivative is continuous this is a compact set there is a maximum which is attain that maxima is of course always less than equal to the maxima of f over whole Rn you are taking the see the maxima over a any unit ball like this the boundary of the unit ball is always dominated by the maximum over a whole Rn of courses you see because this is a very big set.

So, I am just doing like that and then I have integral over Del B 0, epsilon phi of y mod phi of y ds y. So, if I write it like this what happens here? So, you see phi of y is radial function on this ball or boundary of the ball, what did I say radial function on the boundary, it becomes constant. So, it is essentially, what is phi of y so for n equals to again the n equals two case, you guys have to do it yourself del B 0, epsilon mod phi of y ds y.

Let us do that what happens to this. This essentially the Del B 0, epsilon phi of y mod phi of y is some constant is there I am not writing all that constant. But here essentially it is 1 by mod y to the power n minus 2 ds y. See on this ball on Del B 0, epsilon this is particularly this is essentially a constant.

So, this is phi times 1 by epsilon power n minus 2 and then you have integral over Del B 0, epsilon ds y, which is what it is essentially the integral over the boundary of the ball, the surface with respect to the surface on the surface. So, that will give you the surface area of the unit ball. So, there is some constants again, which is getting accommodated here and that will give you epsilon power n minus 1, is this clear?

See surface area of a unit ball is some constant type epsilon power not unit ball a ball with centre epsilon is epsilon power n minus 1. So, that is what it is getting out there is a less than equal sign sorry I should write it like this and I should have actually, this is also less than equal to there are some constants here you should write it like this sorry here there is no here I just calculated this particular thing. So, there is a constant of course, this is a constant here. So, all of that is this equality is up to a constant, I you write it like that or you can just write a constant. So, this happens this is C phi (())(37:26) epsilon. So, essentially, what is happening here is you can bound mod L epsilon with C epsilon.

So, therefore, I mean please check this part mod L epsilon can be bounded by is less than equals to C epsilon mod log epsilon this is for n equals to 2. And see epsilon for n greater than equal 3, please check this is part. And similarly, in the early part also I forgot to write it here I epsilon is less than equals to of course a constant times epsilon square, if n is less than equal 3, this we already saw this is also constant, I mean, there is nothing to write.

And it is constant type epsilon square log epsilon if n equal to 2 so this part you have to check. So, here also this part you have to check. Once that is there, so L epsilon is bounded. I mean, this is C epsilon as epsilon goes to 0 I mean, this is very small. So, I do not have to worry about all that.

Now, I have to worry about k epsilon. So, what is k epsilon k epsilon integral Rn minus B0 epsilon delta phi of y f of x minus y dy minus integral del B 0, epsilon del phi del gamma of x minus y f of x minus y ds of y. So, this is actually equals to minus Del B 0, epsilon Del phi by Del gamma of x minus y f of minus y dy.

See why this is happening because here see this is the fundamental solution this solves Laplacian of phi solve the Laplace equation, Laplacian of phi is 0, delta phi is 0 for all x, which is for x not equal to 0. So, essentially here, why cannot (())(39:52) 0 in there . So, this is outside the ball. So, Laplacian of phi 0 and in that case, the whole thing is 0. So, in K epsilon I only have this particular expression to deal with.

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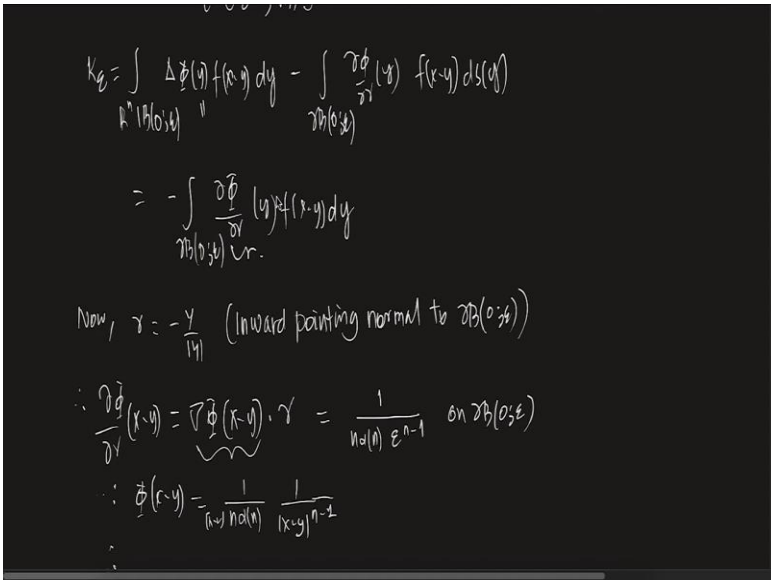


