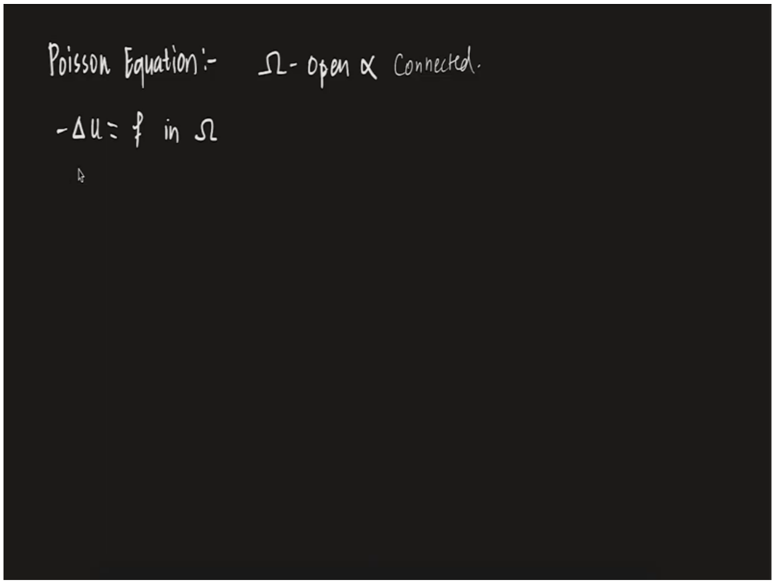
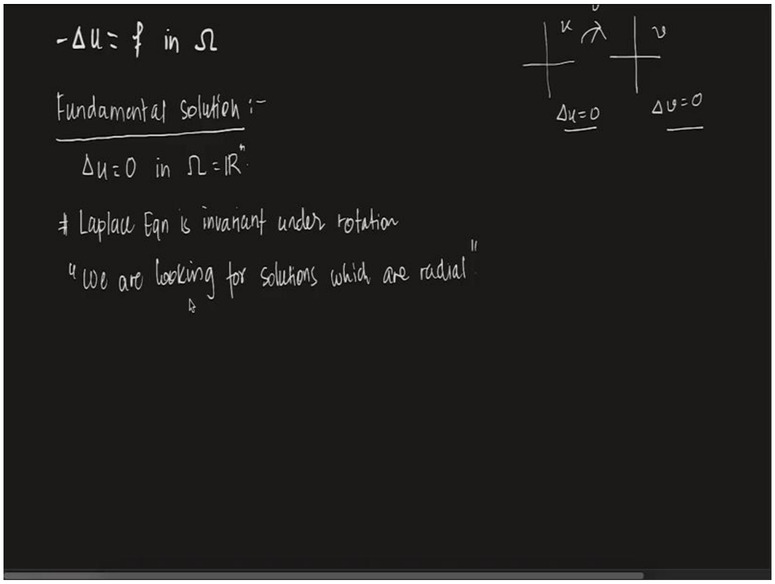
Advanced Partial Differential Equations
Professor Doctor Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Lecture 13
Fundamental Solution

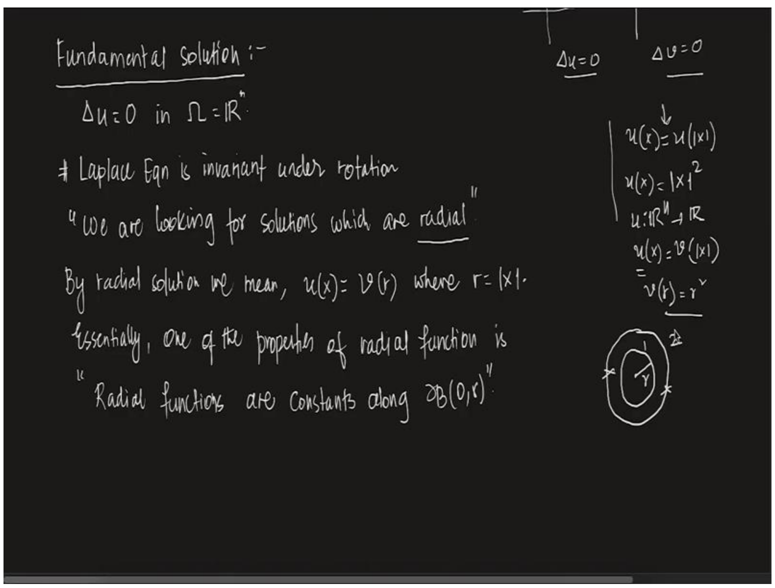
(Refer Slide Time: 00:22)

