Advanced Partial Differential Equations
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Lecture 12
Harnack Inequality

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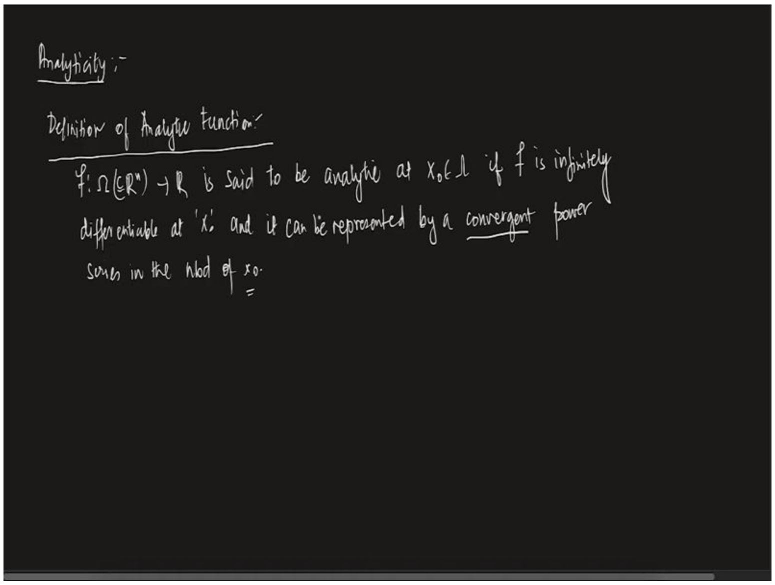

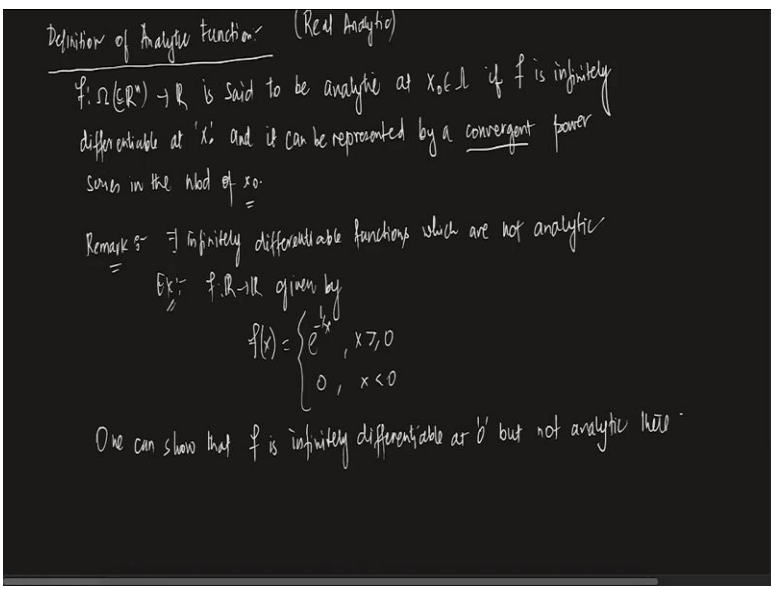

So, let us talk about another property of harmonic function which is given as analyticity. So, analyticity, what does it mean? First thing first, I guess all of you guys know what analytic functions are, but still let us just remind ourselves what those are. So, let us say so this is small definition of analytic function.

So, let us say f from omega subset of Rn to R is said to be analytic at a point x naught in omega, if f is infinitely differentiable at the point x naught and there exist and so let me put it this way and it can be represented by a convergent power series in the neighbourhood of x naught, is this clear? So, essentially you see, it is not only C infinity function the of course any analytic function of infinity not only C infinity, but more it is actually such a function which can be represented using a convergent power series (())(02:26) convergent very important power series.

So, you can represent the function let us say, if it is analytic at the point x naught, you take a small neighbourhood x naught in that neighbourhood you can always represent the function with help of a convergent power series. So, that analytic function. So, what is the difference between an infinitely differentiable function and analytic function?

So, just a small remark, this is just I am quite sure all of you guys all know this part, but still just a reminder. So, remark there exist infinitely differentiable functions which are analytic which are not analytic and for example, you can just take this function, I am not going to do this, but f from R to R, let us say, given by I am not going to prove this thing, I mean you guys can do it yourself if you are interested.

