Advanced Partial Differential Equations
Professor Doctor Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Lecture – 11
Properties of Harmonic Functions

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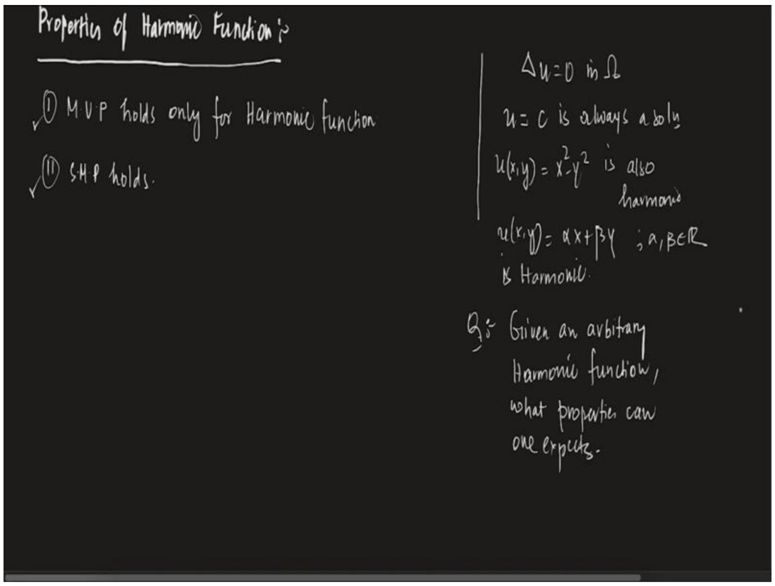

Welcome students, in today's class we are going to talk about some more properties of harmonic functions; so, properties of harmonic functions. Up till now we have seen certain properties of harmonic functions; you have I think you have idea about this kind of. They are very very special in the sense that I know mean value property; that property is something which only harmonic functions chooses. So, first property which we have seen and we have seen that this is a very important property called the mean value property. Mean value property holds only for harmonic functions, only for harmonic functions.

This is what we see harmonic functions and this is very important; because we have seen that you can prove strong maximum principle out of it. And you also saw that the strong maximum principle holds, this also holds. So, essentially you see and two too strong maximum principle we have again used mean value property. Today we will see more applications of mean value property, and essentially that will give you more properties of harmonic functions. So, one of the main idea, about the harmonic function, so let see we did not exactly solved any Laplace equation to find the harmonic function. We definitely you see Laplacian let say Laplacian u equals to 0 in some domain of omega.

Of course, without solving this thing, you guys already know that u is equals to constant is always a solution; that is always there, irrespective of what omega is. And of course, you can also find the solution which looks like this x square minus y square; that is sort of things will also work. This is also harmonic, is also harmonic harmonic and similarly single function let say alpha x plus beta y; that sort of linear function are. So, essentially if here alpha, beta is in R, so if you take alpha to be 0; u x, y if it just y beta times y. And if you take alpha to be 1, beta to be 0; is u x, y is basically just x coordinate.

So, essentially what I am trying to say is these are all harmonic functions; we do not have to worry about where they are harmonic, they are harmonic. Now, the point is this. So, we have already found out three different sort of functions; so this is like a constant functions, this is linear function, this is a quadratic one. And you can go on doing this thing, but the thing is so you do understand. There is like a huge variety of harmonic functions; now the question is this how does without solving this thing. So, it is we do understand this definitely not possible to find all possible harmonic functions like this.

So, without solving it how can we say what are the properties that harmonic functions posses; an arbitrary harmonic function, not a special one. So, the question is basically was down to this, given an arbitrary harmonic function an arbitrary harmonic function harmonic function, what properties can you expect, what properties can one expect? And it turns out that harmonic functions are very very special in that sense.

Because we saw that there is a something got the mean value property which holds; and there is something called the strong maximum principle which holds. There are other properties are also like this, and we have also saw that there is something called positivity; which is application of small maximum principle, that holds for harmonic functions too. Today we are going to prove something different, different form of property; so maybe I can do it here, maybe I can do it here.

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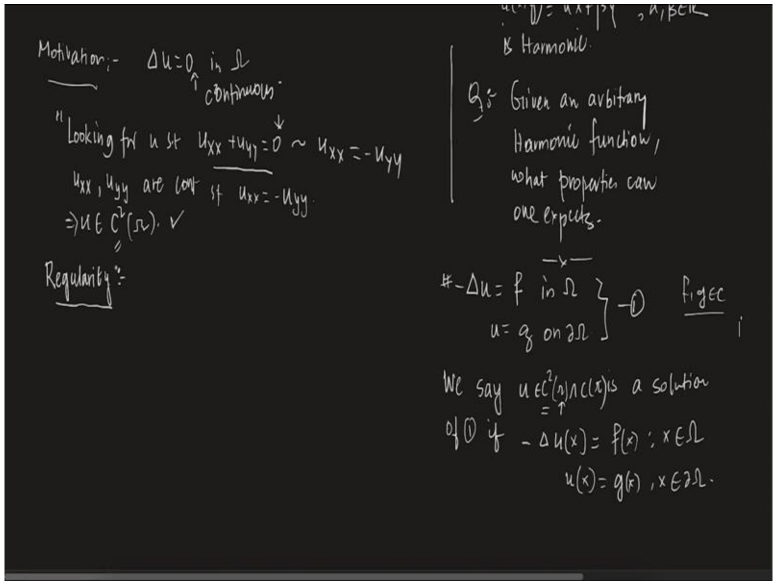

So, the thing is this, this is very important is called the regularity. What it means you see; let me do the motivation over there. Now, think of a function let say Laplacian u equals to 0; whatever it is in omega. 0 is what sort of function is this? This is c2 function it is a continuous function; you do not worry about all that, this is basically a continuous function. So, whenever you are solving this problem, you are saying that you are looking for, we are looking for u; such that u xx plus u yy equals to 0. So, essentially you are or you can also think of this like u xx equals to minus u yy; so, if we if we just look at this property, what it say.

This you are looking for a function such that the double derivative like this; u xx plus u yy that is 0 which is a continuous function. It is a continuous function, so basically you are looking for a function whose double derivative; the sum of double derivative u xx plus u yy that those derivative which is sum it up, that is a continuous function. So, definitely without loss of generality, we can safely say it is not like always going to happen. But we can say that we can assume that we are looking for a function, such that u xx and u yy are continuous, are continuous we can safely say that.

And such that this happens, such that u xx is equals to minus u yy; this is what we need to find. Now, if something like this happens, so what does that say about u? Just says that at least these two derivatives is must exist. Double derivative must exist and they are continuous; that is what used to say. So, essentially you can think of this u to be c2 of omega; that implies that u has to be c2 of omega. Of course, it may not be c2 of omega; it can be much less than that. But we can safely say that at least u is in c2 of omega; and that is where our definition, the our motivation of solution. So, let say if we say that there is a solution through this problem Laplacian u equals to 0, what does that mean?

It means that you are looking for a c2 function; you are basically looking for a c2 function, which satisfies the equation. So, maybe I can do it here a little bit solution. So, Laplacian of u equals to f in omega and u equals to g on the boundary; we say that is your 1. We say we say u in c2 omega is a solution alt c2 omega bar; or you can also take c2 omega intersection C omega bar. But, that is not a now let say, maybe be you can do it later; let us let us do it later. So, we say u in c2 omega intersection C omega bar is a solution of 1 of 1, if let us take minus Laplacian minus Laplacian; there is nothing special of minus you understand; it is can be positive negative just as just is there.

