Advanced Partial Differential Equation
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Lecture 1
Preliminaries and Notations

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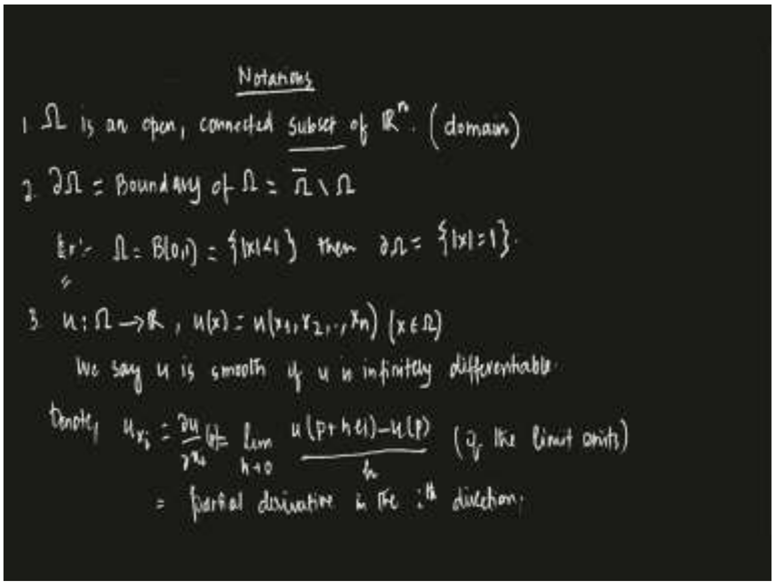

Today, we are going to talk about notation and some definitions which you are going to frequently use in this course. So, first of all let me begin by assuming that omega, I will write omega like this and this is an open and connected subset of Rn. So, n will be the dimension and omega, we will always have the, in this course whenever I am saying omega just assume that omega is open, connected subset of Rn.

So, definitely omega can be Rn, but generally speaking it is a subset of Rn and this open and connected. We also call this as the set as the domain, we also call it as a domain. Now, with this omega what we can do is we can define many simple notations and the first notation which I am going to use is, you see we are going to use Del omega, this is the first notation in the second notation, so basically we are talking about notations here.

So, Del omega, Del omega is denoted by the boundary of omega, so essentially what I mean is it, you take the closure of omega and subtract omega from it, so minus omega, so basically sets which are in omega bar, but not in omega that I am calling as a boundary. So, for example, let us say that if say omega is B 0, 1, so which is set of all those mod x less than 1, and then your boundary Del omega will be the set of all those mod x which is equals to 1.

So, that is an, this is the notation which I am using and we will also follow this thing. So let us say, we will say that u, so u, whenever I am writing this will be a function, this is different from omega to R. So, whenever I am writing omega, I am saying it is a subset of Rn open, connected set, so u for omega R, I will just write and this is given by this, so you see u of x, x I am taking from omega, so these actually looks like this, u, x1, x2, x of n and this x is the omega.

And we say u is smooth if u is infinitely differentiable. So, whenever we say infinity differentiable, we mean that you can, I mean derive this thing, you can take the derivative of this thing infinitely many times. Now, coming to that what we can also say is we denote u xi, we write xi to be Del u, Del xi, so this is the partial derivative of u in the ith direction. So this is essentially, I mean you guys already know this is limit x tends to 0 u of, so let us say this is at the point where some point p, so u of p plus h ei minus u of p by h. So, if the limit exists, so this is the partial derivative of u, partial derivative in the ith direction.