So, it is basically exponential 1 by x let say and x is greater than equals to 0 and 0 if x negative, this function I mean you can show that this function this is sorry minus you can show that this function is infinitely differentiable, but it is not analytic. So, basically one can show that f is infinitely differentiable at 0 but not analytic there.

So, a one can show that so that is the analytic part what and please remember whenever we are saying analytic I mean real analytic very important. Because you guys already know that complex (())(04:44) fully different thing complex analytic functions are any holomorphic functions are basic analytic functions. So, you do not have to worry about all this I mean technical details in a complex case. In a real case, there are infinitely functions which are analytic which are not analytic so for example this one.

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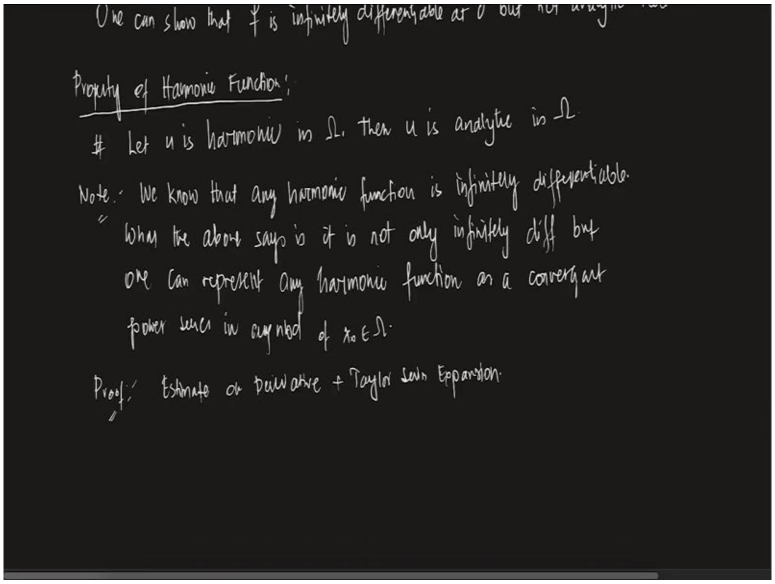

So, the property which I was talking about is this so this are property of harmonic function, another property of harmonic function. Now, this thing for this course, what I am going to do is I am going to skip the proof of this. So, this is called analyticity it says that so this is another property it says that let u is harmonic in omega, then u is analytic in omega.

So, basically what am I saying is this initially you have seen. So, let me put it in a small note, we know that any harmonic function is infinitely differentiable this we have proved infinitely differentiable. Now, what this property says if it is so, what the above says is it is not only infinitely differentiable, but one can represent any harmonic function as a convergent power series in our neighbourhood this is the how do I put it in any neighbourhood of x naught containing omega.

So, essentially if it is harmonic you take any point x naught in omega and you take a neighbourhood around that point you can represent any harmonic function with respect to a convergent power series expansion. So, very very important it says initially we have seen that harmony functions are only infinity differentiable here we are saying that is not only infinitely differentiable, but it is more that you can represent any part of the function with respect to I mean you can represent it with the help of a convergent power series.

The proof of this thing will actually so proof I mean we will not do the proof I am just giving you some idea what we need to do is we need to use the estimate, if you remember the estimate on derivatives this, we have to use estimate on derivative and with the help of this we want to and plus some Taylor series expansion.

So, as I have told you earlier, we are not going to prove this particular theorem for this class for this course, but the thing is if you are interested in proof try to do it yourself. So, essentially it is not a very very difficult proof you just have to use the derivative estimate on derivative and (())(08:45) Tylor series expansion, but I am not anyway suggesting that this is going to be a very very easy proof it is not a very easy proof.

And I am going to skip this (())(08:56) for now, but the thing please remember this thing whenever you are talking about the harmonic functions a real harmonic information they on over analytic. So, basically they are C infinity functions and cans be represented with respect to convergence power series expansion.

