Advanced Partial Differential Equations
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Lecture – 10
Strong Maximum Principle

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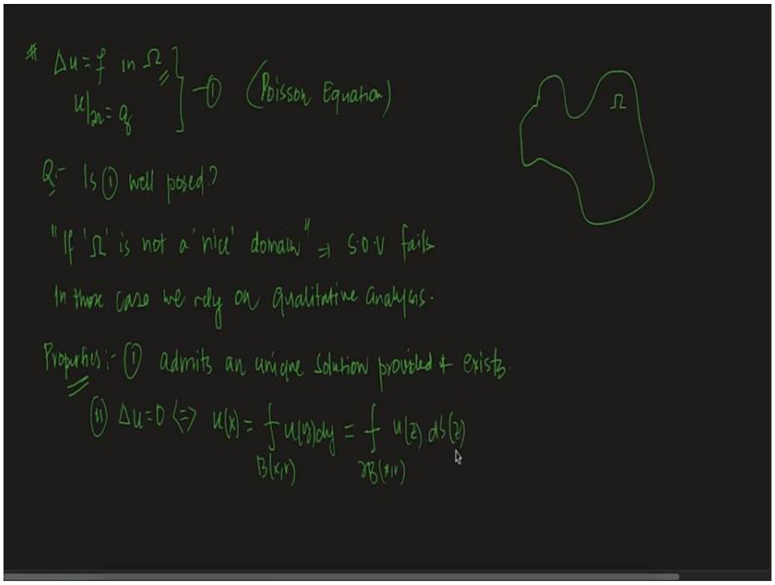

Welcome to this lecture; in today's lecture we are going to talk about some applications of mean value (theorem). So, essentially what we are doing is this; let us do just a small revision. We had this equation the Laplacian of this equation is equal to 0; this is simple harmonic it is called the Laplace equation; the solutions are called harmonic functions. So, let us this is in omega and u restricted to the boundary is some function g. Now, the question was this, so the question was is it well posed? Or maybe I can do that more general problem. Let say take a Poisson equation, let say f. Now, the question is, is 1 well posed well posed?

Now, you have more or less some idea, of what to do. See, essentially this omega is can be any omega; it is may not be particularly a rectangle or a circle or sphere that is sort of thing, so it is not a very nice. So, if let me put it in this way, if omega is not a nice domain; you do understand what I mean by nice. So, you know a usual domain, not a nice domain; then what happen? Separation of variable fails. So, essentially, I give you a crazy looking domain, I mean something like this let say something like this. Separation of variable, you cannot use separation of variable here; so, what do you do then? Let see the point is this.

For this sort of domain for there are most of the domains, which you can think of separation of variable will not work. And essentially what do you do, in those cases in those cases, we rely on we rely on your qualitative analysis qualitative analysis. So, let me give you a small remark here, you may have heard that Green's function; you can actually construct as Green's function. And you can use it to solve this sort of Poisson equation; so, this is called the Poisson equation if you remember. And so, you may have heard about it; we did not talk about this in this course. But you may have heard that there are functions called as Green's function, which you can use to solve this problem.

Of course, there are and for arbitrary domain not any arbitrary domain, but smooth arbitrary domain; we will talk about it later, what are the domains we are we can. But, for generally most of the domains we can actually construct Green's functions; you can show that there exist Green's functions which solve the Poisson equation that can be done. The problem is this that Green's function you can only show that that exist; you cannot actually find it to this function. So, here also the same thing is happening, you cannot find an explicit solution of this equation in any case. So, what do you do then?

So, in this case what we are doing? Without solving the equation, we are trying to find some properties of this solution. So, let say any this problem has some solutions; I want to find some properties of this solution. We have seen what are the properties we have seen till now properties? We have seen that this 1 admits a unique solution provided it exists. So, if you essentially see did not prove that there are solutions; but if you can show that there are solutions. Then we have showed that the problem admits a unique solution; this will remember we did in the earlier week. And moreover, for this problem for Laplacian u equals to 0; so harmonic functions.

If a function is harmonic, so any solution of this equation; what does it do? It satisfies this, mean value property. So, f over B x, r u of y dy. Please do not underestimate this property. This is probably B is most fundamental property; you can think of some particular class of functions, so u of z ds z.

So, this is the mean value of property; this we have proved in last class. So, is now what we are going to do is I am going to show some more properties of this harmonic function. Once you understand harmonic functions for Poisson equation, dealing with Poisson equation is much easier; so, we will only concentrate on harmonic function for now.

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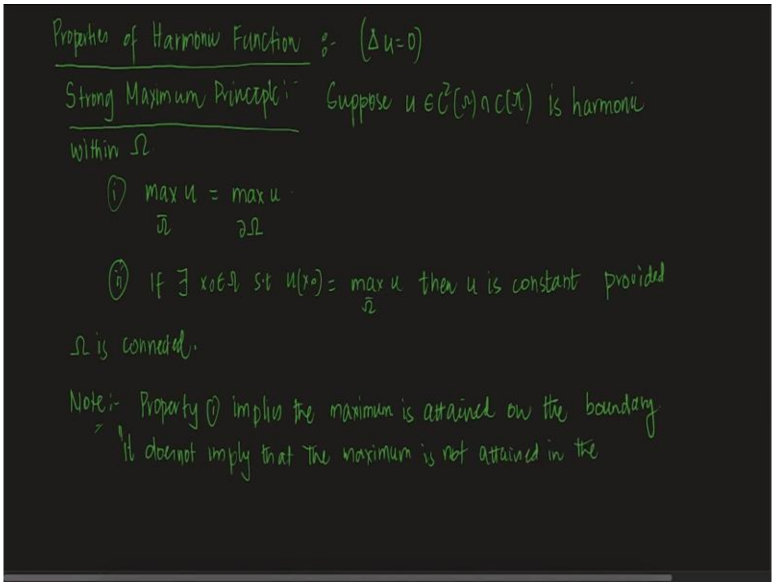