If this happens, this is equals to f of x for all x in omega, and u of x equals to g of x, for x on the boundary. So, you are looking for a c2 function in the interior of the domain omega; so omega in omega, this is has to be c2, on the boundary it is continuous. So, its value that f and g are continuous functions; we are always assuming f and g are continuous. So, essentially whenever we are saying something is a solution, we are always assuming that it is c2 solution. Now, what does regularity says? Regularity is a very important property in (())(08:54); what it says is this. So, it says that is it really a c2 function, can you comment more on how smooth it is C?

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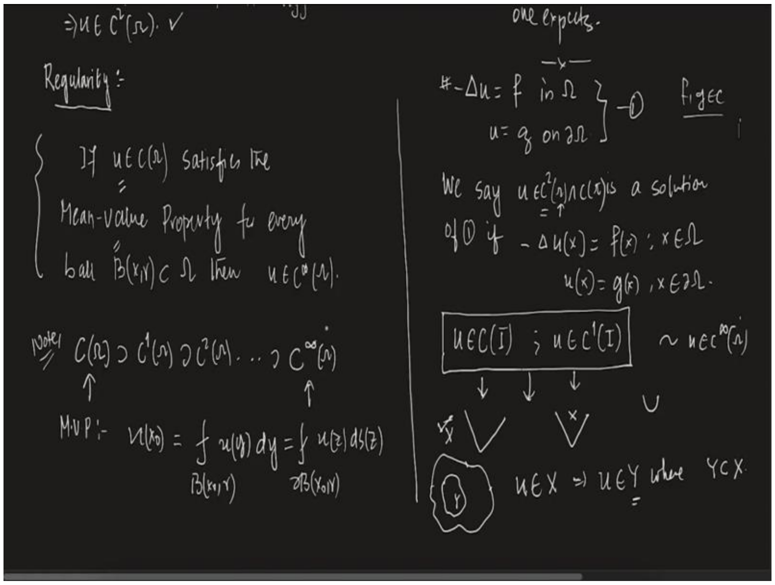

From intuition it is clear that let say a function which is u is in some continuous interval I; and let us assume that u is in c1 interval I. I mean deadly speaking not very mathematically; but deadly speaking; just think of the regularity of this. Regularity means think of this as a how smooth these functions can be.

So, you realize that a function which is continuous may look like this; whereas a function which is c1 be never look like this. So, a function which is continuous may look like this; but the function which is c1 will not look like this. So, you understand that there will be some some small the smoothness is there; it will be smoothing, it will not suddenly turn around and you understand what I am trying to say.

So, basically at every point there are some tangent vectors, and now you realize that there are more regularities. A function which is C infinity of omega that is very regular; so at every point there are tangent vectors. And you can do all sort of things; so every derivative have tangent vectors and that sort of things.

So, they are very smooth functions; so essentially the question is this. Whenever we say something is a regular, will talk about regularity of PD; we mean that it is in certain space. Let say, u is in some x, can that imply that u is in some y; where y is contained in x, is it clear. So, essentially what I am trying to do is this.

Let us assume that you have whenever you are starting to find the solution, you are starting to find it in a bigger set x of course. If you want to find something is better to take the step big and big; so that the chances of getting a solution is much more. Now, once you have found it in a bigger set, now the question is this. Is it also in the smaller set? The smaller set is a most exclusive set. We also wanted to be in a smaller set, so that it enjoys much more properties. So, first of all the idea is this; first you prove it for you show that the thing is for bigger set, and then you show. First you showed that it holds for a bigger set, and then somehow you show that if it works for the bigger set.

I mean does it work for a smaller set also which is contained in a bigger set? Because you understand smaller set is much exclusive; it enjoys much more properties. So, this is the idea of regularity; so, what does regularity says? So, if u which is continuous in omega satisfies the mean value property, mean value property. Mean value property you remember for every for every ball B x, r is contained in omega; then you can say this u is in C infinity of omega. Now, what I wanted to do is this. For now, just take 5 seconds, then think about what it says. It is saying so let me explain you and then think about it.

It is saying that first of all you start with a continuous function; no c2 nothing, you just start with the continuous function, the bigger set which you can think of. So, you remember this thing, continuous function C in the omega is the biggest set; which contains c1 of omega, here which contains c2 of omega. And all of this contains C infinity of omega; so, function which is in C infinity; it is also in all of these sets, so note.

So, essentially what am I saying is this, you start with a function which is continuous in omega. Of course, not every function which is continuous is in C infinity; f equals to mod x is an example. That is in C that is continuous, but not in c1 on minus 1-1; if you take omega c2 minus 1-1. So, what am I saying is this.

You start with a continuous function, such that the mean value property please remember this thing, this is so important. Whenever we say something mean value property, you guys already know; once mean value property is also going to be a harmonic function. So, it says that we are not talking about inharmonic function there; we are saying that it starts with a continuous function, such that it satisfies the mean value property. What is the mean value property? Let me again remind you what is the mean value property. It says that u at the point x naught, this if you want to evaluate.

So, basically let say given a function we say that the function satisfies the mean value property; so maybe I can just put it here. So, essentially let me write it down, we say that the function satisfies the mean value property is for any x naught in omega; we can write it like this. B x naught, r u of y d of y, or you can also write as the boundary x naught r u of x ds of z; where this is the surface measure we say, if you do not know measure theory.

You can think about it as the integration with respect to the surface. So, essentially just take 5 seconds and think about what this property says. It says that the continuous function with this property, then it will take skip inside this set; once you skip all that. Once you sense that the mean value property, continuous functions satisfy will end up in this set. Just think about it for 5 seconds and will comeback.

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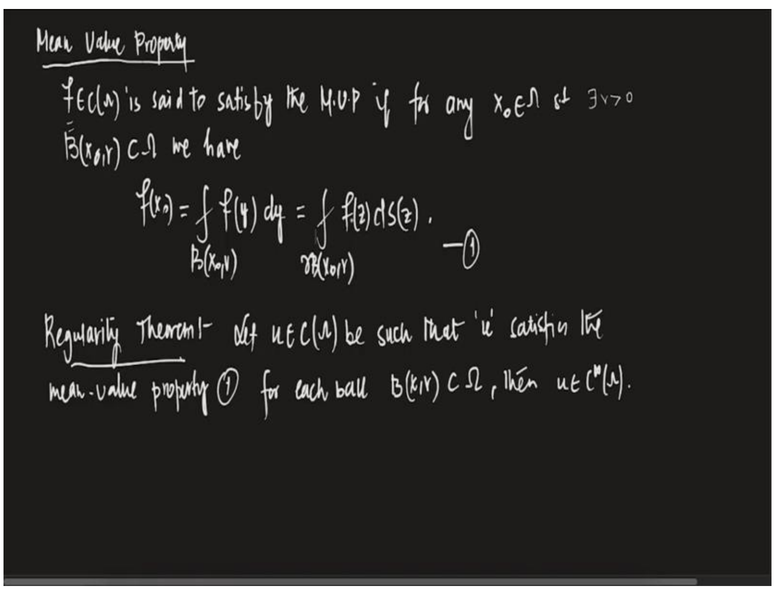

What is the mean value property? The mean value property says that if f is in continuous in omega. So, we start with a continuous function f and we say that satisfies the mean value property, if for any x naught in omega such that you have a small ball, which is contained in omega C. Of course, for any x naught you will have something like this, why? Because of omega is an open set, and every point is an interior point. So, essentially any point you can just put a ball inside omega; so you take any ball. It does not matter whatever it is, such that there exist r positive for which for which B of x naught r is contained in omega.