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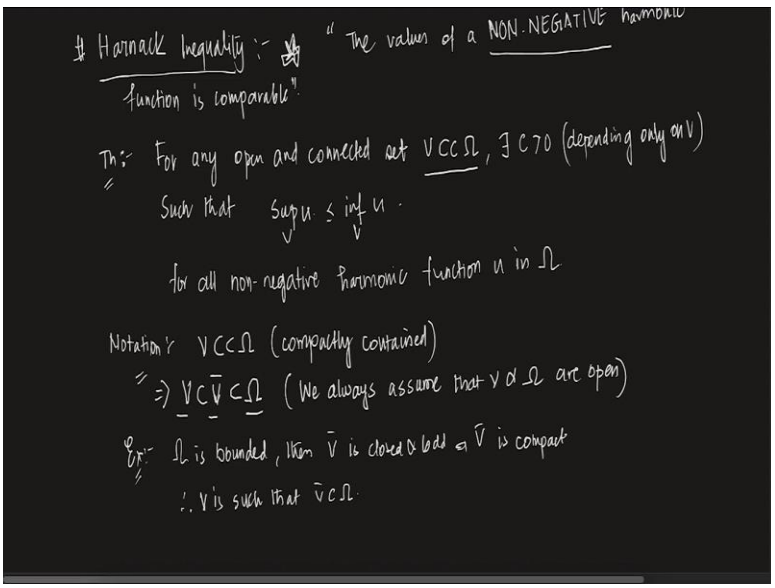

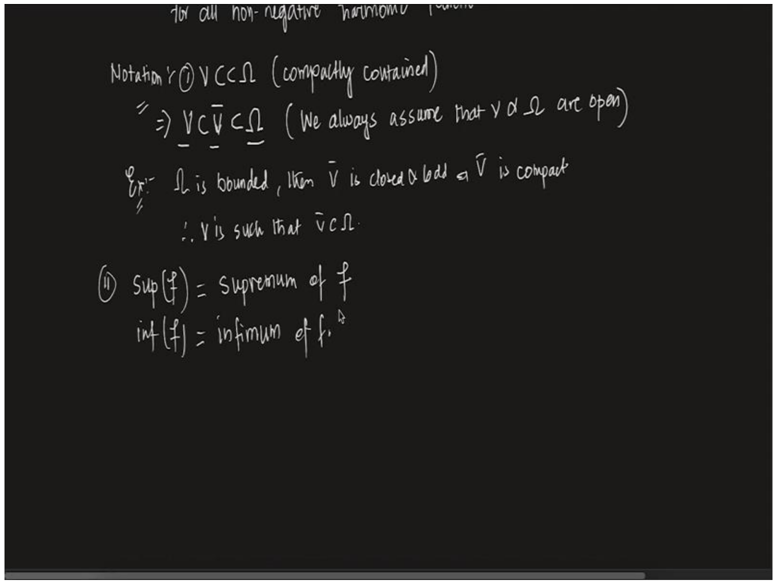

Now, what we are going to do is we are going to prove another property of harmony function which is called the Harnack inequality. This is extremely important this is one of those star properties. Let me put it in like this star properties very very important property. So, Harnack inequality. So, what does this say? So, essentially this actually compares the value of a non-negative harmony function.

So, essentially why is this important? So, basically with this says that the values of a very very important, this one a non-negative important without this it will not work the value of a non-negative harmonic function is comparable. So, this is what it says I mean (())(10:21). So, whenever you think of harmonic Harnack inequality just think of it like this it is saying that if you are giving me a harmonic function I do not care what sort of function it is at least if it is a harmonic function, if it is a non-negative harmonic function this is very important non-negative without this condition it we will not work. If this is a non-negative harmonic function, then the values of the function on a domain omega they are comparable.

So, what is the theorem let us just write down the theorem properly. So, it says that for any open connected open and connected set V which is contained in omega u there exist a constant C positive such that so of course depending on only the V. So, you fix a V which is compactly containing omega and depending only on V such that the supremum of u so over V is less than equal the infimum of u over V.

And this holds for all non-negative harmonic functions u in omega. Here there are some particular things which I want you to understand the one thing is this whenever I am writing V is content so some notation which I need you to understand is this notation. So, first of all I am saying V is compactly contain in omega, what does that mean? Let us just understand that V is compactly contain in omega this is means compactly contained. What it says is this, its says that, so here I am just writing V you can just take anything you want.

So, essentially whenever I am saying is compactly contain in omega what I mean by this is V of course V is contain in V bar that is always true V is contain in V bar this implies V is contain in v bar and that will be contained in omega is this clear this implies is that V is contain in V bar. So, v bar see this is again whenever we are writing this thing we always assume that V and omega are open, is this clear.

We are always assuming V and omega are open just think about it for 5 seconds. What I am trying to say think of omega open V open and V bar is such that V is such that V is contain in V bar of course this is always true, but such that V bar is containing omega take 5 seconds think about what I just said.

So, I am quite sure you have more or less some idea now that what exactly does it mean see essentially it means that it will be away from the boundary. So, essentially what it is saying is the V bar so just think of this as the let us say this is bounded if it is bounded V is bounded of let us say omega is bounded for now just for example let us say.