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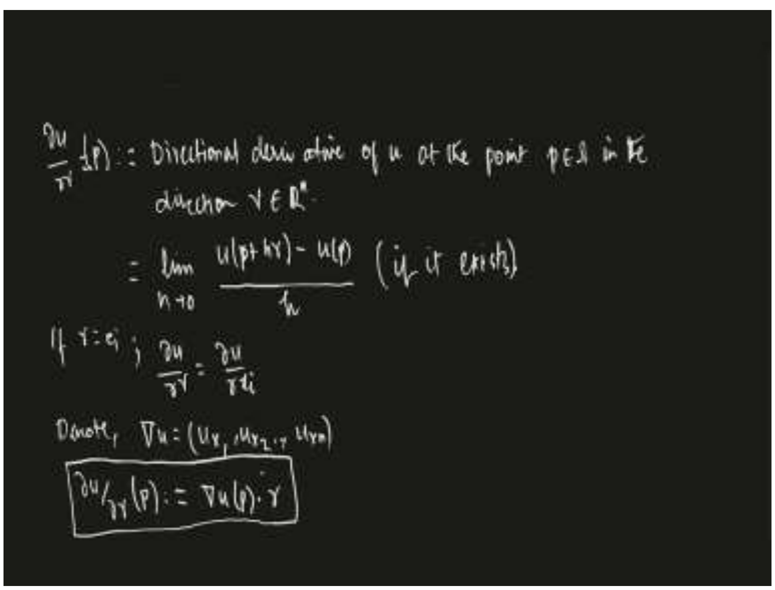

You can also define similarly Del u Del gamma, so this is the, let us set the point p, so this will which is the directional derivative, we call it a directional derivative of u at the point p in omega in the direction gamma, gamma is also a direction in Rn. So, essentially this, I mean, you can define it like this limit n tends to 0 u of x plus h gamma minus, this is at the point p, this p minus u of p by h, obviously if it exists, imagine that, so I mean the set of change of u in the direction gamma. So, if gamma is ei, so if gamma is ei then of course a Del u by Del gamma is Del u by Del xi.

And we also denote gradient of u to be u of x1, u of x2, u of xn. So, you take all the partial derivative together in a vector like this and that will be generate by gradient of u. With this notation Del u by Del gamma at the point p can be written as gradient of u at the point p dot gamma. So, this can be achieved using the definition like this. So, I mean you can also write Del u by Del gamma as this obviously, I mean, you need the regularity on u, but I mean we left in that for now.

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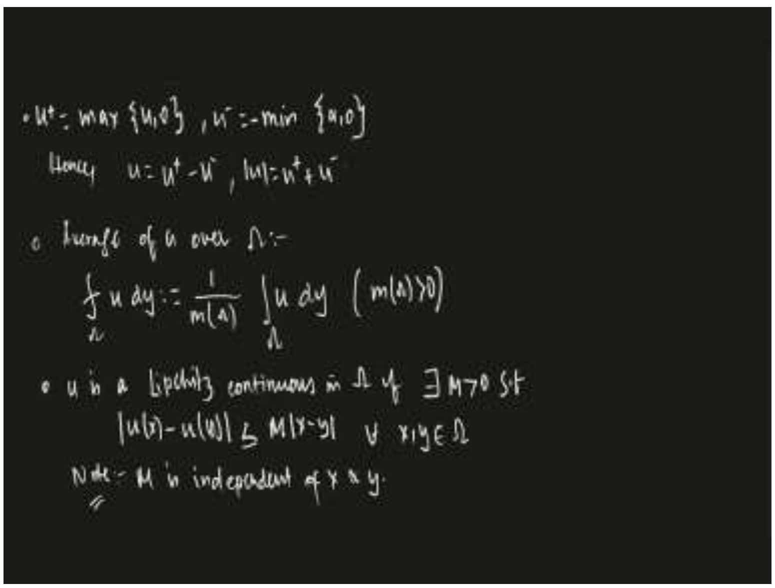

So, let us look at some other things, we define u plus, u plus is defined as the maximum of u and 0 and similarly u minus is defined as the minimum of, sorry, minus minimum of u and 0 and hence, u can write u to be u plus minus u minus, both of these are positive functions and it can be written as a product, sorry, sum of two positive function. Similarly, mod u will be u plus plus u minus, you can check this it is very easy check. So, this is another thing and we will also define average, so this is the average of u let us say over omega.