We have the value of f at point x naught is the average is the average of f over B x naught, r; or the average of f over the boundary. So, that is your mean value property you already know. And the regularity theory says that once you have a u which is continuous in omega, and it satisfies the mean value property, for every ball which is contained in omega. Then you can show that u is not only continuous, but it is infinitely differentiable; this is a huge claim here. So, again let me emphasize it is saying that a continuous function with this property is infinitely differentiable, so it is huge property. How do you prove this thing? So, the proof is not very difficult, will use mollifiers to do this proof.

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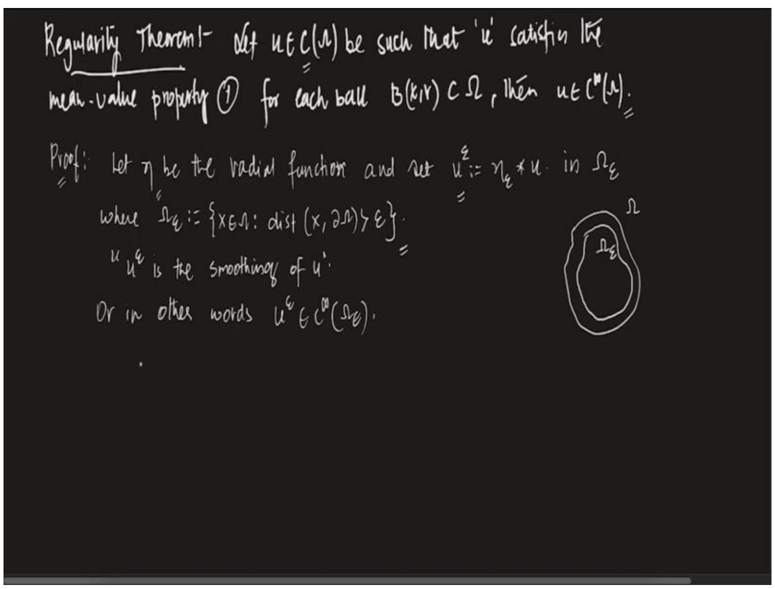

Proof, how do we prove it? So, let Eta be the radial function radial function, but which we have proved earlier the radial functions. And set u epsilon to this, so this is the definition which I am writing; this is the Eta epsilon convolution u. So, essentially what am I doing is this, you see I am you have to show u is in C infinity of omega.

So, you take you take, you show that u is actually what you will do with this? You are not going to directly attack u. What we are going to do is we are going to show that you can actually identify u with a C infinity function inside omega. And what is that C infinity function that is u epsilon; this is what we need to find.

So, what we are doing? We are mollifying u in omega epsilon. So, where omega epsilon is the set of all those x in omega, such that the distance between x and the boundary del omega is greater than epsilon. So, essentially what am I saying? I am saying that omega epsilon, it should be such that it must contain. So, essentially let say that is your omega and omega epsilon is the domain, which is like an epsilon distance from omega; so, this is your omega epsilon. What am I saying is I want first of all I have proved; I will define set a function u epsilon which is defined on omega epsilon. And that is defined as Eta epsilon convolution u, and we say if you remember it is now u epsilon is the mollifier; sorry is the smoothing of u.

So, using a mollifier so you remember Eta is start with a radial function Eta; and we define Eta epsilon as a mollifier. Eta epsilon as a mollifier and you see u epsilon is the smoothing of u, so u epsilon if you define it like Eta epsilon convolution u. Then what happens to the u epsilon? You know that the u epsilon is a smooth; we say it is a smoothing of u and essentially u epsilon is. So, that or in other words u epsilon is in C infinity omega epsilon; is this clear what am I saying? First of all let me again explain to you. We are starting out with a radial function, we did it earlier; we started out with a radial function.

We will always denote it by Eta and we write it and we checked, this Eta is a radial function; which sometimes they call it a mollifier. But, for now we will call Eta epsilon as a mollifier, it does not matter but. So, mollifier is what? It let say you have some problem somewhere; so, like sudden bend or something like this.

What Eta epsilon does is when you convolute u to Eta epsilon; that particular function which is here we are defining at u epsilon; that particular function will be defined in this space, u omega epsilon. It is not defined everywhere, it is defined in omega epsilon such that this particular function is extremely smooth, it is C infinity. Now, what we will show is in fact this u epsilon is equals to u in omega epsilon, and this epsilon is arbitrary. It can be as smaller you want, what we can do is, we can show that u epsilon is equals to u; so, let us prove this thing.

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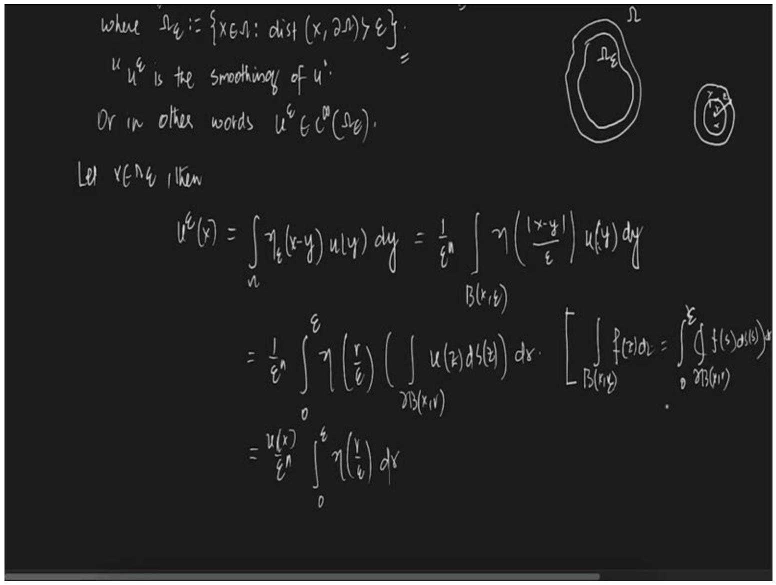