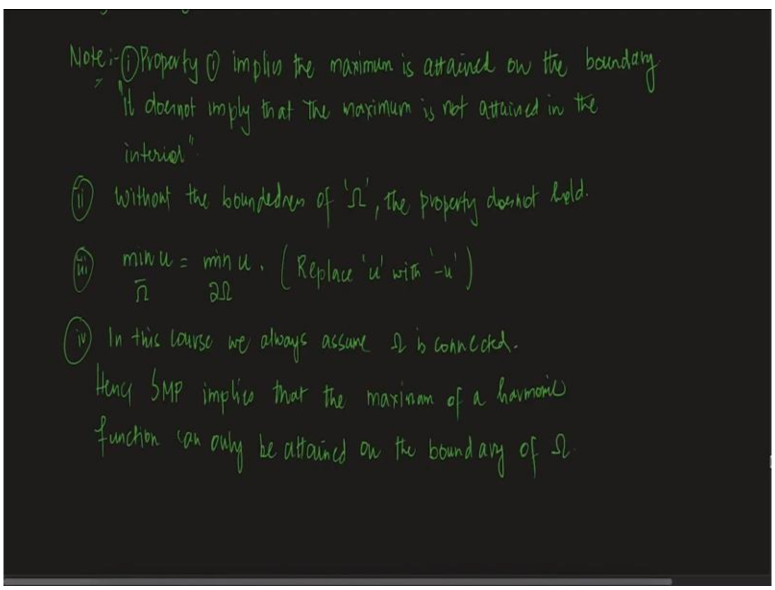

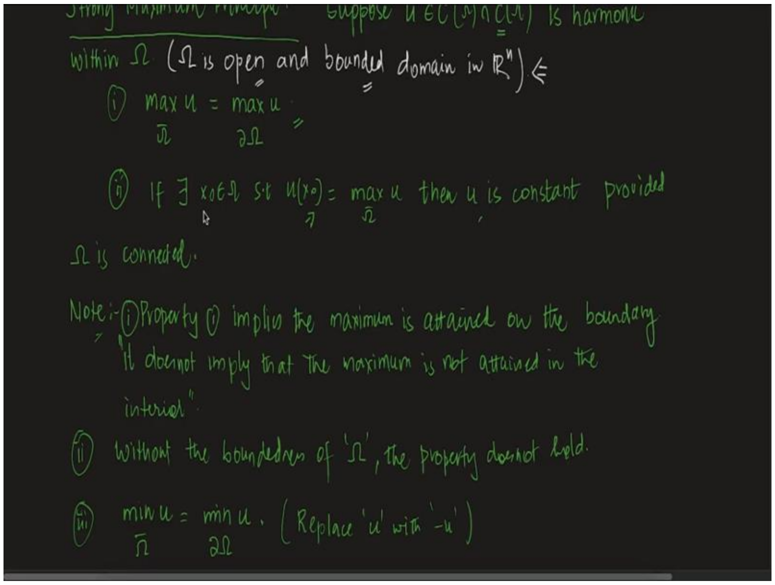

So, what we are doing is we are going to prove some properties of harmonic functions. If you remember what are harmonic functions, these are functions for which Laplacian of u is equals to 0; so, properties of harmonic functions. So, the first important property very very important property is called the strong maximum principle strong maximum principle; so, what it says is this. It says that suppose u is in c2 omega intersection c omega bar. So, what we are doing is we are saying that the function, let say that is your omega; that is your omega where we are talking about.

And so essentially here Laplacian of u is 0, so we are looking for properties of u. See, we have not solved this problem; of course you can find some solutions but not everyone. But the point is this, let say any u is there which in this Laplacian we can show Laplacian equals to 0. I am writing down some properties of those in some domain; so that is your omega. Now, we are using u to be c2; so, inside the double derivative exist and they are continuous inside. What about those on the boundary? Boundary double derivative may not exist, but boundary it actually coincides with the continuous function, c omega bar.

So, let say this is harmonic, harmonic within omega; so this is what we are assuming. Now, let me give you this, and then we will talk about that remark. First of all, what happens is if this happens, you can say that the maximum of u over omega bar equals to the maximum of u over the boundary. And number 2, what you can do? Is you can show that if here exist f x naught in omega, such that u of x naught is equals to the maximum of u over omega bar. So, essentially what it is saying is the maximum of u is attained anywhere in the interior of the domain; then u is constant then u is constant provided omega is connected.

So, what exactly is this saying, let us understand first theorem. This is a very very important theorem; the proof is not very difficult. Once you prove the mean value property, but I mean later understand what is being this saying. In the first assumption, we have not assumed that the omega is connected, we did not assume that.

So, what we are saying is the maximum of u over omega bar is equals to the maximum of u over the boundary. So, basically what it is saying is the maximum u attains its maximum on the boundary. If you see that the first thing note, property 1 implies implies the maximum is attained on the boundary.

Property 1 does not say anything about the interior, it may happened that it is also there is a point where there in the interior also taking maximum. But you can always guarantee that there is a point this is the t; one is not saying that t cannot attained the maximum in the interior, please understand this. It is saying that you can always find the point on the boundary of the domain, where the maximum is attaining; this is what we are saying. It may very well be happening, so it says it does not implies, that the maximum is not attained in the, let say it not attained in the interior.

So, it is not saying that the maximum cannot be attaining the interior; it may be, but it is saying that that you do not know whether that happens or not. All we know is it is always attaining on the boundary; this is being the first property note. The second property is this is very important; here I did not assuming anything on omega, so important thing.

So, let me put it I do not know maybe let us put it in quite, omega is open and bounded domain in Rn. Here, please understand for a strong maximum principle to hold, this has to be opened and bounded. This is very important why? Because we see if it is open, omega bar is closed; and it is a bounded domain.

So, you are looking at a continuous function on a closed bounded domain, which is a compact set. So, continuous function on a compact set attains its maxima; so that is why this maxima is here. Do you understand what are things? It is try to understand this thing; without this condition this is very important. This condition you cannot guarantee that this happens; because it cannot even guarantee that the maximum is attained. So, let me put it this way without the boundedness boundedness of omega without the boundedness of omega; the property does not hold, so that is true.

Now, let us give you an example, so let me give you an example; I will. Before that there is another property which we want to talk about, number 3. See, here I am saying the maximum is attained on the boundary that is what I am saying. I am not saying it is not attaining on the interior; I am still talking about 1.

You can also say that the minimum, see u is a continuous function; u is a continuous function on a closed bounded set. U omega is open, u omega bar is closed; so it is a continuous function on a closed and bounded domain. So, it attains a maxima and the minima, now the question is this, is the maxima is attained on the boundary; what does the minima is attained?

So, the minimum of u over omega bar on the whole domain including the boundary omega bar, is attaining on minimum of u over the boundary; this is also true. So, what you can say is the maxima and the minima is always attained on the boundary. How do we prove it? Just replace u with minus u. u satisfies the harmonic equation sorry the Laplace equation; so minus u also satisfies the Laplace equation. So, the Laplace equation the set of the space of the solutions are it is a vector space. So, minus u is also satisfy this and you can just replace u with minus u; and we can say it is a maximum can be replaced with a minimum.

Now, number 2, this is what is so important about the property 2? Property 2 is important because what is says is this; it connects the whole thing. So, it is saying that once you understand that omega is connected most of the times; so please remember this thing I am not going to repeat this again. In this course in this course essentially we always assume omega is connected; this is always assumed. But generally why so? You may ask that one why I am putting this two conditions separately. This condition the second property, what does it say? It says that if there is a maximum.

If maximum is attained in the interior, what it is saying is, in the earlier case it is saying that the maximum is attained on the boundary; it does not say anything about the interior; it may or may not happen on the interior. The second property is saying that if there is a point what the maximum is attained; then the function has to be constant. And this is true, if the domain is connected; but for our course we are always assuming omega is connected. So, for our course we will always assume that the omega is connected. So, basically hence the strong maximum principle, I will write it like this SMP Strong Maximum Principle.

SMP implies implies that the maximum the maximum of a harmonic function can only be attained on the boundary of the domain, boundary of omega. So, clear is this clear, see unless the function is constant; is this clear. So, what it is saying? It is saying that it is saying if the maxima is attained on the somewhere in the interior; then the function is constant, provided that omega is connected. Now, our course we are always assuming omega is connected; so, connected or not is you always know that there is always the maxima on the boundary. And here it is saying if it is the interior point where the maxim is attained and then it is constant.

So, you can plus these two together and it says that the maximum of a harmonic function. The maximum of a harmonic function can only be attained on the boundary of the domain, until unless the function is constant. So, if we just show away that constant functions; all other harmonic functions have to attain.

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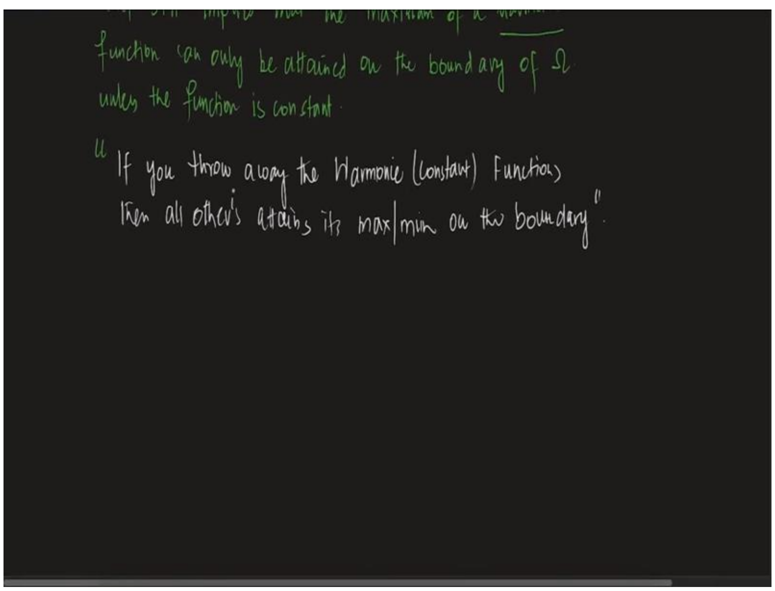