So, how we define, we write it like this, average of u over omega, so let us say this dy, so this, u of y, u of y, I am not writing of y, so this means that this is, you write the, let us say this is measure of omega, you take the integral of u dy over omega, take this whole integral and then divide it out with the measure of omega that will give you the average of u over omega. So, this is the average of u over omega. And obviously, I mean generally we assume that the measure of omega is positive in this case, otherwise I mean this will blow up.

And we also write, we say that u is if Lipchitz continuous in omega if, so basically Lipchitz continuity is not in a single point, but in a omega, so this is a non-local concept. If there exist M positive such that mod u x minus u y this is less than equal to M times mod x minus y for all x, y in omega and note this M is independent of x and y. So, this depends on omega, of course, but I mean, it does not, it can be less than equals to, it does not depend on x and y, so basically you can change x and y, but this M is not going to change.

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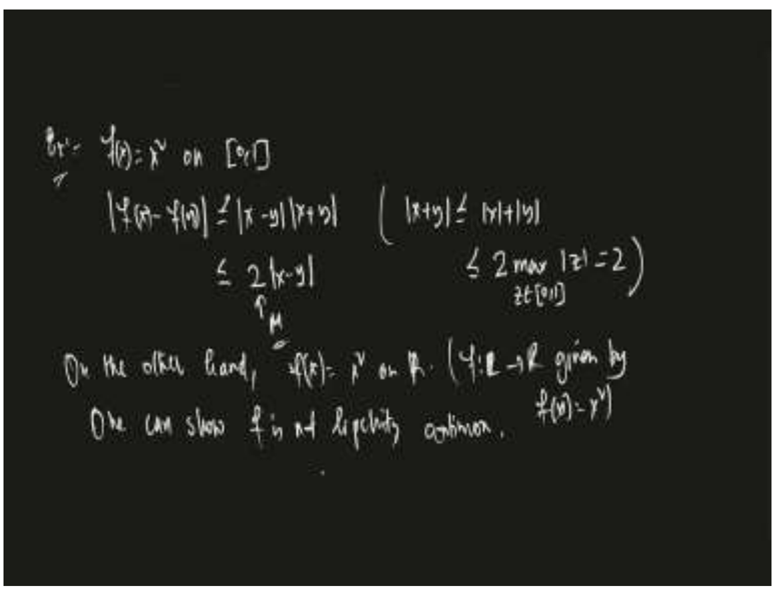

So, as an example, let us take an example, look at an example, example, you can take f of x to be x square on let us say 0, 1. Of course, you can see that f x minus f of y this is, can be written as less than equal x minus y times x plus y which is less than equal 2 times mod x minus y and this 2 this is your M here. So, this is efficient here. On the other hand you see if f of x is x square on R, so basically what I mean is x square given by f x is equal to x square.

So, basically x from R to R given by f x equals to x square then what happens is one can show f is not Lipchitz continuous. Because you know in that case you cannot get a bound on this thing here I am bounding, so here what we are doing is mod x plus y is less than equal to mod x plus mod yi. And this is less than equal two times the maximum of let us say mod z such that z is in close 0, 1 and this is definitely 1, so this is 2, so this becomes 2.

And that is why this is 2. But in this case what happens if you are (())(12:52) your domain to be R, I do not have a control on x and y, it can be as big as possible, that cannot be a M which dominates this whole thing for any of y. So, now we want to talk about notations for derivative.