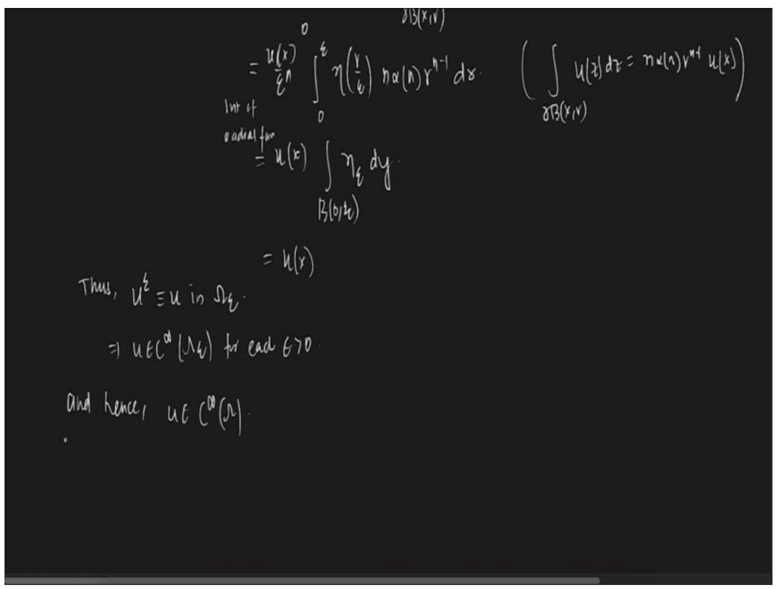

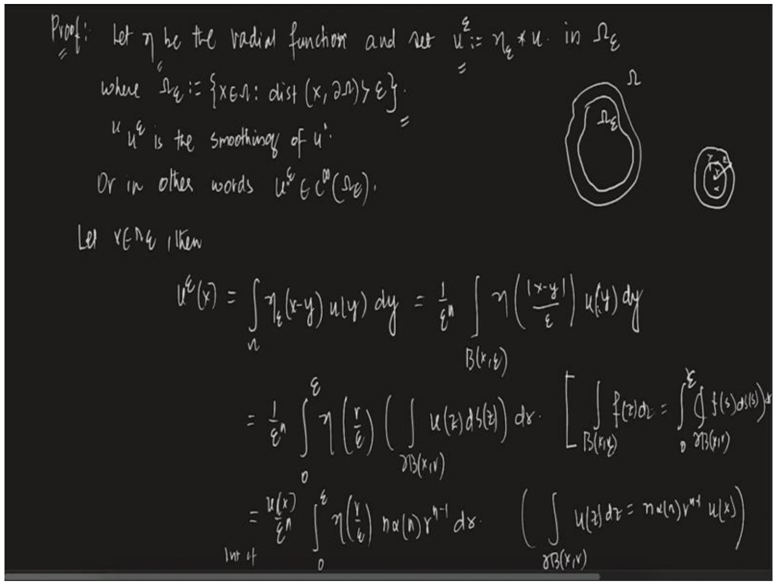

So, let x is in omega epsilon, then u epsilon of x it is the convolution if you remember. So, this u epsilon of x is u defined; so, it is Eta epsilon of x minus y, u of y dy; so that is given by 1 by epsilon power. So, this is just I am writing what it as epsilon is here; there is nothing special about it. B of x epsilon Eta of mod x minus y by epsilon, Eta of y dy.

This we can do this is equals to I can write it like this; so you remember this is a radial function. Please, sorry I have made small mistake here, this is u of y; now this is a radial function Eta. And I am integrate it on u of x, r; you remember for any function f, I can a function when you integrate on u of x r is it means that you integrate between 0 to epsilon Eta of mod x minus r.

So, let me write it like this r by epsilon; I will explain to you what we are doing. Integral del B x, r is u of z ds of z, and d of r. If you remember see essentially what we are doing is for a integrable function f integral if you want to do between this; f something this x dz. What we can do is first of all we are integrating between B x naught, r.

So, think of this like if you want to know measure the width of a trunk section of a trunk of a big tree; what you do? We look at one ring. First of all, integrate on one of the ring, and after that extend the ring from 0 to infinity. So, essentially what am I saying is this something like this. I want to integrate in this whole region what and this center here. What I can do is let us look at the value of the function in this particular ball, for an arbitrary ball like this; which is inside this with the same center. And after that radius of the ball from 0 to whatever the radius of this particular thing is then, I can map the whole thing.

I can expand the whole thing; so essentially this will turn out to be 0 to r del B x, r you understand; f of s ds of s and the dr. Maybe I can write it as epsilon, there is nothing special; let us write it as epsilon that will be better. So, essentially what am I doing? I am taking the integral over this the surface, and after that I am extending the surface from 0 to infinity; that will give you the whole volume. So, this is what we are doing. Now, you see where is x? x is in the center; where is y? y is on the boundary. And what is the distance between x and y that is basically r. So, that is why this r I can put r here, mod x minus y is r. Is it clear?

X is in the center, y is on the boundary, y is on the; so, this thing I can write it as B x, r. And x, r what is r? r is varying between 0 and epsilon. So, for every fix for every fixed r what is essentially happening is y is on r; initially this is your r. Think of this, this is your epsilon, this an arbitrary r; this r is varying between 0 and epsilon.

So, r can be here, r can be here, r can be here it extends; for a fixed r, this is x center and y is here. So, x minus y is r, so that is why I put Eta of r by epsilon; and again, integral over this ball u of z ds of z, dr. That will give you 0 to epsilon, if you remember what is this? For any r x is on the domain omega, and r is lies between 0 and epsilon.

So, essentially del B its lies in omega, and that is why this is basically from mean value theorem; the value this is u of x. So, I can write it as u of x and this is Eta of r by epsilon; this whole thing is u of x, so I have took, it outside d of r. I am missing something I am missing the value here; so, this is n alpha n r power n minus 1 d of r. That will come if you remember, so this is the that is the average, the average part should with the. Why? If you see del B x, r u of z dz, this is the average is there. So, that gets multiplied there, so that will be the surface area of the ball which is n alpha n r power n minus 1; and what is it? u of x.

So, that is what I am just putting it. Now, once I do this what happens? It can calculate this thing. This will become B 0, epsilon whatever we did here, we are just reversing it back, Eta epsilon dy. If you are not convinced, start with this thing Eta epsilon you can integrate this thing. Eta epsilon is a radial function, you integrate it on B 0, epsilon; you get this thing. So, please check this part, please check this; this is radial integration of radial function from here to here, so integration of radial function. So, this happens and we already know what is the integral of Eta epsilon over a ball, which is 1.

So, essentially this is transfer to be u of x; so thus, what we proved? Thus, we proved that u epsilon is equivalent to u. u epsilon of x is equals to u x in omega epsilon; therefore, u is C infinity in omega epsilon, for each epsilon positive. Why? Because u epsilon by definition u epsilon is a mollifier, it is smoothing of u. So, essentially u epsilon is C infinity of omega epsilon, so if u is equals to u epsilon in omega epsilon; that will give you that u is the C infinity in omega epsilon, for each epsilon. And hence and hence u is C infinity of omega.

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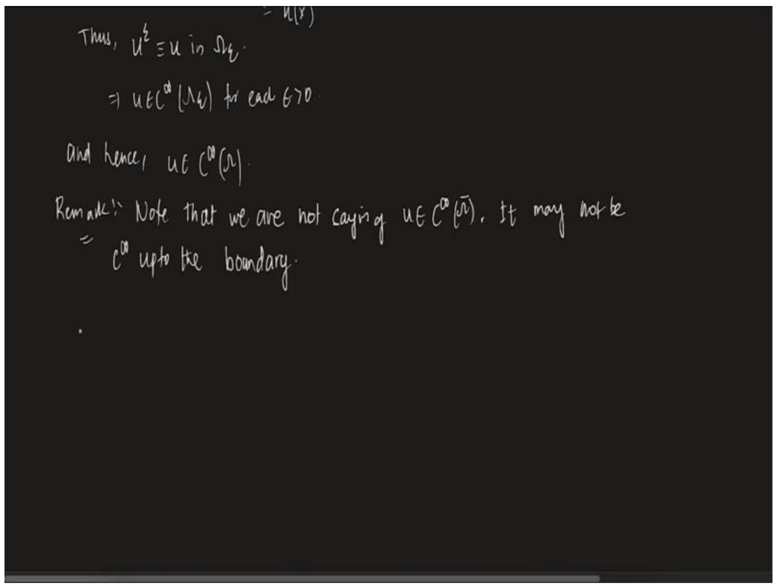

So, one small remark which I want to make is this. Note that we are not saying u is C infinity of omega bar; we are not saying that. So, basically it may not be it may not be C infinity upto the boundary; is this clear what am I saying? It is may not be C infinity upto the boundary; what you can say is inside, inside the domain. It is always C infinity, on the boundary it may not be C infinity; so please remember this thing. So, now what we are going to do is so essentially what we learnt any functions, continuous functions satisfying the mean value property is C infinity. Now, we are going to prove an estimate which is an extremely important estimate; is called the estimate on the derivative.