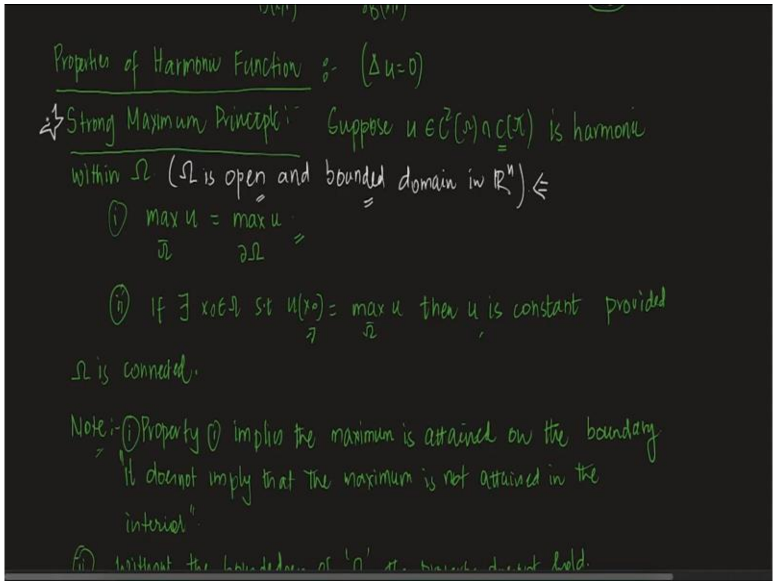

So, it says let me put it this, if you if you throw away the harmonic constant harmonic functions constant harmonic functions; then all others attains, its maxima plus minima, on the boundary. I hope this is clear and you may think that I have taking too much time explaining these things. This is the most fundamental property of c z; so mean value property which is always only satisfied by harmonic functions. This maximum principle we are using we are proving maximum principle using mean value theorem. But, maximum principle can be used; you can find maximum principle not only in Laplace equation, but also heat equation and we have heat equation also. So, let us write it down many other partial differential equation, so this is very important property; so, this a very very fundamental property. So, let me put it in a star kind of thing, so it is a star property, let us put it like this.

Very very important please remember this thing. So, this is strong maximum principle, here we are just doing it for a Laplace equation; later we will also used it for a heat equation also. But, for heat equation we will not do any not exactly the mean value property which we did. So, let us look at the proof of this.

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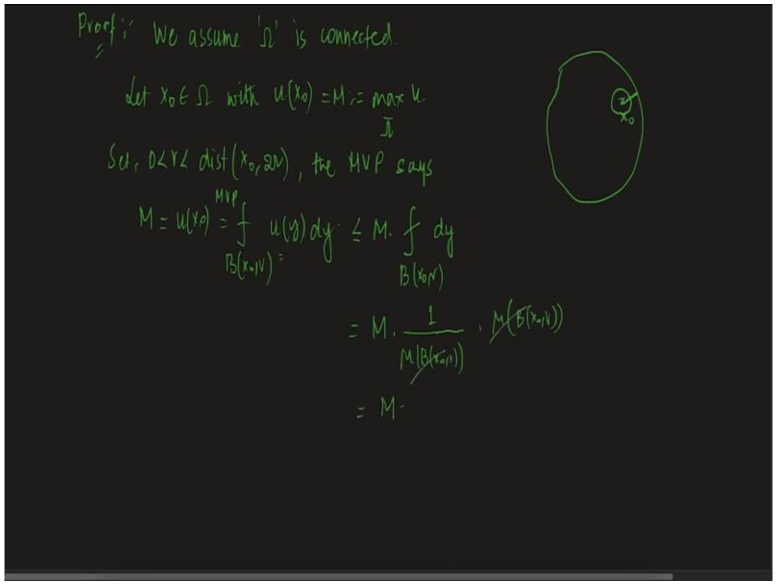

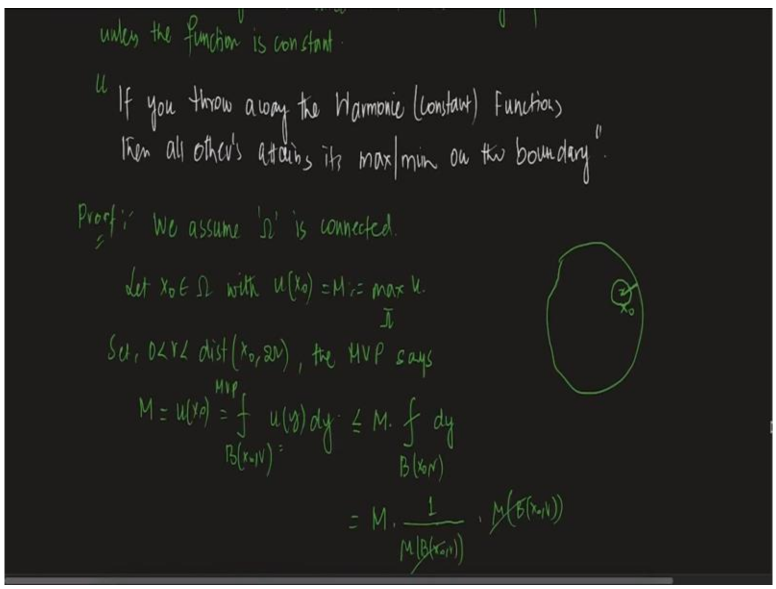

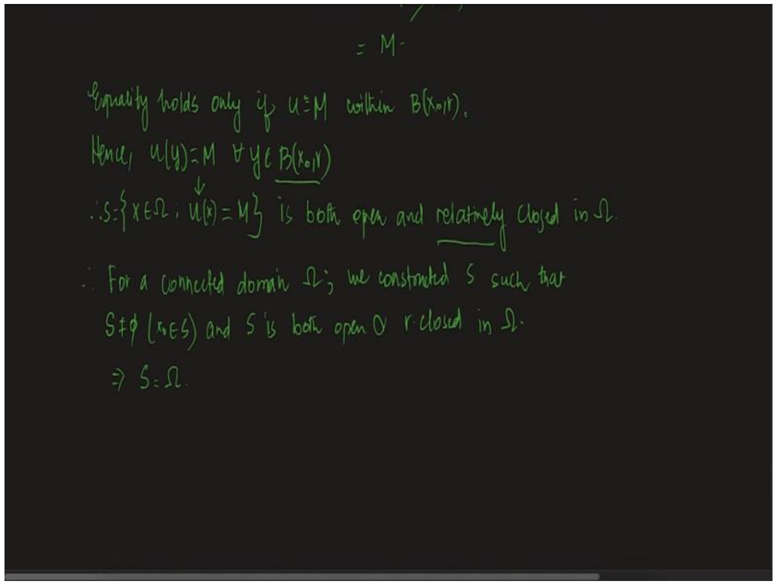

Proof, what is the proof of this is the proof is very easy. So, let us assume that so we are not we are not really interested in all this, you know omega is not connected, no no omega is connected. So, will assume we assume omega is connected omega is connected; and hence if omega is connected. What we are going to do is let will assume let x naught is in x naught is in omega with u of x naught, with u at the point x naught is M; which is given by the maximum of u over omega bar. So, what we are saying is there is a point in the interior; so essentially what do we have to show?

I just have to show that u is a continuous on omega bar; so definitely u attains its maximum. Maxima and minima is always attained, somewhere either it is on the boundary or in the interior. We have to show it does not attaining the interior that is all; and then definitely it has to be on the boundary. So, let us just assume that there is a point x naught, where it is taking the maxima. So, the maximum of u over where omega bar is M, where u of x naught u; that is the value of u at the point x naught. Now, what we are going to do is this set r, which is basically between; what we are going to do is this is something like this.