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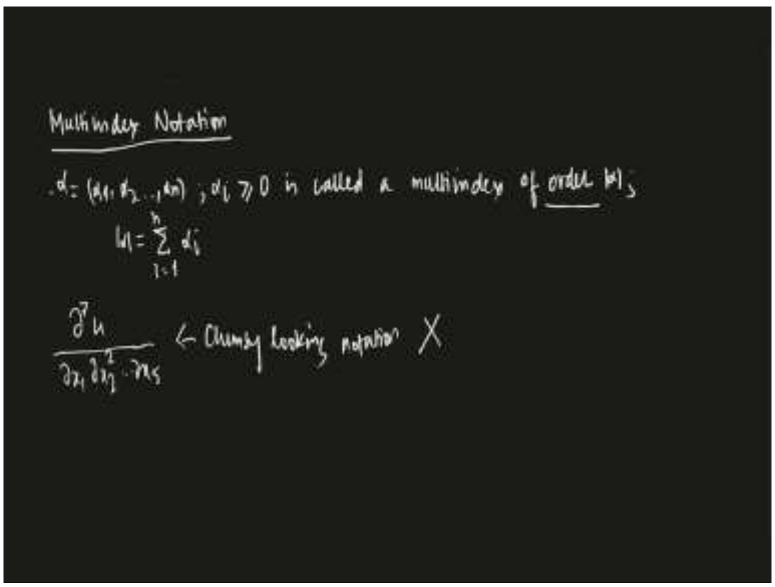

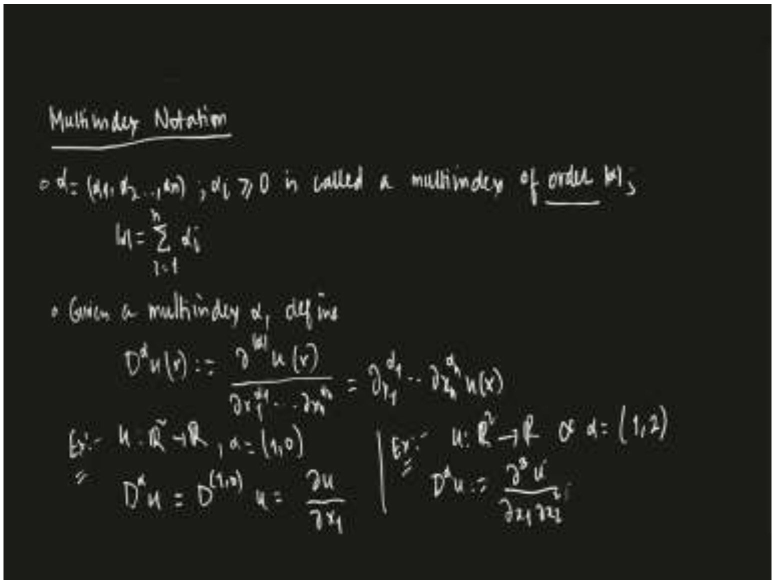

So, we in the last part we saw that you can define the partial derivative of u and now what happens is we define a sum new concept called a multi index notation. Now, what is the multi index notation? So, we say alpha equals to alpha 1, alpha 2, alpha n this is a vector, a vector of this form, so that alpha i is always greater than equal to 0 is called a multi index of order, whatever you want to call it let us mod alpha model mod alpha.

So mod alpha is summation alpha i, i equal to 1 to n, this is called a multi index. So essentially what we are talking about is any vector alpha is given by alpha 1, alpha 2, alpha n, but the condition is all the alpha must be non-negative and that is called a multi index essentially; a vector with this form, of this form and the order of the multi index is given mod alpha, where mod alpha is the sum of alpha.

Now, with this notation, the question is this, why do we suddenly introduce a notation like this? This is the introduced because let us say I want to talk something like this, so let us say Del 7, something u by Del x1, Del x2 square something like this 5, I want to talk to about this and I mean, you can understand that the more partial derivatives we have the more clumsy, so this is a very clumsy looking notation and you can think of it like let us say if I am changing 7 to 10 or 11, 12 this gets more bulkier and clumsier, we do not want this thing.

So, this is not recommended that is why we need to do something called a multi index notation. So, let us erase this thing and we come back to the multi index notation. What as we extend multi index is a vector which looks like this and with this we are going to find a new thing, so given a multi index and alpha, define D alpha of u x, this we defined as Del mod alpha of u x by x1 alpha1, xn alpha n, so essentially what happens is basically you are looking at the mod alpha partial derivative and x1 has to be alpha 1 times and xn has to be alpha n time.