Now, $\nu = -\frac{y}{|y|}$ (inward pointing normal to $\partial B(0;\varepsilon)$)

$$\therefore \frac{\partial \Phi}{\partial \nu}(y) = \nabla\Phi(y) \cdot \nu = \frac{1}{n\alpha(n)\,\varepsilon^{n-1}} \quad \text{on } \partial B(0;\varepsilon)$$

$$\because \Phi(y) = \frac{1}{(n-2)\,n\alpha(n)} \frac{1}{|y|^{n-2}}$$

$$\therefore D\Phi(y) = -\frac{1}{n\alpha(n)} \frac{y}{|y|^{n}} \quad (y \neq 0)$$

$$\therefore K_\varepsilon = -\frac{1}{n\alpha(n)\,\varepsilon^{n-1}} \int_{\partial B(0;\varepsilon)} f(x-y)\, dS(y)$$

$\mathbb{R}^n \setminus B(0;\varepsilon)$

$$\overset{\text{I.B.P}}{=} -\int_{\mathbb{R}^n \setminus B(0;\varepsilon)} D\Phi(y) \cdot D_y f(x-y)\, dy + \int_{\partial B(0;\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y) := K_\varepsilon + L_\varepsilon$$

where ν is the inward normal to $\partial B(0;\varepsilon)$.

$$\therefore |L_\varepsilon| = \left| \int_{\partial B(0;\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y) \right| \le \int_{\partial B(0;\varepsilon)} |\Phi(y)| \left| \frac{\partial f}{\partial \nu}(x-y) \right| dS(y)$$

$$\le \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0;\varepsilon)} |\Phi(y)|\, dS(y)$$

For, $n \ge 3$ $\displaystyle\int_{\partial B(0;\varepsilon)} |\Phi(y)|\, dS(y) = C \int_{\partial B(0;\varepsilon)} \frac{1}{|y|^{n-2}}\, dS(y) \le C \cdot \frac{1}{\varepsilon^{n-2}} \cdot \varepsilon^{n-1} = C\varepsilon$

$$\therefore \Phi(y) = \frac{1}{(n-2)\, n\alpha(n)} \frac{1}{|y|^{n-2}}$$

$$\left[\fint_{\partial B(x,\epsilon)} f(y)\, dS(y) \to f(x) \text{ as } \epsilon \to 0, \quad f \in \text{Cont.} \right]$$

$$\therefore \frac{\partial \phi}{\partial \nu} = D\Phi(y) = -\frac{1}{n\alpha(n)} \frac{y}{|y|^n} \cdot \quad (y \neq 0)$$

$$\therefore K_\epsilon = -\frac{1}{n\alpha(n)\, \epsilon^{n-1}} \int_{\partial B(0;\epsilon)} f(x-y)\, dS(y)$$

$$= -\fint_{\partial B(x;\epsilon)} f(y)\, dS(y) \to -f(x) \text{ as } \epsilon \to 0.$$

Hence u solves, $-\Delta u = f$ in Ω.

If $v' \neq 0$; $\left[\log(v')\right]' = \frac{v''}{v'} = \frac{1-n}{r}$. (Check)

and for $r > 0$;

$$v(r) = \begin{cases} a \log r + b & (n = 2) \\ \frac{a}{r^{n-2}} + b & (n \geq 3). \end{cases}$$

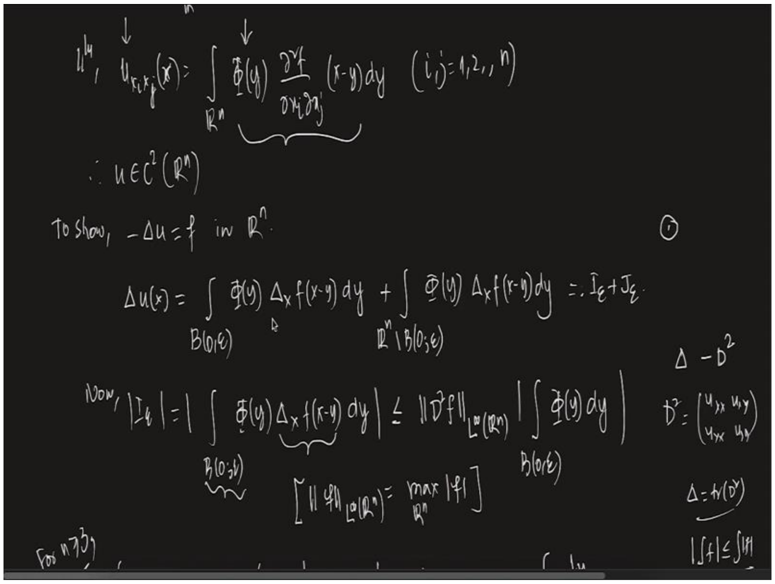
where a, b are constant

"Check".