Let say x naught is some point here, the distance between x naught and boundary is this point. And we are choosing r which is like smaller than this; so, you can choose r like d by 2 let say, you can do that so that is your r. So, I am choosing a small ball, so essentially that ball will be on the domain like this. So, if that happens the mean value property the mean value property says

M; this is equals to u of x naught that is what we have to assume. And that equals to integral over B x naught, r u of y dy.

U is a harmonic function this holds for harmonic function; we want to show that the strong maximum principle holds for harmonic functions. So, you see u of y dy on the on a ball B x naught, r any ball; here r is a very small, you do not realize this thing. That is equals to u at point x naught, this is mean value property.

And this what happens what is the value of u on this ball? So, the maximum of u is always M; I do not care whether if it is this ball or some other ball, it is always them. So, I can always dominate it is M times B x naught, r dy; this we can always do. So, what is this? This is M and this is 1 by the volume of B x, r; so I will write it like this.

You guys whoever for, if you guys know measure theory; then this is the measure of the ball. If you do not know, do not worry about it; just think of this as a volume of a ball. So, when you do it like this that 1 by volume of ball; and integral of dy that is the again the volume of the B x naught, r. So, this gets cancelled out and this is equals to M; so this is clear. So, now what we are getting is we are going to get from here; that is integral of u over this ball; this is equals to M and less than equals to M. Thus, this so at this equality when does it holds? So, equality holds only u equals to M, within the x naught, r.

This is a point where u is less than M, then this I mean you can please show this. You can dominate that thing with this thing; so, this will be greater than equal that M minus epsilon let say, at the measure of the ball. This cannot happen, you cannot show this is equals to M; so please check this part that holds in only if u equals to M within B x, r. So, hence u of y is equals to M for all y in B x naught, r; is this clear? So, please check that u is equals to M, u has to be equals to M; so, you can show that there are no points u less than M. So, let say there are for some epsilon u is M minus epsilon on this ball. You can actually dominate that you can use this thing to dominate that particular thing.

So, basically this will be greater than M minus epsilon times that whole thing; so, please do that part, you can show that this is it cannot be equals to M in that way. So, u of y equals to M, this is clear; now, what is happening is this. So, therefore the set x in omega such that u of x equals to M; this set is both open and relatively relativly closed in omega. Let me explain what I mean by this thing. When I say it is relativly closed, of course this u is a closed set; sorry u is a continuous function and u is there looking at the values of u, level set of u, for the height M. So, this is in its own rights a closed set in Rn; it is a closed set in Rn.

But, at the intersection of it is relatively closed, because this is we are looking with the intersection of omega this set; so that is why it is relatively closed in omega. And why it is open because we have shown that if we take a point there inside this set; then you can always have a ball of radius r when u is m; so that is why it is open. So, basically you see here therefore for a connected domain for a connected domain omega; we constructed this set, this set this s let us call it S. S such that S is non-empty of course it is; because we have assumed that this is there is a point where interior where this is happening.

We have assumed that there is a point x naught, where M is attained; so essentially this x naught point is always in S, x naught is always in S. Further this happens and S is both open and relatively closed in omega; so, all of this is happening. And since omega is connected what can you say, this will imply that S is equals to omega. Because you guys is, already know that in a connected domain, if you have a subset; it goes open and closed. Then the subset has to be either c or the whole set; it cannot be phi, so it has to be the whole set omega.

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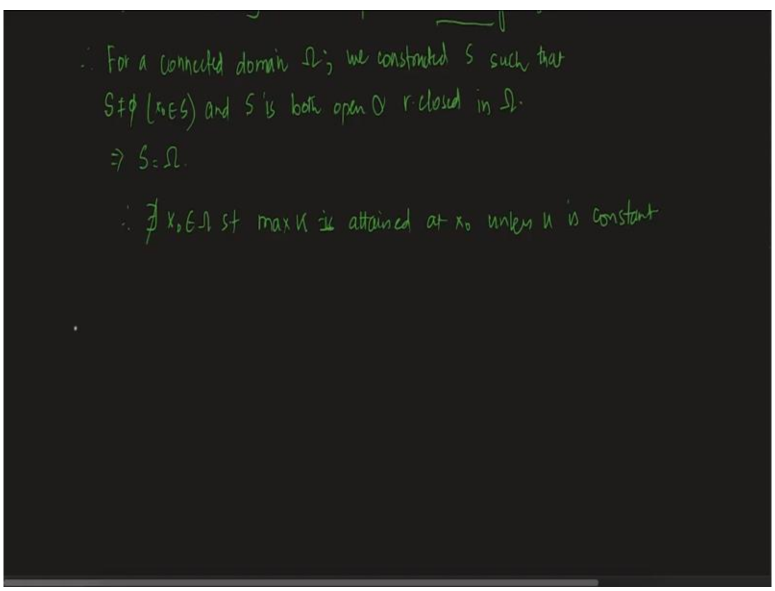

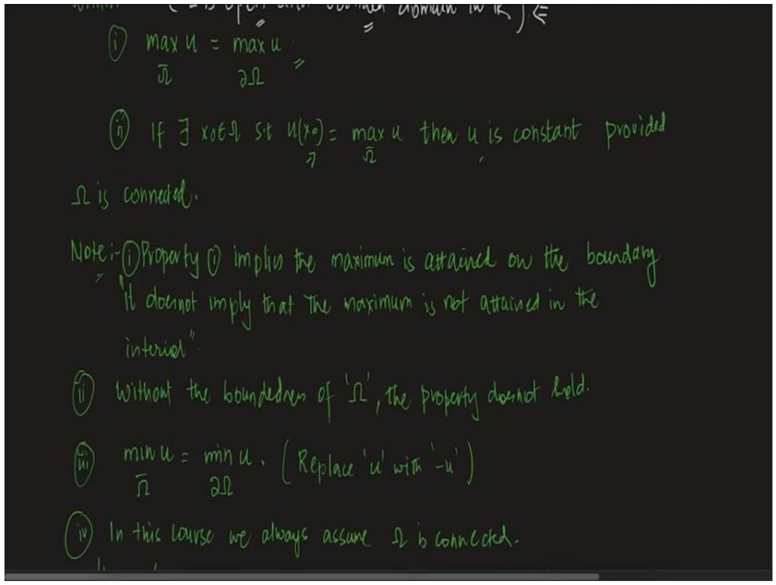

So, hence there cannot exist a point x naught in omega such that the maximum of u. So, what we have shown? We have showed that S is essentially omega; so basically we have showed that it is omega is the, we proved this second part. It proved that is there the minimum maximum is attained on in an interior; then it has to be constant that is what we showed. S equals to omega, it has to be constant; so basically u is equals to M is in on the whole domain omega, so maximum of u. These are does not, so we showed that there does not exist x naught in omega such that maximum of u is attained at x naught unless u is constant. So, now so this is this is very good, so we learned this thing; as I have again explained that this is also true, if we assume that if we replace maximum or the minimum; so, the same thing happens.