So, this is all linked, alpha 1 is linked with x1, alpha 2 is linked to x2 and alpha n is linked with xn. So, basically the ordering of this vector is very important and we can also write it like this, Del x1 alpha 1 Del xn alpha n of u of x. So, that is your multi index. Let us take an example and see what it means. Let us say u is from R2 to R, u is form R2 to R and alpha is 1, 0, let us see what happens to D alpha of u.

So, you can say this is D of f 1, 0 of u, this should look like Del u because mod alpha is 1 plus 0, this 1 by Del C, the first 1 is for x1, so this is x1 and 0 is for Del x2 which is 0, so that is not there, so we are not going to write it, this Del u Del x1. Similarly, you can take another example, let us say u is from R2 to R and alpha is 1, 2, in this case D alpha of u, it should look like Del 1 plus 2 is 3, Del x1 Del x2 square, so it should look like this. So, that is your multi index notation.

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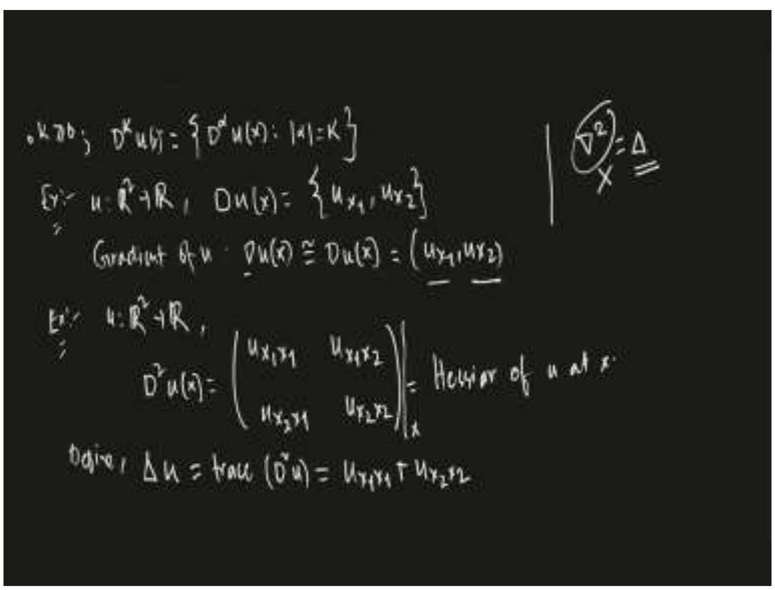

Now, we can continue with this thing, so let us say k is greater than equal to 0 and we can write Dk of u, so what is it? This is the set of all, this is the set of all D alpha of u x such that obviously this is very-very fair, I mean it does not matter but you can just write it like this mod alpha is equals to k. So, basically Dk of u is the alpha, so when the order of alpha multi index is k, all those partial derivatives we are looking at it and putting it in a set that is your Dk of u x.

So, for example let us say, for example u again, I am starting with R2 to R and Dk I am talking about, I want to write what is Du of x, Du of x in this set will be a set which contains u of x1, u of x2, it should contain like this. But generally what happens is we also write the gradient of u, if you remember it is gradient of u of x, we write it like gradient of u of x this is D u of x, and that is given by, we also write it like this, ux1, ux2.

So, you see, whenever we talk about gradient we write it like a vector ux1 and ux2, this is the vector and we identify these, this is an identification, we identify these it D of u which is like a set which contains u x1 and x2, this is an identification that is all. So, we can just think of it like this. And another example let us say u from R2 to R and I want to talk about the second derivative, so D2 of ux, how should it look like? It should look like ux1x1, ux1x2, ux2x1 and ux2x2. So, that is called the Hessian of u.

So, this is obviously evaluated at the point x, so this is Hessian of u at x. And you can actually define the most important operator which we are going to study in this course which is the Laplacian of u. We write it like this, so note in some books this is written like this, Laplacian is given by this, this notation is wrong notation, this is the right notation.