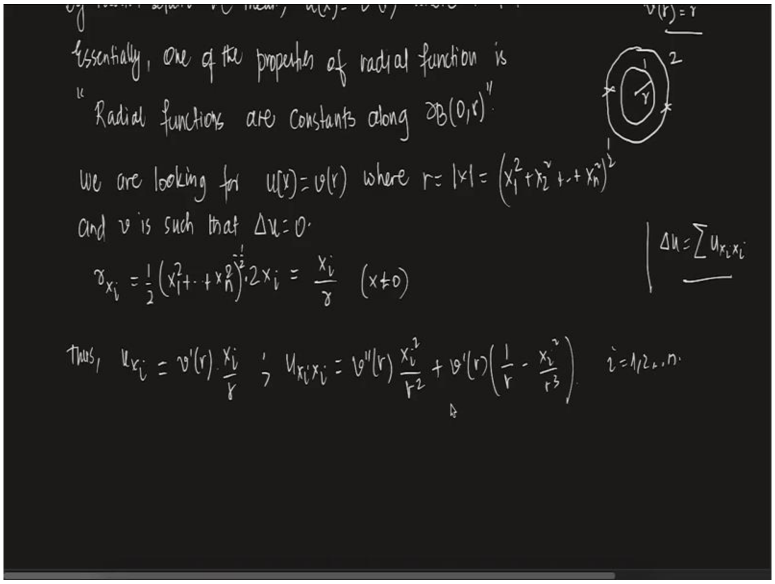


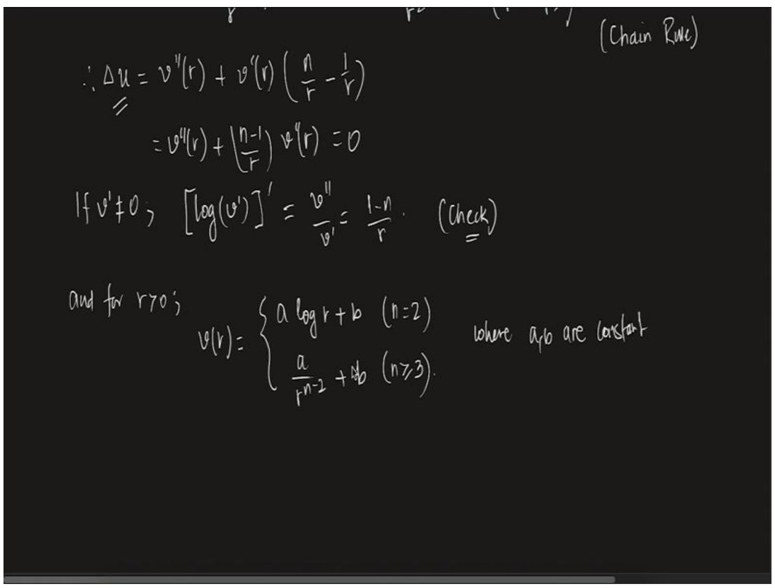
Welcome to you into this class, we are going to solve the Poisson equation. So, essentially, if you remember Poisson equation. So, let us say that you have this equation Laplacian of u equals to f in omega and what is omega of course, omega is assumed to be open and connected. And minus Laplacian of u equals to f in omega, this is the problem, which is called the Poisson equation. Now, how do we solve this problem? To do this, we going to start with something called a fundamental solution. We are going to solve the homogeneous problem let us see if we can do that.