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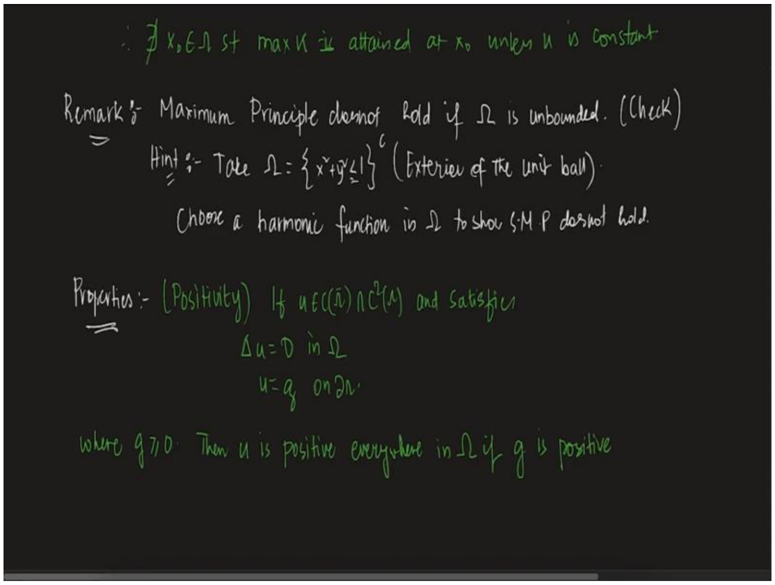

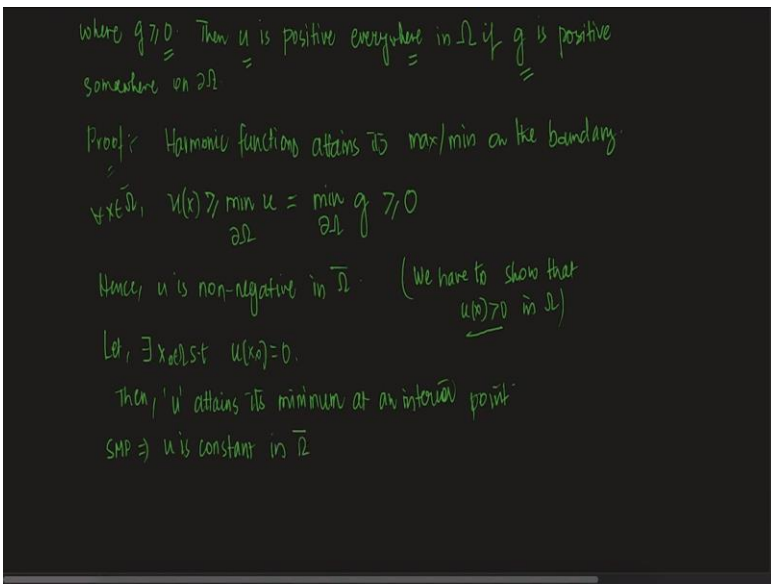

Now, let me make a small remark here, remark so remark. Maximum principle does not hold does not hold; whenever I say maximum, see there is nothing maximum here; which you can call it a minimum principle also, same thing. Maximum principle does not hold if omega is unbounded; so of course maximum may or may not be attained that is one point.

Even though maximum may be attained, then also it may not hold. So, let we have to check this part. It is a very simple example; take the exterior of the domain; so it will give some hint. I could have gave you the example, but you have to do it yourself. Hint: take omega to be the exterior of the domain x square plus y square less than 1 compliment of this thing. Take that to be your omega; so essentially what I am doing? I am taking the exterior of the unit ball. So, this is the exterior of the unit ball exterior of the unit ball; that is your domain. Now, choose a function choose a function harmonic function.

Now, choose a harmonic function, of course not a constant; harmonic function in omega to show maximum strong maximum principle does not hold. Please do it yourself, find a harmonic function; it does not for which there will be a similar problem in assignments like this. But, when you are doing this thing, this thing try to find.

Now, let me give you another property so of harmonic function. So, now we are going to use this maximum principle to show some other properties, so properties. This property is called positivity, so what does it says? It says that if u is in c omega bar intersection c2 omega; and satisfies satisfies Laplacian u equals to 0 in omega.

And u equals to g on the boundary; let say this it satisfies this. So, basically you are looking at a Laplace equation, but on the boundary it is g; where g is greater than equals 0. Then you can say u is positive everywhere in omega, if g is positive somewhere on the boundary. So, what it is saying is this, it is a very easy property. It says that you look at a Laplace equation and u equals to g on the boundary. Of course, it is given that g is greater than equal to 0, what it is saying is this please realize this thing. You take a point, if you can show that there is one point on the boundary, where g is positive.

It is saying that just showing one point on the boundary where g is positive; it is enough to show that u is positive everywhere in omega. So, this is the very important property you realize this thing; how can you do that? Take 5 seconds, think about it. So, I think it is most of you got it; so let me give you a short proof; there is nothing to prove but. So, let us look at the proof of this theorem; what it does it take to prove this thing. So, essentially you see this is a harmonic function harmonic functions attains its maxima plus minima on the boundary; that is what we learned boundary.

Now, g is greater than equal to 0, so u x for for all x on the closure of omega, u of x is greater than equals to the minimum of u on the boundary dell omega, which is definitely true. Because you see u of x is always greater than equal the minimum of u over omega bar. And that is minimum of u over the boundary; because the maximum is always attaining the on the boundary using the first property. So, now this is equals to the minimum of dell omega over g, because u is on the boundary; so, and g is always greater than equal 0. So, the minimum of a non-negative function is always non-negative.

So, u of x here there hence, hence u is non-negative in omega bar; now you have to show that if g is positive, somewhere u is positive everywhere. So, let there exist x naught such that u of x naught is 0 in omega; so x naught in omega such that this is 0. What I am doing is this I have to show that u is always strictly greater than 0 in omega.

So, this is what, so we have to show we have to show that u of x is strictly greater than 0 in omega; this is what we need to show. So, let say there is a point x naught where u of x is equals to 0. If that happens then what you can say is this; then u of x is always greater than equals 0 in omega bar. And there is a point where u is taking 0 in the interior.

So, again u attains its minima minimum at an interior point interior point; so, this is contrast about the statement. We are assuming that if u is; we have to show u is greater than equals to 0. Let say there is a point x naught in omega, where us is equals to 0. If that happens then u is in non-negative, it means that u is attaining minima which is 0, at a interior point x naught. What is that therefore strong maximum principle that will imply u is constant in omega bar; that is what strong maximum principle says in that case. So, u is constant in omega bar.

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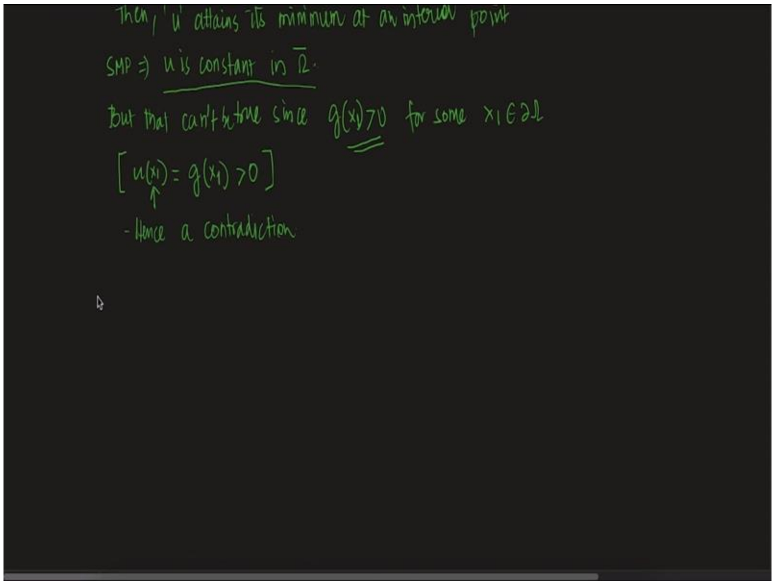