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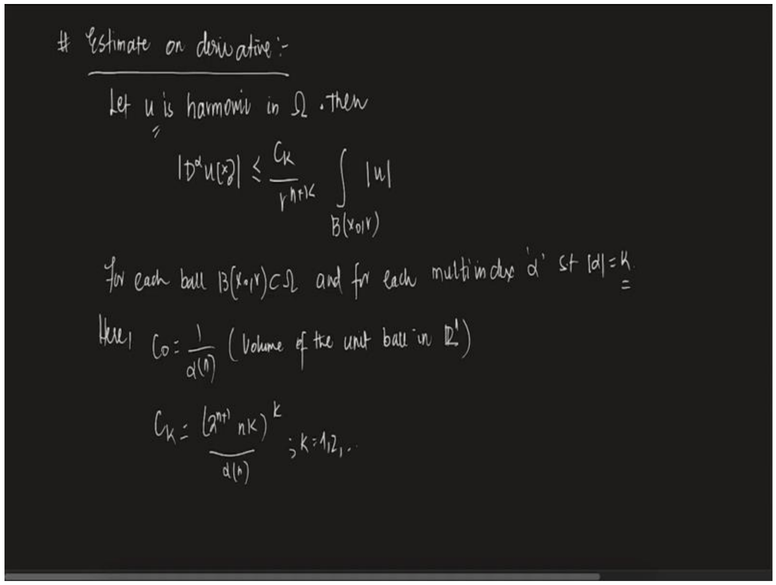

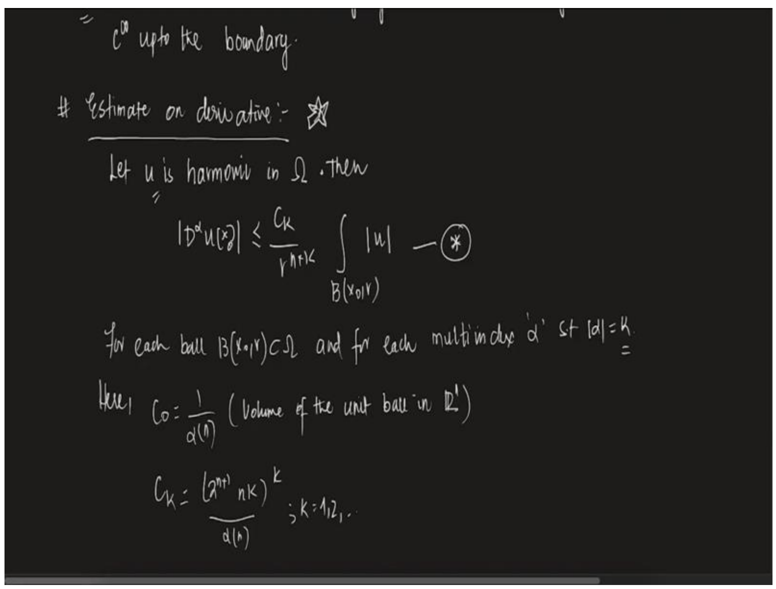

So, another property estimate on derivative estimate on derivative; what does it says? It says that assume let u is harmonic in omega; then the following happens it is a very very important property. Then what happens is D alpha of u of x, here see is less equals to Ck by r power n plus k integral B x naught r mod u. Of course, you can write u of x dx, but anyway I do not care; you can write it if you want. Let us write it if you want is better, I am not writing all the time g of x and all that; and please understand that. So, essentially what am I saying is this it is actually x naught; what am I saying is from the earlier estimate what do you know?

Any harmonic function a particular function satisfies the mean value property is saying harmonic function; so, any harmonic function is C infinity. So, it says u is harmonic in omega, then you can take any derivative D alpha of u x naught, definitely there is true, which is C infinity. So, you take any derivative D alpha of u x naught; and that you can bounded by Ck some constants ck by r power n plus k. This is very important and the integral of u over B x naught r; and this holds for each ball B x naught r which contains in omega.

And for each multi-index alpha such that mod alpha equals to k, if you remember, multi indexes. So, those those are used to write down the derivative. So, for all multiple index up till k; so Dk of u if you write it like this Dk of u; that is always bounded by the D kth derivatives of u. That is always bounded by Ck r power n plus k; so integral of over u, integral of u over the ball. Now, do you think the integral is depends? Of course it is depends, why? Because it is the C infinity smooth function over a ball, so the integral is defined. Here one thing what is Ck? So, C naught is very important; this is 1 by alpha n.

And what is 1 by alpha n? 1 by alpha n is the if you remember, this is the volume of the unit ball in Rn. It is the volume of unit ball in Rn, and what is Ck? Ck is 2 power n plus 1; it just a calculation. You do not need to get intimidated by anything; I will explain to you why this all of this is coming; and what is so important here, K equals to 1, 2. So, essentially what it is saying is this, in PD this is a very important aspect of PD; so, let say given any PD, not any PD most of the PD's. What we want to do is you need to let say without knowing the solution; what exactly the solution. We wants to know the we want to know how the solution behaves.

From here let us assume that this is true; later on, we did not prove this, let us assume this is true. If this is true, what can you say think about it? What can you say about the behavior? So, if you can take the balls as bigger as possible; you can take the balls as bigger as possible. So, in Rn if you are working, you can take the balls as biggest as possible. What can you say about the behavior of D alpha? That it is always bounded by this. So, the growth what it means this? The growth of u at that point. If you look at the derivative the alpha derivative; so, this is the kth derivative of u. The growth you cannot is always controlled by 1 by r power n plus k.

So, it depends on the ball in which ball you are looking at what is the radius of the ball; and with that radius there is a direct relation between this two. The growth is always controlled here that is what is this saying. So, let us prove this thing and the proof is very easy; we will also see a very very important. This may not look much, but let me put it this way; so, this is a star candidate, let put it like this. Here, most of the things are very important, but this is a star candidate very very important this estimate. So, how do I prove this thing? So, I have to prove it for all k, k is a natural number. So, the most natural thing to do to prove something like this is to use the instruction; so let just do it for k equals to 0.

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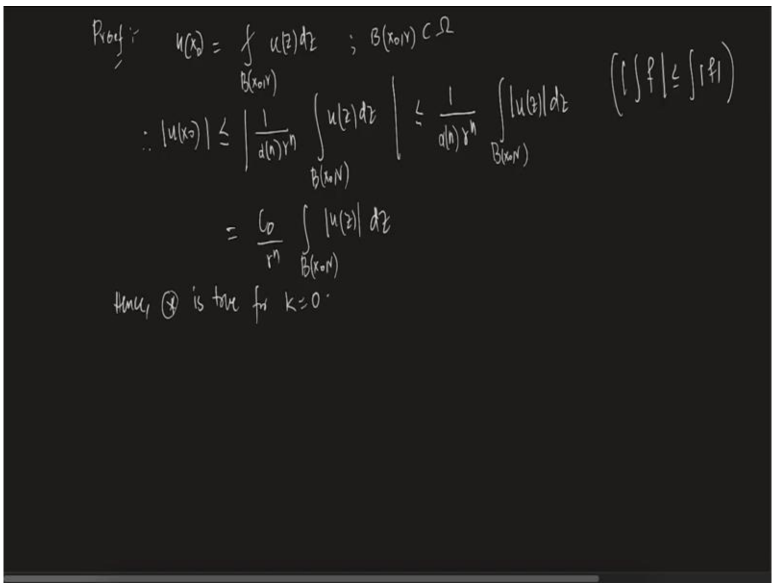