(Refer Slide Time: 01:01)









So, essentially, we are going to find something called a fundamental solution. Now, so, what is a fundamental solution, we will explain all of that later. But for now, what we are going to do is we just I mean, without thinking much, we want to solve this equation Laplacian of u equals to 0. We want to solve this equation and solving this equation and you can think of it right now, as I mean, we are solving it in omega, which is in Rn.

Let us, just do that. Once we do that, then we will from there we will go on about the solving the Poisson equation. So, how to solve this equation. See in one of the problems in that (())(01:52) if you can I mean figure it out, there is a problem where you are asked to show that the Laplace equation is invariant and translation.

So essentially, you see Laplace equation is invariant under sorry rotation (())(02:14) translation, but it is invariant under rotation, what does that mean? It means that the let us say I mean, if you have a coordinate axis and if you rotate it I mean you through some angle whatever it is let us say some angle theta and when new coordinate axis.

So, let us say u is define here in the new coordinate axis that will be v. So, this is a new coordinate axis after rotation figure, it should not look like this I mean, you do realise there is a change in theta angle, but the point is this. So, for that change, you may get change to v. But in this coordinate let us say Laplacian of u equals to 0 then in the new coordinate also Laplacian of so v is getting change to v so Laplacian v we will also be 0.

So, basically if u is harmonic v is also harmonic under a rotation. So, since this is I mean invariant Laplace equation is invariant under rotation what we can think of is, we can look for this problem as you know that there are many solutions for this problem. For example, if you take the coordinate axis so, u of x1, x2, xn let us say you take any xi any coordinate function x then Laplacian u is equal to 0 any linear function.

So, let us say x1 plus x2 plus x10 if you are taking that will also solve this problem, any constant will solve this problem, but we want something more fundamental more non-trivial let us say. So, how do you get it? So, basically we are looking for let us put in (())(03:46) we are looking for solutions, which are radial. Why we are suddenly looking for radial solutions?

Because we know that Laplace equation is invariant under rotation. So, if you rotate the this u the resulting function which you are going to get out of the new coordinates, that is also solve the Laplace equation, and hence, I mean, you can think of the solutions the problems should admit radial solutions. Radial solutions if you I mean, what are radius solution? I am sure you guys know, but still let me just write it down.

So, by radial solution, we mean u of x must look like some v of r, where r equals to mod x. So, basically, see radial solutions are those solutions so essentially, u should look like so basically u of x should look like u of mod x you understand. So, (())(05:02) for example ley say u x equals to mod x if I am taking mod x square let us just take that here.

So, that is the simple problem. So, u from where to where just a this is from Rn from that your u. So, do you think this is a radial function of course, it is a radial function because u of x, you can write it like u of x and u of mod x are essentially same, I mean, so, it is u of (())(05:27) it will be some function v of mod x.

You should just have use of notation, it should not write it like. So, it will basically u of x is v of mod x some other function v of mod x. What is v here so where v of r let us say, r as mod x I am writing v of r is r square. So, basically, what it says is this function. So, essentially, what radial functions one of the properties of radial function is the following. So, I am not writing the following. So, radial function are these are constant along they will be let us say 0, r. This is the one so basically, if you take if you think of I mean radial function in rn.

You are taking radial function of rn you take the point origin and if you take origin and think of ball disk let us say in two dimension, you think of the disk, it takes a boundary of the disk. So, basically, mod y x minus y is r so this distance is r. If it is radial, then this function, this is basically the function has to take a constant value over this ball.