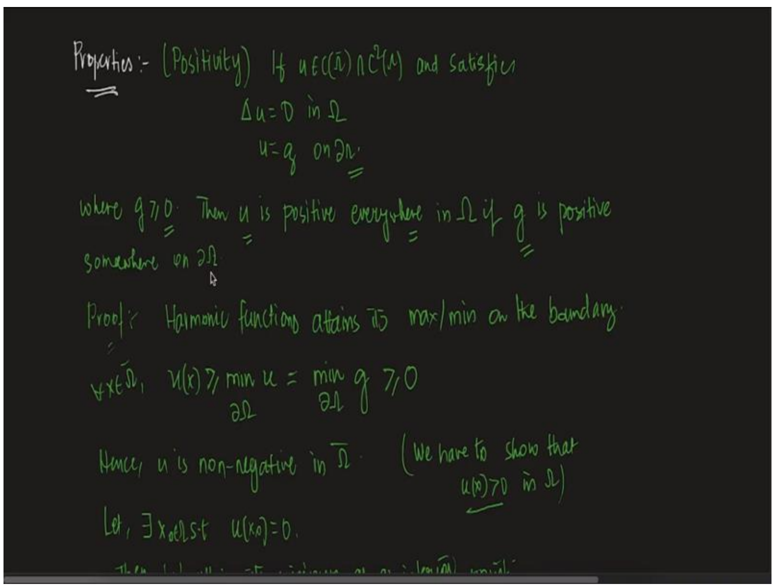

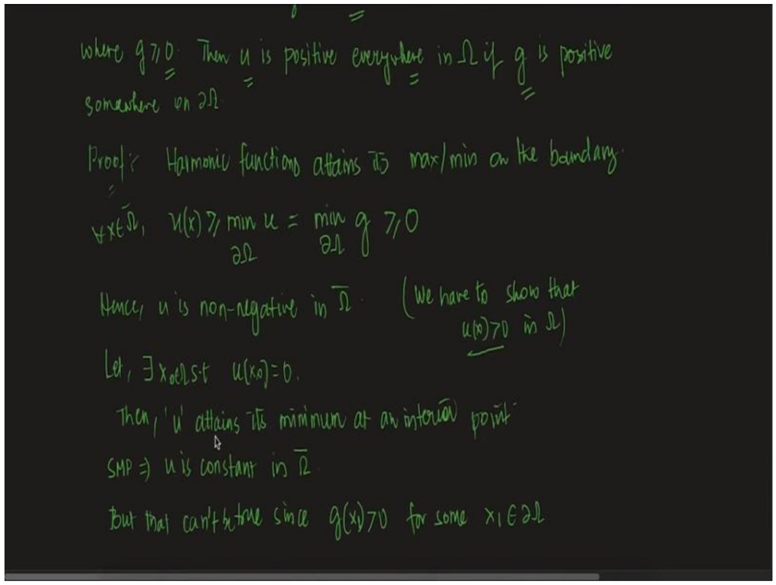

But, that is not true since that cannot be true that cannot be true; let me put it in this way. Cannot be true since g of x is greater than 0; let say g of x1 is greater than 0, for some x1 in the boundary. What is happening is this. We already know that g is positive somewhere on the boundary; so let say there is a point x1, where g is greater than equals greater than 0 strictly greater than 0. Now, if g is strictly greater than 0 at point x1 on the boundary; so why because if g is strictly greater than 0 on the boundary x1, then u at the point x1 is g at the point x1. Because u and g are same on the boundary; so, this is strictly greater than 0.

You are saying that u is constant on omega; yes, u is constant along in omega bar. There is a point on the boundary where u is positive; so, u has to be positive everywhere. But again you showed that on the there is a point on x1 where u is 0; so that is a contradiction, is it clear. What we did is this, let me explain again. We have to show that this is positive everywhere in omega, strictly positive; let say there is a point where u of x naught is 0 in the interior of the domain. If that happens strong maximum principle says what? It has to be constant in omega bar. If it is constant in omega bar, our condition is there is a point x1 on the boundary where g is positive.

If that happens what happens at the u at the point x1 is g at the point x1, which is positive. So, basically you showed the point x1 on the boundary where u is positive; and you have showed a point x naught in the interior, where u is 0. So, but you are always strong maximum principle is saying that u is constant in omega bar. So, that cannot happen a constant function cannot take 0 and 1; 0 at some point and 1 on that point, hence a contradiction. I hope this is fine contradiction and so we have proved that if g is positive somewhere at one point; then u is positive everywhere, very very important theorem.

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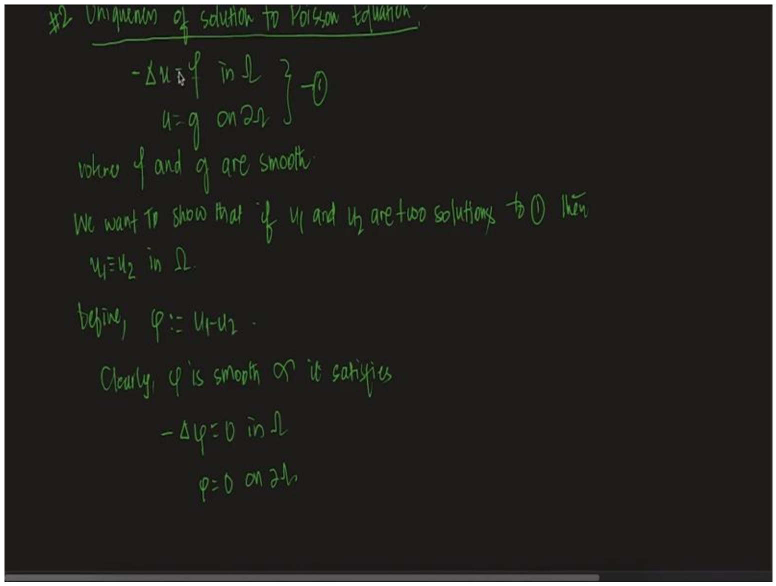

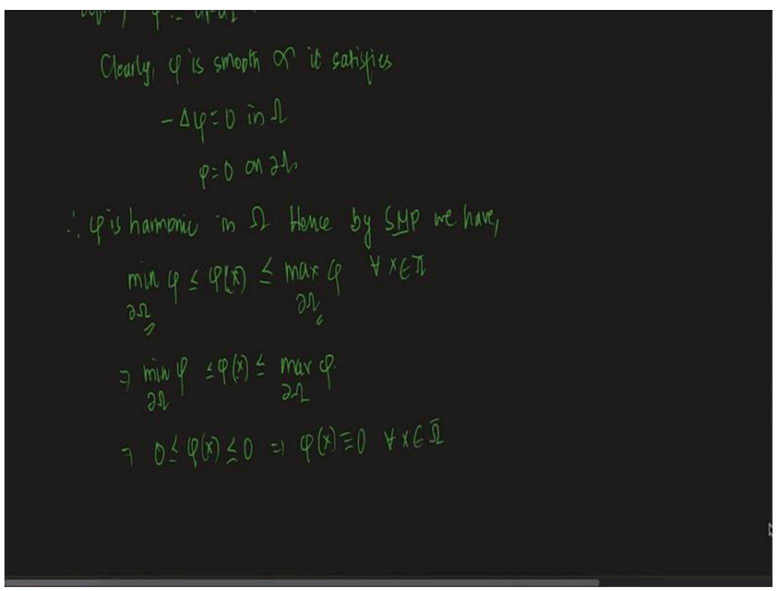

So, another very important property is the uniqueness of solution to Poisson equation. Here what I am doing is this, so this is property 2; that is your property 2. This uniqueness of solution for the Poisson equation what I mean by this is? Let say you are given this equation in omega and u equals g on the boundary. Of course, where f and g are smooth assume that; so, basically infinity differential let us assume this thing. We want to show that we want to show that if u1 and u2 are two solutions, distinct solutions, or different solutions to 1; then u1 is equivalent to u2 in omega.

So, if you remember last in one of the lectures, we have already covered this thing; that there is a uniqueness solution for this problem.

But now will do it again in a very, we proved earlier it easy in the integration by parts; here, will use the maximum principle to prove it. So, let say so define this is just an application of maximum principle that is all and nothing else. You just define phi to be u1 minus u2; so clearly phi is smooth and so phi is u2 in this case. If u1 and u2 are phi2, phi is u2 and it satisfies, this is, and it satisfies. So, let say let me put it like minus or plus; it looks nice, it just a technical thing, nothing problem no problem; you can use Laplacian also. But it satisfies minus Laplacian of phi is 0 in omega, and phi equals to 0 on the boundary. I hope this is fine.