Example omega is bounded then V bar is closed and bounded that will imply V bar is compact. So, basically it is a compact set which is contained in omega so there is some distance between these things. So, in this case, therefore, V is such that is a open set such that v bar is contained in omega, omega is an open set v bar is a compact. So, essentially you can think I mean you do realise that there is a distance between those two. So, essentially it is quite comfortably V is quite comfortably inside omega you can just think of this as a vaguely. This is what it is says.

Now, let us talk about another small details which we are going to improve here and that is called the small idea which we I am going to write it down here. So, this is called the supremum Here I am writing sup, here I am writing the supremum. And what does this say? See, here I wrote sup of u and inf of u, what does that say?

Let us, understand this is the first notation to take a notation supremum of a function f, so this is basically supremum of have f and whenever I write infimum of f it means that it is infimum of f. So, what is the sup and f sup of f and inf of f, sup of f is supremum of f, inf of f is the infimum of f. So, once we know this, what we are going to do we are going to start with the proof of this theorem.

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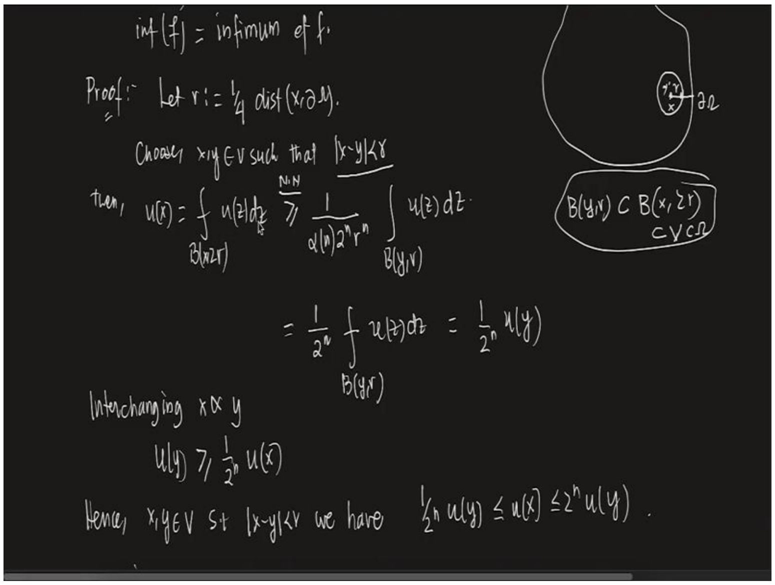