Definition: The function

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & ; n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & ; n \geq 3. \end{cases}$$

defined for $x \in \mathbb{R}^n \setminus \{0\}$ is called the Fundamental Solution of the Laplace Equation. ($\alpha(n)$ = Volume of the unit ball in $\mathbb{R}^n$)



Now, let us deal with this expression. So, now what is gamma? Gamma is here it is minus y by mod y I hope all of you guys know this thing. So, see why this minus is there because it is inward pointing normal to Del B 0, epsilon that is why the minus sign is there and what is y by mod y see this is r by mod r essentially. So, you do realise I mean the unit normal (())(40:49) x by mod x sorry it is not r by mod r it is x by mod x. So, that is always there, but since it is inverse there is a minus sign.

So, gamma of y is like gamma y and mod y and therefore Del phi by Del gamma of x minus y what it is it. This is gradient of phi at the point x minus y dot gamma. What is gradient of phi at the point x minus y. So, these you guys have to calculate yourself? It is let me write it down. It is 1 by n alpha n epsilon power n minus 1 this is a Del B 0, epsilon. See phi of x minus y please remember what is phi of x minus y, phi of x minus y is 1 by n alpha n r power n minus sorry there is n minus 2 there in r power n minus 1. It is not r it is mod x minus y present it properly 1 by x minus y whole power n minus 2 sorry n minus 1, n minus 2 sorry n mins 2.

You see n minus 2 (())(42:25) mod x to the power n minus 2 so this is there now see x and y, I am taking y on the boundary y you see in this integral y is on the boundary of the ball and x is I mean x is not equal to y in (())(42:52) x is not equal to y. Because since phi f minus y look like this there for if you take the derivative of this thing d phi of y.

So, maybe I can just write it like this sorry I made a mistake here this fine. And then here it is phi of y this is the mistake I did sorry sorry this is a mistake I which I did so here and this also here also, sorry you want see this is delta phi of y so this phi of y which is why I could not understand. So, this (())(43:33). So, in this case you see this let me write it properly so let me this this is at the point y this is y dot gamma so this is y phi of y is equal to this kind for mod y to the power minus 2 phi of y, in phi of y is this you do realise that I can just take the derivative of this thing.

So, D phi of y, why I can do this thing see y is on the boundary of this ball. So, definitely I can take the derivative here because y is not equal to 0. It is on the boundary can never be 0. So, I can take this thing and this is minus 1 by n alpha n, y by mod y whole power n so please check this part, is this y not equal to 0. See y is on the boundary so y cannot be 0 and hence I can take the derivative here because except at 0 everywhere it is every well define so I can just take a derivative and it is become like this.

So, once that is there then what is K epsilon? K epsilon can be written as minus 1 by n alpha n epsilon power n minus 1 this one integral Del B 0, epsilon f of x minus y ds y. I can write it like this see this I just calculated and after that Del phi Del gamma at the point y on the ball is given by this thing, it is given by this things I just calculated I mean Del phi Del gamma, which is this why this is a case because gradient of phi, which is this this is gradient of phi gradient of phi is this and after that gamma is minus y by mod y if I put it together on the ball it is becoming 1 by n alpha and epsilon power n minus 1.

So, k epsilon if you write it down it becomes this and this is equal to the average of Del B x epsilon f of ds y. What am I doing here I just change the variable so I just shifted the origin from 0 to x so here the centre is at and that is why this f become f of y you just change of variable n. So, and what the I mean you see if we do the change of variable the volume of the unit ball does not changes so essentially this become this this remains the same and I am just writing it like an average.

So, it becomes minus average of this now this goes to where if you remember minus f of x at epsilon tends to 0 please remember I am using mean value theorem here but this is actually true I mean this particular thing as epsilon tends to 0 is always true given a fixed (())(46:50). So, I mean you can find you can check that the average of over omega let us say f of y, ds y this converges to f of let us say x in omega x is in omega which is centre at omega basically a ball omega is a ball and x is a centre of the ball so it becomes f of x as epsilon tends to 0.