Laplacian of u, this can be defined as the trace of D2u, so this is essentially what I mean by this is x1x1 plus x2x2, so that is your Laplacian of u, u from R2 to R if it is given, D2u will be defined like the this, this is the Hessian matrix and Laplacian of u can be defined as the trace of the Hessian matrix which is given by this. Now, with this we are going to move on and define some other notations.

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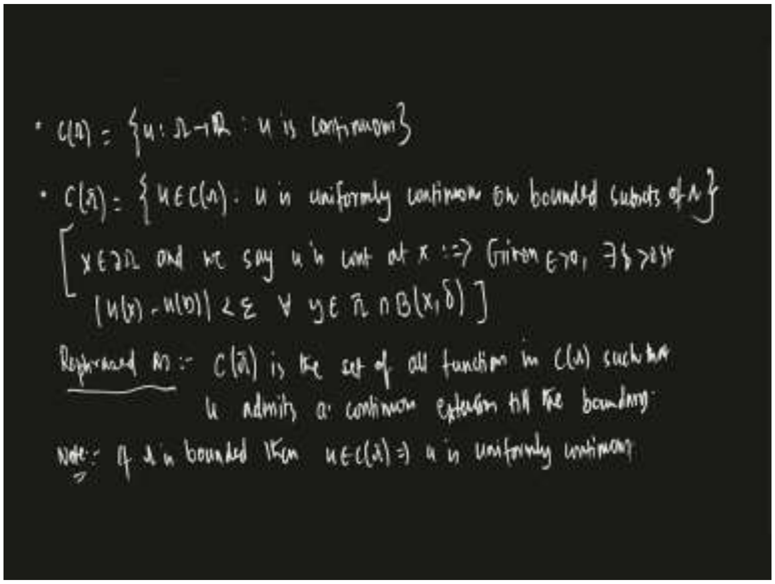

So, we are going to define something called the C omega. C omega at the set of all continuous functions, so this is you let us say from omega to R such that u is continuous. And similarly you can also define C omega bar is the set of all those u in C omega such that u is uniformly continuous on bounded subsets of omega. And what I mean by this is see, first of all let us understand what I mean by, so let us understand, let us say x is on the boundary, x is on the boundary of omega and we say u is continuous at x, what does that mean?

I want to, so what it means is it means that given epsilon greater than 0, there exists delta positive such that u of x minus u of y this can be less than, made less than epsilon for all y in omega bar intersection B x delta. So, essentially I mean if we just look at the ball with centre at x and radius delta and you take the intersection with the closer of omega for all those y's, ux and uy should be very close to each other. That is what I mean by a continuity of u at the point x, where x is on the boundary.

Now, when we say that u is in C omega bar what I mean is u is a function in C omega such that it can be continuously extended till the boundary. So, this can also be rephrased as u C of omega bar is the set of all functions in C omega such that u admits a continuous extension op to the boundary, till the boundary. So, essentially note what it means is, if omega is bounded then this is the set of all uniformly continuous functions. So, then u in C omega implies u is uniformly continuous.

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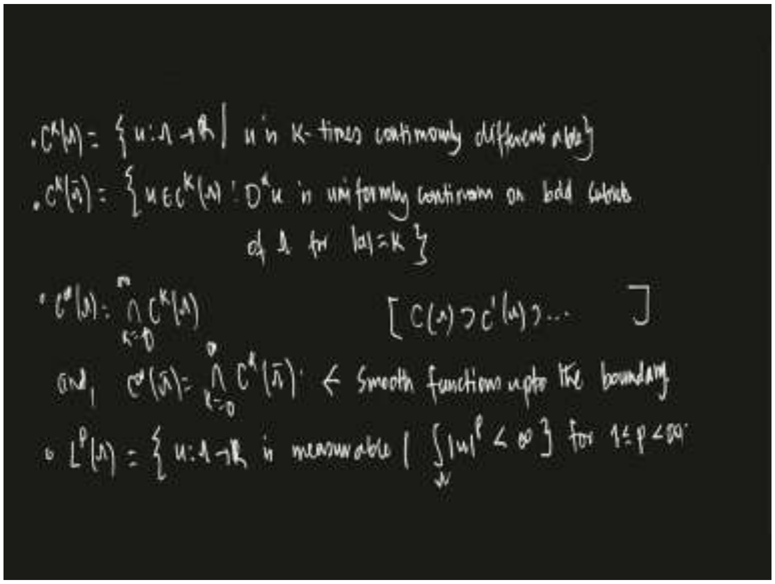