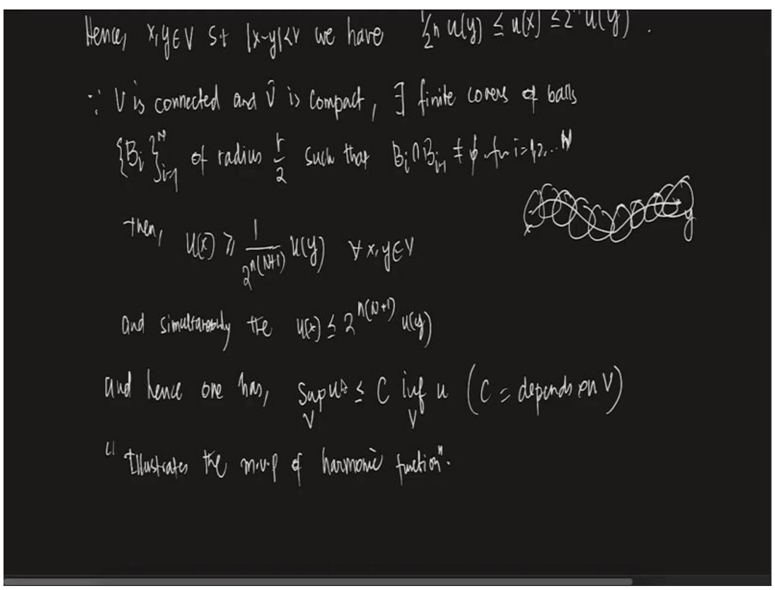

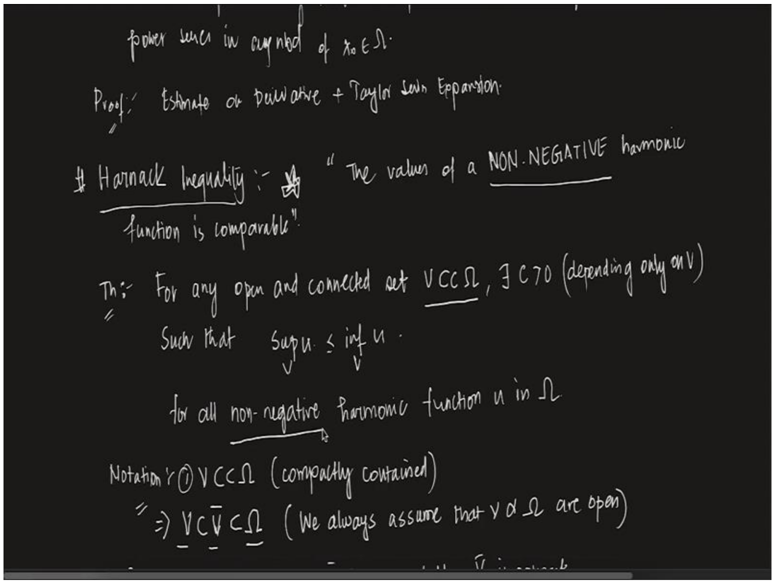

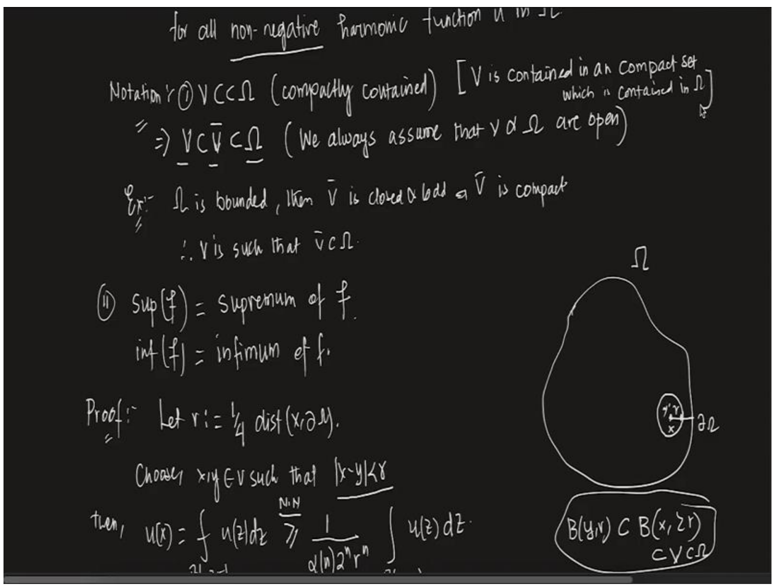

So, essentially, let us say we start with the proof. Let us start with let r be defined as the one fourth times the distance between x and the boundary. So, essentially, we are proving the Harnack inequality and what we want to do is this. Let us, say that you are omega and x is such that the distance between x and Del omega, so I am taking r so let us say this is x here. The distance between x and Del omega let us say this is your sum the distance between x and the boundary of omega.

And let us say that is the one fourth distance. Now, you choose x and y in V such that mod x minus y is less than r we can of course do this. So, essentially think of x in the centre and you can choose a y which is inside the ball of radius r something like. Then by mean value theorem what can you say, you can say that u of x equals to B x2r u of z dz, this is mean while property.

Because (())(18:09) is harmonic inside the omega and V is contain in omega and does not touch the boundary. So, it is away from the boundary. So, this is this and this I can write like this is try and understand what I am doing I will explain but for now just try to understand this one B y, r u of z dz see what am I doing is this this notation I am just writing it together the integral over the ball that can be written as 1 by alpha n r power n r is 2r here.

So, it is basically 2 power n and r power n this I am writing and here I am dominating this integral of B x2r of u by dominating B y, r with B x2r integral over B y, r with integral over B x r. Why I can do this? Because, you see here y of r this is a smaller ball then B x2r, Bx2r is a much bigger ball and since the B y, r is contained in B x2r.

If we take it x minus y to be less than r of course, B y, r it containing B x2r and this is again contain in V of course, this is containing V which is again containing omega. So, I can do this thing this set is bigger, the integral of this set is always getting to be will always dominate this. So, that is there.

Now, this can be written as 1 by 2 power n integral B y, r u of z dz I can always dominate (())(20:11) like this I am just writing this notation like this and I am taking the volume inside. So, that is equals to 1 by 2 power n and what is it this is u of 1. So, for any x, y you have u of x any x, y (()) (20:29) that this happens in V or z mod x, y makes minus y less than r, u of x (())(20:34) is greater than equal 1 by 2 to the power (()) (20:36) y.