Again, if you are taking another ball, it will take another value, but that will be constant. So, basically in all this ball, let us say the value is 1, I am just using one example on this ball, this 1 again this ball, this can be 2 but the thing is over the ball you take any point on the ball it is always going to be same. But when we change the ball it of course changes but do understand what I am trying to say. So, on the boundary, so they are going to be constant.

So, let us find some solutions here some functions. So, essentially, what we are going to do we are looking for u of x which looks like v of r and what is r where, as I have already mentioned r equals to mod of x. So, what is it? So, this is x1 square plus x2 square plus xn square whole power r. It is just a norm of (())(08:17). And of course, we choice v such that Laplacian of v Laplacian (())(08:26) and v is such that Laplacian of u is 0. So, essentially, we are assuming that u is a solution of the Laplace equation, and since ux is also vr we choice v as a this will happen.

So, if we do that, then, so let us write this equation, see this equation is in terms of u let us write it in terms of v and see what happens. So, if we do that, first of all a little change of variable r with respect to xi, if you differentiate r with respect to xi that will be half x1 square xn square 2 xi. So, that will give you xi by all of these sorry this is minus half, that is a half is there so half minus 1 (())(09:23) minus half that to xi.

So, this is 1 by x1 square plus x2 square plus xn square whole power half which is r, of course, we are assuming that x non-zero where assuming that. So, once we have the thing, where thus u xi see here I have to find u xi xi then they have sum it up. What is Laplacian? If you remember Laplacian of u is equal to summation u xi xi.

So, first of all, I find u xi and then again differentiate it and (())(09:58). So uxi let us calculate that first, what is it? This is chain rule it is v prime of r r xi which is xi by r. So, that is there. Now, and u xi xi is equal to v double prime of r xi square by r square plus v prime of r 1 by r minus xi square by r cube. So, how am I getting it is very easy. You take this particular thing you differentiate it. So, does just differentiate this one.

So, first of all this is let me (())(10:52). So, basically I differentiate this thing leaving this thing aside. So, when I differentiate this, it is v double prime of r and then rxi and then xi by r remain unchanged xi by r is there v double prime of r and r xi r xi is a xi by r, so that is a v double prime of r, xi square by r square.

And again, I leave v prime of r here and after that differentiate this thing with respect to xi. So, I mean, you do realise that first of all, it will be 1 by which is coming 1 by v prime is common 1 by r differentiation xi with respect to xi is 1. So, this why 1 by r minus again, you keep this thing and differentiate 1 by r if you differentiate 1 by r, which is minus 1 by r square and after that r xi r xi xi by r so this is minus xi square by r cube so that is (())(11:44). So, this holds for i equals to 1 to n, this is for any i equals 1 to n, I hope this is very clear here, this is just chain rule nothing so by chain rule.

Now, therefore Laplacian of u is what Laplacian of u is equals to V, this is this, you take the sum, that is Laplacian of u, so you take that sum from i equals 1 to n, when you take this what is happening, this is independent of r, i this is independent of y you getting summation here. So, that is r square by r square, that is becomes v double prime of r, plus v prime of r let us see what happens there.

If you take the sum 1 by r if you take n sum it is n by r minus again if you take this thing, this summation is xi square r is r square by r cube, which is 1 by r. So, what happens here, if you just write it properly it is v double prime of r plus n minus 1 by r v prime of r and this is equal to 0.

Why? Because u is harmonic see if you u is harmonic Laplacian is equal to 0 Laplacian u is this so essentially, this is this particular expression is 0.