So, phi is equals to u1 minus u2; u1 and u2 both satisfies this equation. So, the difference of those two will also satisfies this equation because of the linearity of the harmonic function. If that happens therefore phi is harmonic in omega; here what is happening is these functions are not harmonic. This is not a harmonic function, but since I am taking the difference of those two; so, the difference is harmonic.

So, phi is harmonic function such that phi is 0 on the boundary. So, what do we hence by strong maximum principle we have; one thing again I am saying this thing again and again, we will always assume omega is open, bounded and connected. This is always assumed; in this course we are always going to assume this thing omega is open, bounded and connected. So, my strong maximum principle what do you have? You have the minimum of u over del omega is less than equal; so over omega bar, but then that is also in the del omega. So, this is u of x, u of x is always lies between this; maximum of u over the boundary, I hope this is fine. U is always in between the maximum and the minimum of u over; so, this holds this holds for all x in omega bar. So, maximum of if you replace this thing with omega bar, this thing is omega bar.

So, u of x will lies, between the minimum of u over the omega bar and the maximum of phi over omega bar. Strong maximum principle says that the minimum of u over omega bar is on the boundary; and again, the maximum is also on the boundary. So, u is lies between minimum of u over the boundary and the maximum of u over the boundary. So, what is the minimum, so this means that this is basically minimum of phi over del omega, less than equal I think it is already done. So, maximum of phi over del omega; because u is phi on the del omega. Now, you see phi is identically equals to 0 on the boundary; so, this is less than equal u.

Sorry, I have to change, this is not u, we have to do it for phi; I am sorry for this thing, this is phi. Phi is harmonic so this is phi of x, maximum of phi; so, this is again phi and that is phi. So, what does it means? It means that phi of a x, it means that phi of x is greater than equals 0; and again it is less than equals 0. So, that will give you phi of x is identically equals to 0, for all x on the closure of omega.

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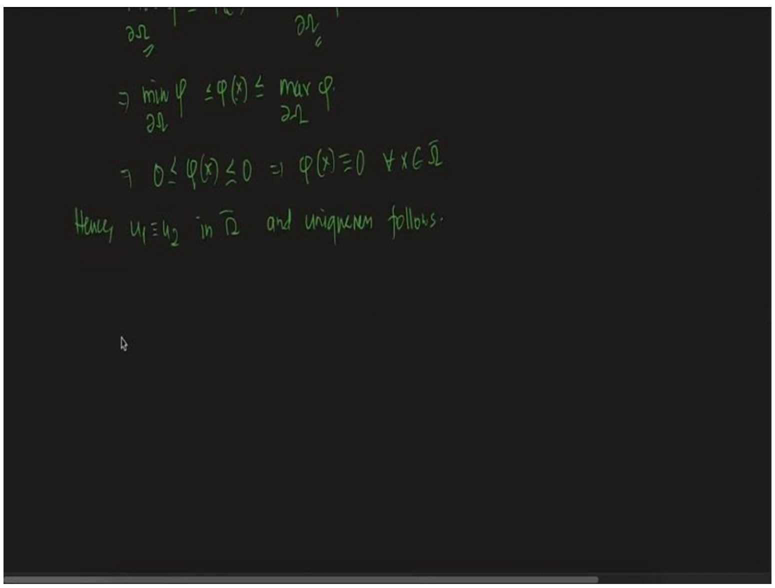

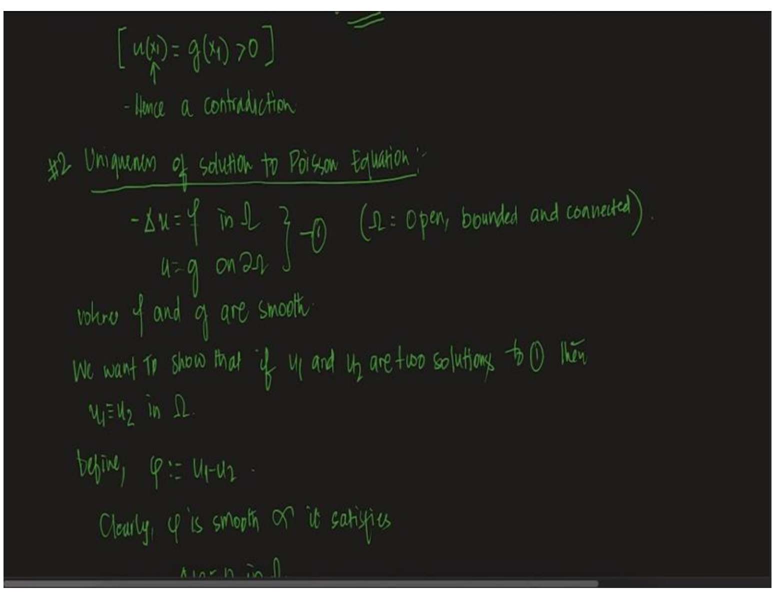

And hence u1 is identically equals to 0 is identically equals to u2 in omega bar and uniqueness follows. So, why I gave you this application of this strong maximum principle uniqueness, because we already did uniqueness; but again I am doing it, because I want you to understand that for other problems of the phi.

Most of the times other methods which I showed you using integration by part showing uniqueness that may or may not work. You understand what I am saying? That may or may not work all the time. Maximum principle is for operator for some particular equation, which can show maximum principle; then there is a good chance that this sort of equation will hold. I mean you can cut out the uniqueness provided the function is linear; function is the operated with linear.

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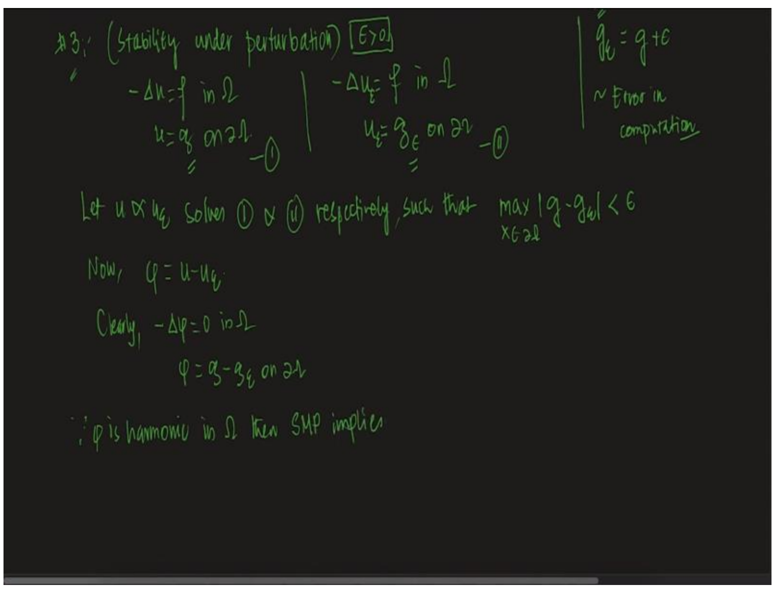