And similarly you can also define Ck of omega, Ck of omega is the set of all those u from omega to R such that u is k times continuously differentiable and Ck of omega bar is similarly u in Ck of omega such that Dk of u is uniformly continuum on bounded subset omega for mod alpha plus equals to k, so for all partial derivatives of kth order.

And of course we can also define we can also define C infinity of omega to be the intersection of Ck of omega k equal to 1 to infinity and C infinity of omega bar can define as intersection k equal sorry this is k equal to 0, 1, k equals to 0 to infinity, Ck of omega bar. So, essentially you see, of course, you can understand that C of omega contains C1 of omega and it goes on like this, this is a nested thing, nested subset.

So you take for the intersection of that thing and that will contain all functions which will (())(27:59) differentiable and you call all those functions as smooth functions, which is C infinity omega and C infinity omega bar as smooth function pick up to the boundary. So, these are smooth functions up to the boundary, clear.

We also define, this thing you guys already know, but let me again define it LP of omega, this we define as u from omega to R is measureable such that integral over omega mod u to the power p, this is less than infinity. So, basically you are taking all the functions, p (())(28:58) functions, for p greater than equal 1 but less than infinity. So, those are Lp function.

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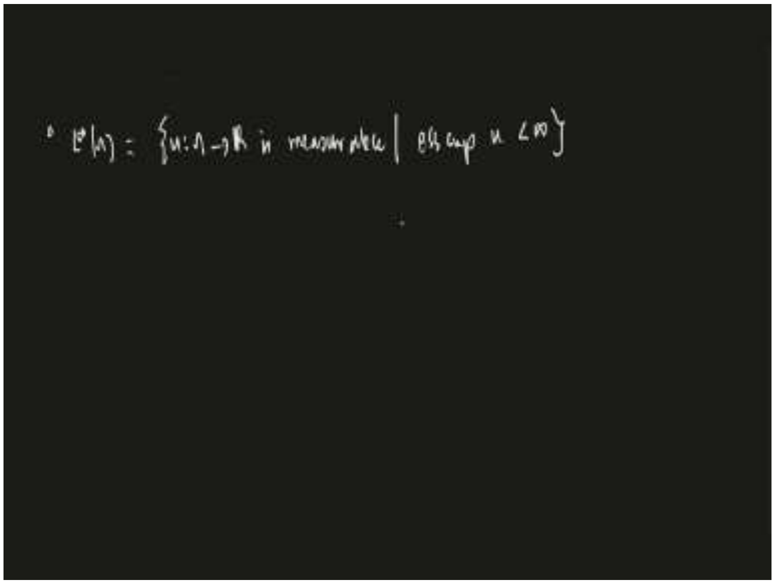

And we also defined similarly L infinity omega, so this is the set of all essentially bounded function, what I mean by this is, this is set of all those u from omega to R is measurable such that essential supremum of u is less than infinity. So, this is L infinity function.