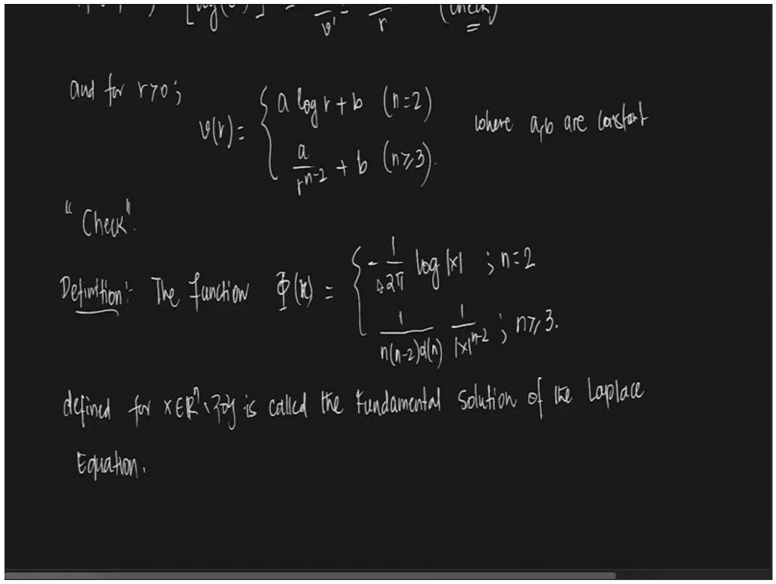
Now, if I want to say let us say, I want to find the explicit function, if I asked to find explicit function, I just have to solve this particular problem. And do you really know how to solve it? Of course, we know because this is just one variable function, this is so beautiful about this change into radial function.

So, here, you see this just a function of one variable r. And now I know this is just an OD, which I know how to solve. So, let us say if, if v prime is not 0 then you can just solve this problem and you can get log of I hope all of you guys can do I mean, solve this equation it is very, it is not very difficult to do.

So, the derivative of this thing, this is equal to I am just writing it down. I am not wasting my time on how to solve the thing absolutely certain all of you know this I am just solving this problem. So, once I do this, I get this thing. So, in terms of v prime I get this and for r positive, so please solve this problem.

And you can get so this is you guys have to check it yourself. I am not doing this this is very easy check it for yourself. For r (())(14:17), what do you get? You get v of r what is v of r it is essentially this is in two parts. So, a log r plus b, this is for n equals to 2 and you also get a by r power n minus 2 plus b, this is for n greater and equal 3. So, for n equals t get a different expression for n greater than or equal 3, you get this expression, where a and b are constants. This is just your OD usual OD.

(Refer Slide Time: 14:57)



So, again, emphasise, please check I want you to take this and get the (())(15:03) done. Now, so for all physical purposes, so we will define a new thing from here. So, this is our definition. So, this a function which you got this is a very special function this function the function phi of r it is always called phi of r. So, sorry it is called phi of x.

So, it is see this phi this is like a universal symbol whenever I write capital Phi of r x like this it is all a fundamental solution in this kind of context not in every context I write it like this 1 by minus 2 pi log of mod x n equals to 2 and it is 1 by n minus 2 alpha n 1 by mod x to the power n minus 2 n greater than equal 3. So, this is defined for x in rn minus 0 and is called the fundamental solution of the Laplace equation.

So, this particular function which we get see this function, so, let us take some time and understand what is going on here. So, I have got a u of x, which looks like this and what I am going to do is I am going to define a new function Phi capital Phi. So, this capital Phi will look like so, I am taking b to be 0 and a to be minus 1 by 2 pi there is our reasons for doing this thing.

So, for now let us just write it down. And we will later you will see why we do all of that. So, minus 1 by 2 pi log of mod x and this is 1 by n alpha n minus 2 alpha mod 1 by mod x to the power n minus 2. So, this for n greater than 3. So, for these two, you can always see this is also radial function. So, this kind of thing is called a fundamental solution of the Laplace equation.

Now, you can see that as x tends to 0 as x is near 0 this actually it is a blows up this also at x equal to the 0, I mean, of course, it is this does not blow up but it is not defined at x is equal 0, it is also not define as x equal to 0. So, essentially, if you just leave out the 0 part, this is defined everywhere, and it solves the Laplace equation everywhere except at 0 and we call this thing as a fundamental solution. So, with this, we are going to end today lecture.