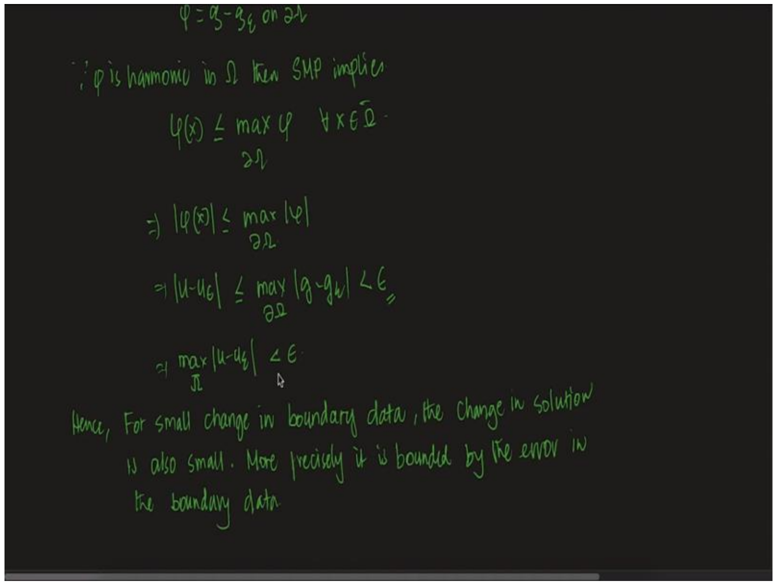

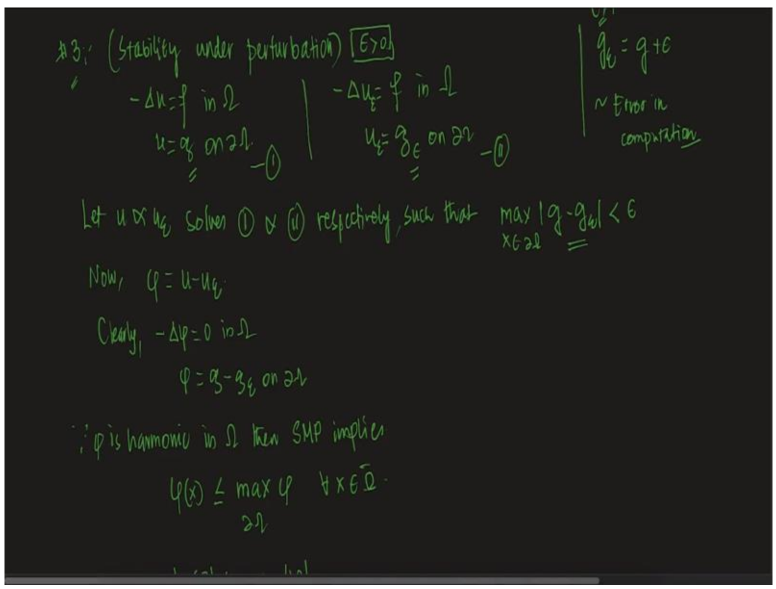

Let see give you another very important property number 3; so the stability under perturbation. What stability under perturbation says is this; says you have equation in omega and equals g on the boundary. So, this is the equation which you already had; so let say we have a phenomena, which you have somehow in a modeled; and you have got something like this and let say why measuring. The next time again you are doing the same thing; you guys know that experiments have to be carried out many times not to get a reliable data. So, again you are doing the same experiments but definitely there is difference in measurement; so, there is a slight change in measurement.

So, the source term is same source term is same; this is f in omega. But the data which you are going to get on the boundary that is different. So, let say this g is getting transferred to g epsilon on the boundary; so, we call it u epsilon, so this is an epsilon perturbation. You solved this problem 1, 2; let u and u epsilon solves 1 and 2 respectively.

So, what I am saying? I am saying that both u and epsilon solve the same problem in omega Laplacian. So, they satisfies minus Laplacian u equals to f and minus Laplacian u epsilon equals to f. u and u epsilon are both satisfies the same equation; but on the boundary u is g and u epsilon is g epsilon on the boundary, that is the difference.

So, g epsilon can be for example you can think of g epsilon to be, this for an example; it can be else also something else. So, think of this as g plus epsilon something like this; so, basically you are just taking a small perturbation. This is think of physically as the error in computation, think of this as error in computation.

So, basically you have this respectively such that now this error; you are measuring the same thing. But it may happened that the measurement is slightly off; so basically difference between g epsilon and g is very small, so such that the maximum of g minus g epsilon that is less than epsilon itself. So, which u epsilon positive, let say that is given to you.

So, the maximum and x epsilon del omega; so, if you take the maximum of g minus g epsilon that is very small. So, what I am doing? I am assuming that while doing this measurement on the boundary; the error is extremely small. So, basically the difference between g and g epsilon for every the maximum of that is always bounded by epsilon; we are assuming this thing. Now, phi you are defining it by u minus u epsilon. Clearly, minus Laplacian of u is 0 in omega, and phi is equals to g minus g epsilon on the boundary. So, what this is gives you is the following. U of x it means that the u, so by strong maximum principle this happens.

Phi is a harmonic function, so therefore since phi is harmonic in omega harmonic in omega; then strong maximum principle implies phi of x. This so what I am trying to say is this always less than equal the maximum of phi on the boundary, for all x in omega bar. That is the maximum principle; so it says that the maximum is attained on the boundary. That will imply that phi of x, so mod of phi of x; I can say this thing is less than equal maximum of mod phi over del omega, I can say this. So, you see what will happen is this mod phi of x is u minus u epsilon; this is less than equal maximum of del omega mod phi g minus g epsilon.

So, basically you see g minus g epsilon is on the boundary; so I am just replacing this with this. And this is given to be less than epsilon. So, what this said is this if you change the initial data, data on the boundary a little bit; the change in the solution is also going to be bounded by that similar error.

That implies that the maximum of mod u minus u epsilon, and over omega bar; this is less than epsilon. So, hence what does it says? For let me put it this way, for small change in boundary data boundary data the change in solution the change in solution in this change, the maximum of u minus u epsilon.

Change in solution is also small; more precisely more precisely it is bounded by the error; error is in epsilon error in the boundary data. I hope this is fine; you have understood, so what it is saying is this. If you change your boundary data a little bit and the change is given by maximum of g minus g epsilon is less than epsilon. If that happens then your initial solution the solution u minus u epsilon; what is change in the solution? That change is also bounded by epsilon. So, little change in boundary data also changes the solution a little bit; so that is expected. So, by doing this uniqueness theorem and the stability under perturbation, we have actually did, two things.

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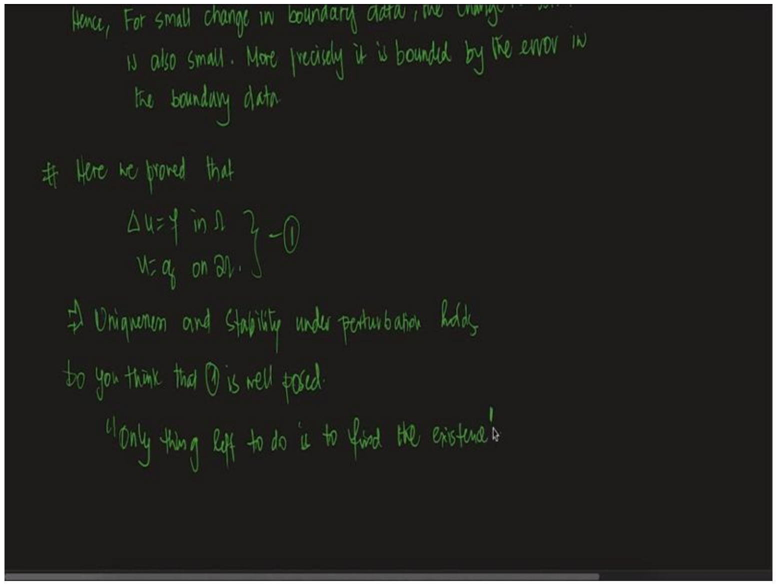

So, here we proved that Laplacian u in omega, u equals to g on the boundary, for this equation; that is 1. So, uniqueness and stability under perturbation under perturbation holds; so we have proved it. So, we proved this particular thing for this equation, for this equation. Now the question is this now the question is this, so do you think do you think that 1 is well posed? So, if you remember what are the properties of well posed. The first of all you have to show that the existence, uniqueness and then the stability. We have proved the uniqueness on stability; but we did not prove the existence.

So, essentially the only thing left only thing left to do is to find the existence; and this is the difficult question in this case. And this will do it in the next week, but this week we continue to talk about more properties of harmonic functions. So, with this we are going to end this particular lecture.