So, this is omega is B epsilon let us say you can show this thing in the average f over B x epsilon so Del sorry let me write it properly del B x epsilon with the average of f of y ds y goes to f of x this is again in a assignment please check so this holds for any f in it continue. Continuous function you can do this thing one thing do this then K epsilon goes minus fx.

Now, if you can put all of this together. So, what is happening here? Let us check, let us go back to this a long way. This particular expression let this expression I epsilon on the ball. The whole thing is going to 0 we have checked that on the ball you see I epsilon is going to as 0. (())(48:12) this goes to 0 and this particular thing on Rn minus B 0, epsilon the first expression you see the first expression is L epsilon if you remember (())(48:28) it is a big calculation one second.

So, this K epsilon plus L epsilon L epsilon is n greater than equal to 3 this is like C epsilon as epsilon goes to 0 this particular expression L epsilon is very small. And the same holds for n equals to 2 also and what about K epsilon this K epsilon this is this goes to minus f of x so hence u solves minus Laplacian of u equal to f in omega and you hence you the proof.

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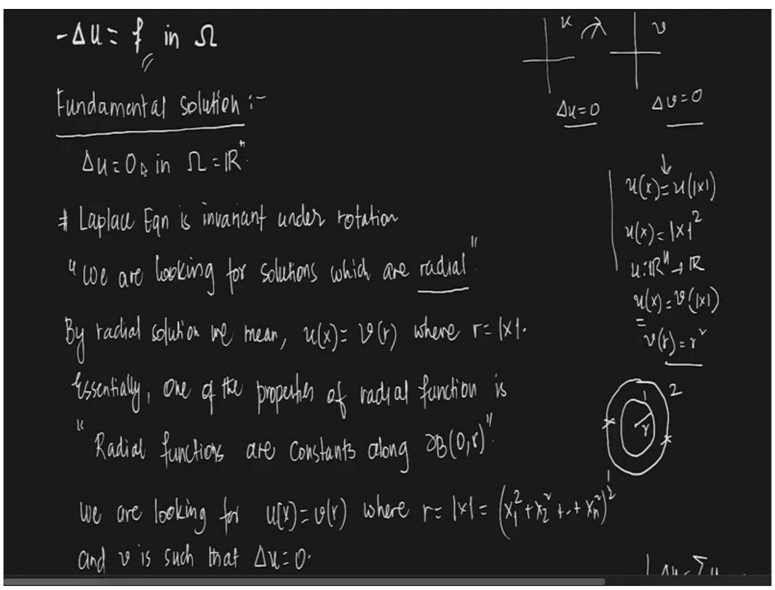

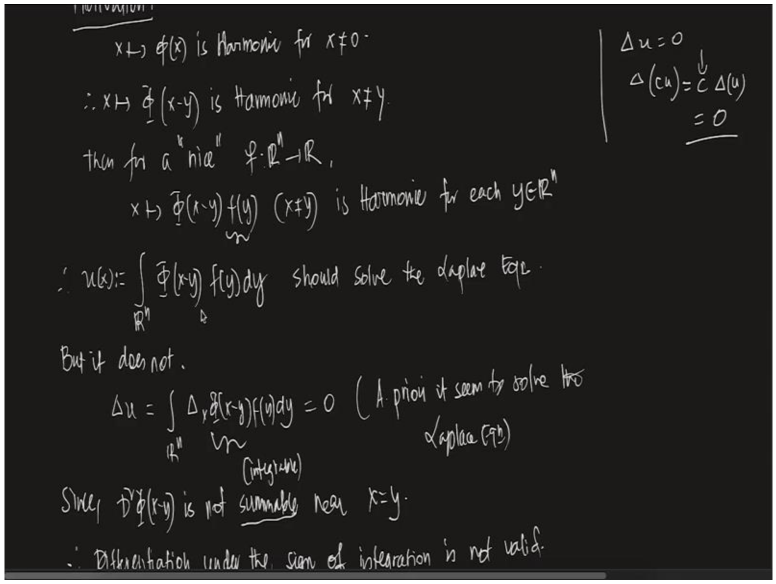

So, essentially what we did so let me recall quick recall. We wanted to solve the Poisson equation for that we found out fundamental solution which is there for solution of harmonic function in Rn minus 0 use that thing and we did something call convolution this called the convolution which we did the convolution. Define the ux and we show that this (())(49:28) actually solve the Poisson equation even f is nice function f is phi phi f is a twice differentiable function with compact support and we have should that. So, with this we are going to end today lecture.