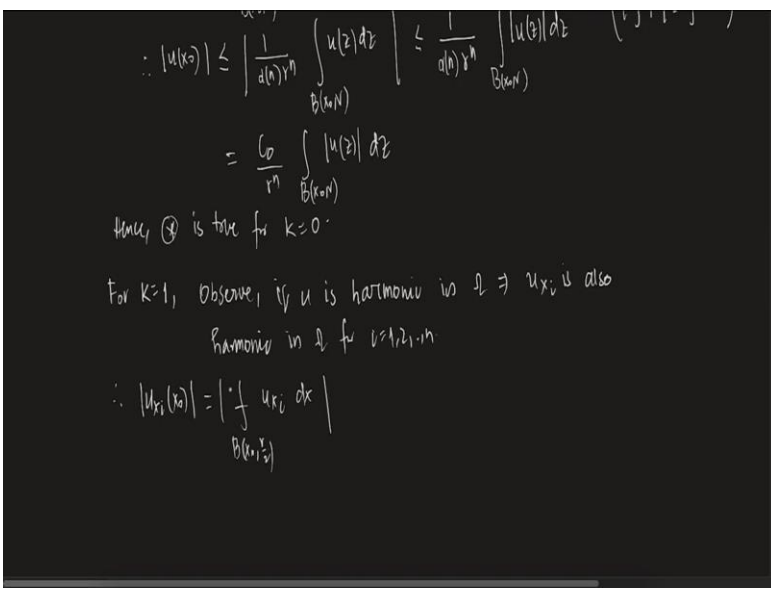

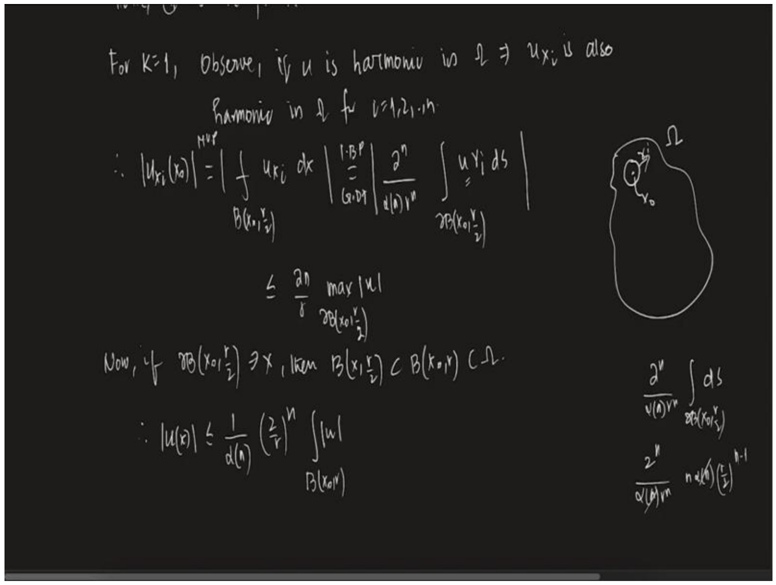

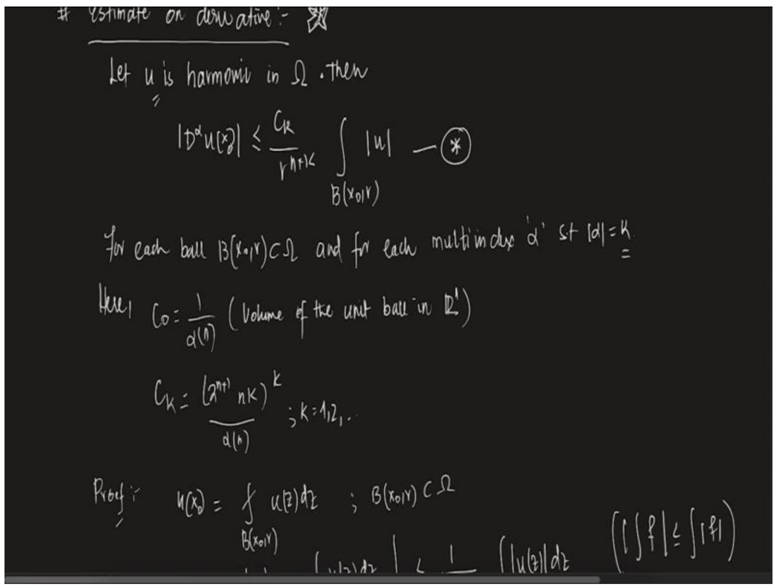

So, proof what happens for k equals to 0, please anyone just tell me what happens; think about it what happens for k equals to 0. For k equals to 0, mod alpha is 0; if mod alpha is 0 essentially you are looking at only u of x. So, x is basically C is harmonic, so write it down. If C is harmonic u of x how can you write u of x? If C is harmonic u of x can be written as, x naught can be written as integral B x naught, r u of z dz. That is what we can write it like this. Now, if you can do it like this and of course r is such that B x naught of r is contained in omega; we are always assuming that.

Now, if this happens then we can say that mod of u x naught; therefore, is less than equals to the volume of this thing, which is 1 by alpha n r power n; and integral u of z dz over B x naught r the mod of this. So, this is an of course always write it as 1 by alpha n r power n integral B x naught, r mod of u of z dz. I am I think you guys all of you guys know, because integral mod integral of f is always less than equals integral of mod f; that is always true. So, if that happens this is true, this is true; now if this is true what is 1 by alpha n? That is defined by C naught. This is C naught by r power n integral B x naught, r mod u z dz.

This is the same thing which we proved, which is k equals to 0 c0 by r power n; k equals to 0 r power n integral mod u over the ball. So, this is what I wrote here, this is what it is. So, for k equals to 0 it is true; so hence hence star is true for k equals to 0. Now, what happens when k equals to 1? Let see. For k equals to 1 for k equals to 1 we have observed this particular thing observe; if u is harmonic u have infinity derivative towards it. So, if u is harmonic observe if u is harmonic in omega, that implies u of xi is also harmonic in omega, for I equals to 1, 2 n. Is this clear what am I saying?

What I am saying is this it is a very quirky result; it says that if u is harmonic, if Laplacian u equals to 0. Then you can take the derivative of u, u of xi, if you think of Laplacian of u of xi; you can show that is 0. If u is harmonic it means that u of xi is also harmonic. And therefore, if u of xi is harmonic, u of xi satisfy the mean value property. So, what you can do is you can write u of xi at the point x naught that is equals to integral B x naught, r by 2, u xi; I do not know maybe I write down dx. So, that is this. Now, if this is true, so this is mean value property; please remember mean value property I am using.

Mean value property why I can use? Because if u harmonic, u of xi is also harmonic. If u of xi is harmonic, then u of xi is also satisfy the mean value property; and that is what I am using on u of xi, not on u. So, from here this you take the integration by part; if you remember your integration by part, this is equals to del to power n by alpha n r power n.

Here I am just writing it down, I am breaking it down; what is that volume of the ball? It is alpha n by r power n. Here r is r by 2, so that is why this is. And that I can write it down as del B x naught, r by 2 and u gamma i ds. I think it is quite clear how this is coming; this is integration by part. If you remember gauss diversion or you can just write it as gauss diversion theorem.