Now, interchange x and y of course we can do this there is nothing special about x and y there. So, I can write u of y can (())(20:52) greater than equals to 1 by 2 power n u of x hence for all x minus y for x, y in v such that mod x minus y less than r we have 1 by 2 power n u of y less than equal u of x which is less than equal 2 to the power n u of y this is always true.

So, once this is true now, we are basically done. So, now see V is connected now here one thing I am using please remember see we have assumed that this is a non-negative harmonic function I said it is it goes for all non-negative harmonic functions, do know where am I using non-negativity here, this here non-negativity is important. Because otherwise this may not be greater than equal this.

Now, since V is connected this is also important since V is connected and V bar is compact, why V bar is compact? Because you remember what am I what I said is u is content so this is the definition v is compactly contain in omega it means that essentially v is contain in v bar which is bounded and it is contain in omega, v is contained in a compact set v bar. So, it means that I did not write it, sorry, so we have to write it compactly contained.

So, basically compactly contented is v content in a v bar which is compact and it is contained in omega that is called a compactly contented set, maybe I can write it here sorry (())(22:55) somehow skip it. So, V is contented in a so omega can be any set does not matter omega does not have to be bounded, but v is contained in and compacted set, which is contained in omega, this is what it state.

Now, so, since V is connected and V bar is compact here V is compactly contained in omega. So, v bar is compact set what we can do we can actually cover it with a change of there are finite cover. So, let us say there exists we can say there exists a finite cover of balls Bi. And this is i equals to 1 to let say n of radius r by 2 such that Bi intersection Bi minus 1 is non empty of course, we can do that.

See, essentially what am I doing is this, this is why compactly contented is very important. Otherwise, we cannot do this compact since (())(24:28) omega can be anything it can be bounded, it can be unbounded, but whenever I am saying it is this holds this property you can compare the values of a harmonic function in a V bar.

It basically says that you just look don not look at the whole domain just concentrate on a part where it is compact concentrate on a compact part V bar and in that part we can do this. So, that is why it seems V bar is compact you can have a finite cover like this. That Bi intersection Bi minus 1 is not equal to phi for i equals 1 to n and this Bi of radius r by 2 that will cover the V bar.

So, then what happened then if this happens for any x, y less than equal here you see for any x, y less than equal r we have proved that this sort of property holds u y greater than equal. So, u y and u x are comparable. So, then what you can say is u x is greater than equals to 1 by 2 to the power to n n plus 1, because for every Bi this is happening.

So, u x it will just get 1 by 1 if you just go on 1 by 1 for every x, y see x and y if it is here and here, y connected is required, because otherwise we cannot connect these two points. Now, you just take balls overlapping this thing, do it like this and after that after a finite number of points, what will happen is, so, let us say N, capital N is that finite number of points, what is happening you can covered the whole thing down.

So, and for every x, y in between these two, you have this property that u of x and u of y satisfies these total property, so for any x, y in v, so for any x, y in v you can just write u x is greater than equal 1 by 2n plus 1 u of y and simultaneously the opposite is also true. So, u of x 2 over N plus 1 times u of y that is also true.

So, once this happens this holds for any x, y in v and hence we can say that we can compare x and y. So, and hence one has that the supremum of Vu over V is less than equals some constant times the infimum of u over V and this C of course, depends on V. So, is this clear what am I saying?

Because, you see if you take the supremum here supremum of u is always getting going to be dominated by these and then you take the infimum on both sides, so, that nothing will change. So, it will remain the supremum of u and that will be less than some constant (())(27:39) the infimum of over u. So, that is all so that is why what we are saying is the supremum of u so cannot be you see these actually gives you so this is the quite.

So, this illustrate this property illustrate the mean value property of harmonic function. What mean value property says is basically in a small ball kind of thing, you can actually average out the value of u that is what it is essentially saying. Here also it is saying the same thing. So, basically, let us say in a small compact set, if at one point (())(28:21) is like an infinity and one point it is very small, then the average will not gives you the value of u at some other point, You understand what I am saying for the average thing to work you have to be quite homogeneous in his boundary neighbourhood.

So, that is what it is saying that u can compare u the value of non-negative harmony function. In a compact set, in a compact set you can compare the values of u, that is what it says and it is a one of the most important properties of harmonic functions. So, with this, we are going to end this video.