So, integral over the ball of u xi is u gamma i over the boundary of the ball. Now, this I can write it as less than equals to 2n by r, the maximum of mod u over the boundary x naught r by 2. Let just think about it, how we can do it. See here you take the maximum u is a continuous function, u is a continuous function in omega; I am looking on only that is your x naught and this is r by 2.

So, I am only looking at this portion of the domain; if u is continuous definitely this ball del B x naught, r by 2. That is a closed bounded set, hence it is compact, continuous function compact set; there is a maxima. I will take that maxima, if I take that maxima out; what do I have? 2 to the power n, alpha n r power n and integral over del B x naught, r by 2.

This is gamma I, if I dominate it, it is dominated by 1; gamma i is the unit outward normal if you remember. So, gamma i this is gamma i, unit outward normal; so, this is this you guys already know, so I am not writing this. Here this is g of x, now 2 to the power n alpha n r power n; this is the volume of this ball; so, the surface area of this thing. What is the surface area of this? It is n alpha n, r by 2, whole to the power n minus; that is what the surface area of this particular thing. So, if you calculate this thing, there is 2 n: alpha n alpha n is getting cancelled; and you have 2n by r and the maximum of u always there, so this is there.

Now, see if del b x naught, r by 2contains x, let say this contains x. Then the ball B is center x and radius r by 2 is contained in the ball with center x naught and radius r; which is contained in omega, which is quite clear. Because, the difference the difference between x and x naught, the difference between x and x naught; the maximum distance is between r by 2. So, I can take a ball with center at x and radius r by 2 which is contained in B x naught, r; and of course that is contained in omega. Therefore, what we can say is u of x is less than equals to 1 by alpha n, 2 by r whole power n, integral mod u B x naught, r; so, this you can say.

I hope this is fine; why we can say this thing? Why we can say this because this is mean value property; from mean value property we can say this. This is proof for k equals to 0; so, k equals to 0 is the proved, so k equals to 0. This is what I am writing for any x, the difference here is this; here proof is for mod u of x naught can be written like this. What I am saying is here for any x such that B x, r by 2 which is containing B x naught, r; we can write it like this. So, once this is true, you combine these two properties; the earlier property this property and this property.

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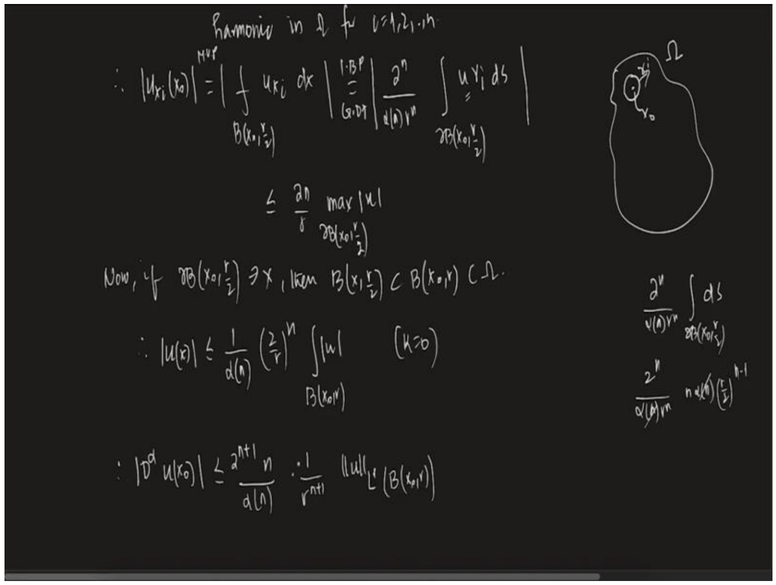

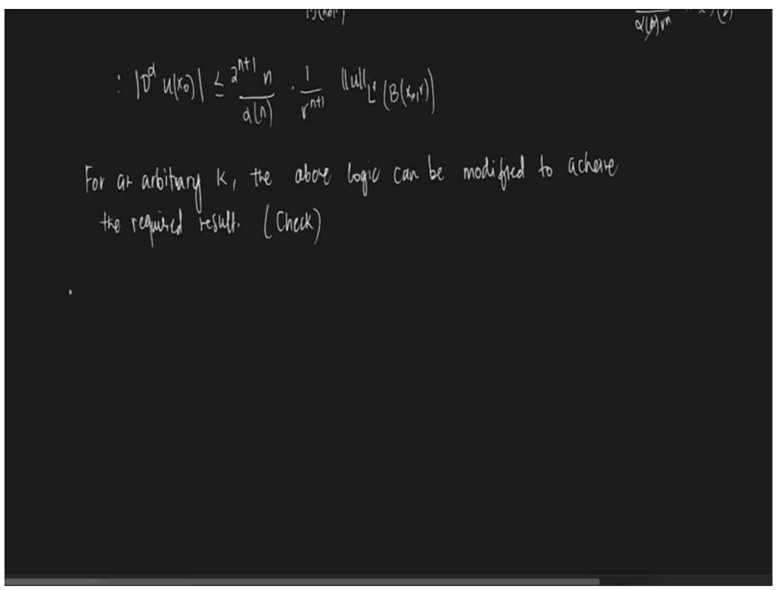

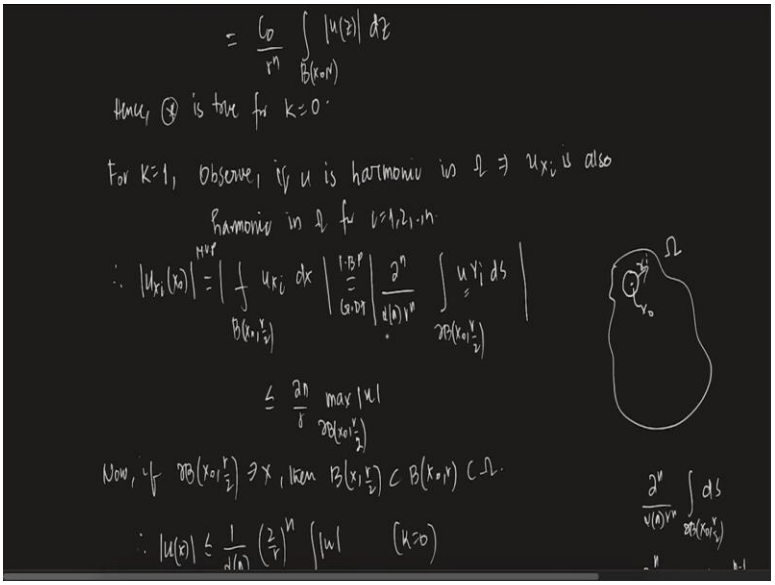

Once you combine this two, we get mod D alpha of u at point x naught; this is less than equal 2 n plus 1 n by alpha n times 1 by r power n plus 1, u L1 of B x naught, r. So, combine this and this for any x such that this happens; u of x can be bounded by this and so we use this property here. So, this is less than, you take the maximum here with the maximum will also be bounded by this.

So, essentially what am I saying? I am saying that mod u of u of xi at the point x naught, can be bounded by 2n by r times this whole thing; and that is why this thing. So, we have proved it for k equals to 1, now what you need to do is you need to prove it for an arbitrary k. M equals to 1 i have proved, M equals to 0 by mean value property equals to 1 will prove.

Now, we will just have to show for M equals to in arbitrary k let say, and this thing is exactly the thing whatever we did earlier; whatever the exact idea, whatever we did for k equals to 1; so exact idea will work. So, I am going to skip this part. So, for an arbitrary M equals to arbitrary K, the above logic can be modified logic can be modified, to achieve the required result. For an arbitrary K, we are doing induction; for an arbitrary K just think of it like it holds for K minus 1, and do it for K. what we need to do is exactly whatever we did for K equals to 1; the exact same sort of thing will work.

You just have to change things a little bit, but this is for yourself; so please check this part, I am not going to do. So, this is not very difficult the same kind of thing and you should be able to do these things.

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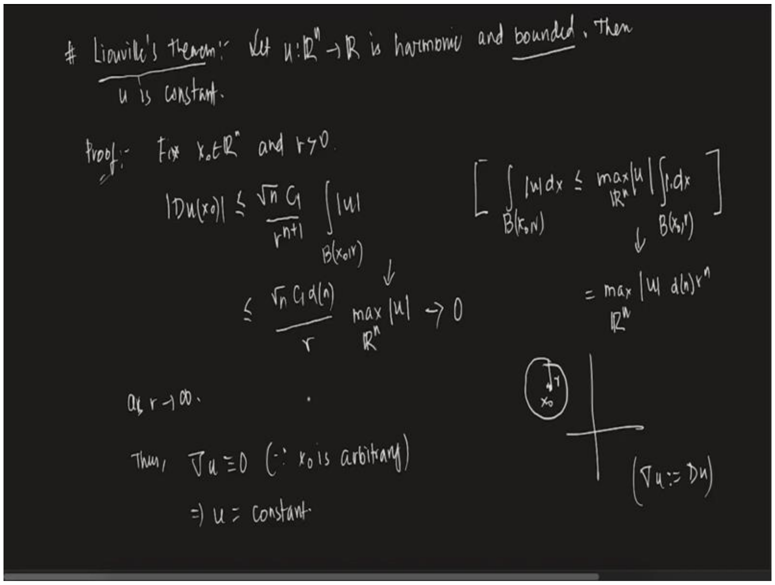

Now, we show another property of harmonic function; this is the very important property and I am quite sure to all of you guys already know this property. This is called the Liouville's theorem. So, by from complex analysis you guys know that it is a function is allomorphic function, then that real and imaginary part all are harmonic.

And you can show that there is Liouville's theorem complex analysis, which says that bounded in their functions are constant. So, exactly the same thing, but only in the real case; so what we are going to show that there is no non-trivial bounded harmonic function in whole of Rn. So, let u from Rn to R Rn to R is harmonic is harmonic and bounded.

So, I am starting with the bounded functions which is harmonic in Rn. Then u is constant, so quite clear. How do I show this? So, for the proof what I do is, you fix x naught in Rn, x naught in Rn; and r positive. Once you do that can we say something like this Du at the point x naught, let look at this thing. This is less than equals to root n C1 by r to the power n plus 1, integral mod u over B x naught r. Can I say this if you remember? See the derivative of u at point x naught can always be bounded by this particular thing; so that is what I am writing here. So, essentially this is always bounded by some constant 1 by r to the power n plus 1; the integral of u over ball.

So, this is one thing I want to clarify whenever I am writing some function, the L1 norm of some function. It is a shorthand of writing this; this is just a shorthand. So, I am getting it from the earlier, so from the earlier theorem we know that the D of u of x naught can be written like this; now, you see this says everything.

Once you do this thing, you can write it if you view on the ball; it has maxima on the ball. You can take some maxima on the ball; you can see u is bounded in whole of Rn; so, the maximum of u in the whole of Rn exist. So, essentially what I can do? I can do this; I can write it as root n C1 alpha n by r. I will explain to you why I can write it like this.

Maximum of mod u in Rn, first thing first; let us let us understand how we can write all of this. B x naught, r mod of u dx, this first of all; you are evaluating on this ball; I can always dominate this thing with the maximum of u over Rn. Here I know that u is bounded in Rn, so I am taking that bounded here, maximum the upper bound; and then I left out with d of x over B x naught r. And this is what it is maximum of sorry this is mod u maximum of mod u over Rn. Here, here what is the integral over the ball of D of x1, integral of 1 over ball; so, it basically the volume of the ball.

So, this is alpha n, r power n; that is what this is equals to this. I am just putting it here that is why that R power n is gone; and alpha in this is getting carried here. So, this is this is how this is some; I hope this is very here. Now, please understand this thing that I can say that this particular thing, it is this term goes to 0 as R goes to infinity.

Here I do not have any restrictions, this ball this ball this is our whole Rn whole Rn; that is the x naught. I am taking a ball with radius r with radius r and in that ball this happening. The gradient is always dominated by this. Now, you this is always six number, I cannot always change this number; all of this is C this is constant.

The only thing which can varies is r, because I can take r as big as possible. I am working in the whole Rn and this bounded independent of r; this bounded on Rn, I am working on whole Rn; I can take whole R as big as possible. So, if you take R towards infinity, all of this is constant; I am what is going to happen? This will go to 0.

So, essentially what is in this? In Rn in the whole space the gradient of u is going to be 0. So, thus gradient of u is identically equals to 0, why? Since x naught is arbitrary. Since x naught is arbitrary, see x naught is nothing special is there about x naught; you can choose any x naught. And you can choose as bigger ball as possible with this (())(55:14) no problem.

So, essentially that is saying gradient of u is 0, gradient of u is D of u; we are always using this notation, it is D of u, D of 1 is essentially. So, gradient of u is 0, what happens if gradient of u is 0? It means u is constant; it is constant, and we are done. So, this is the Liouville's property, you are already know Liouville's property and this is proof is in such a easy way.

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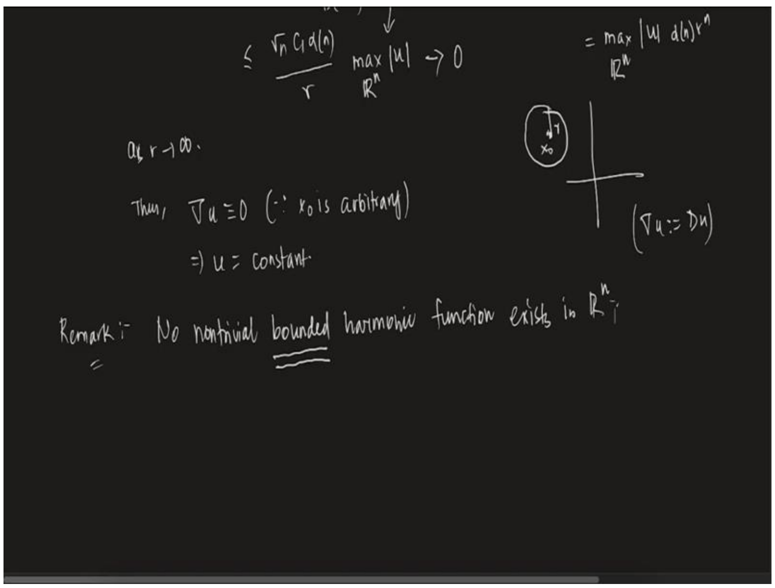

So, what it is saying, so let us a small remark; let me put this in a small remark and will end it here. It is saying that no non-trivial bounded harmonic function exists in Rn. So, the important thing is bounded harmonic function; there are obviously bounded harmonic functions in Rn. But there is no non-trivial, it can be constant; but no non-trivial, bounded harmonic function in whole of Rn. So, with this we are going to end this lecture.