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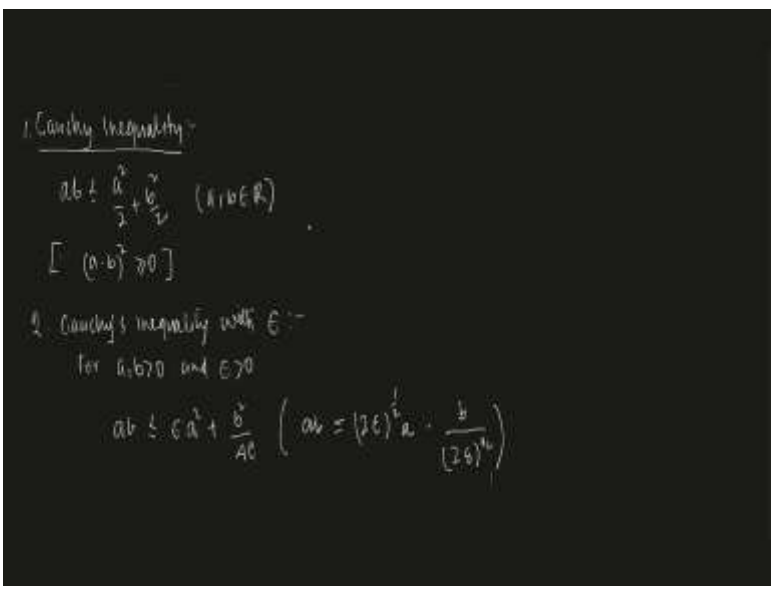

Now, we start with something called a Cauchy inequality and most of you know this inequality because this says a b is less than a square by 2 plus b square by 2, it is very easy to see that why this is true, this is true because we all know that this is (())(30:12) a, b, ER, why this is true? This is true since a minus b whole square if always is greater than equals 0 and that will give you this particular thing.

So, that is your Cauchy inequality is the most basic form and there is another form which is a Cauchy’s inequality with epsilon, what that says is for epsilon greater than 0 for a, b positive and epsilon greater than 0, one has a, b less than equals epsilon square plus b square by 4 epsilon.

This can be proved using this, so essentially what happens is you can use Cauchy’s inequality to prove this Cauchy inequality with epsilon, you just replace so a, b you see a, b, if I write it like this 2 epsilon to the power half times a and dot b by 2 epsilon to the power half. Once you do that you can see that if a, b can like this side and then you just put a square by 2 plus b square by 2 you get this epsilon inequality. So, that is Cauchy’s inequality with an epsilon.

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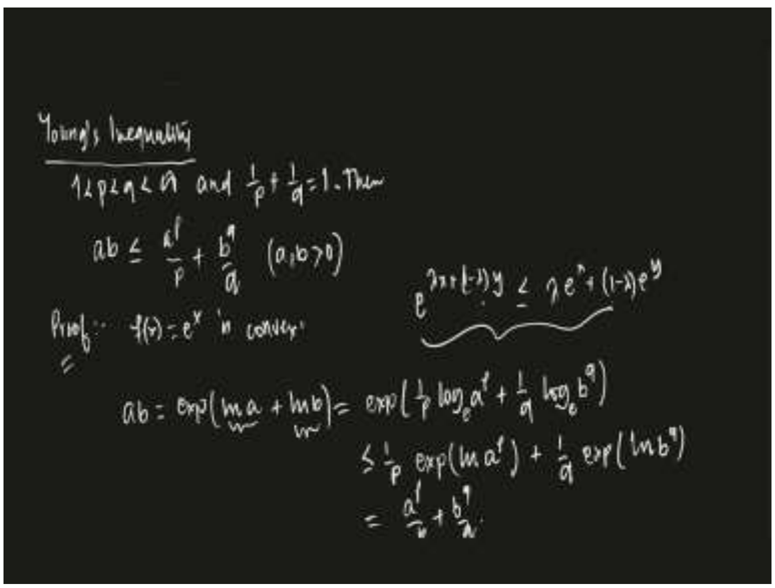

Now, let us look at another inequality, so this inequality is called Young’s quality. What is the Young’s inequality? It says that for p between 1 and infinity and 1 by p plus 1 by q is 1, then ab less than equal, so this is a to the power p by p plus b ot the power q by q and this is obviously we need a and b to be positive, this is very important, for ab positive we can write ab is less than equal a to the power p by p plus b over q by q.

So, let us look at a quick proof of this the proof involves the convexity of exponentially, so we know that f x equals to exponential x is convex, this convex, so we will use this property. So, what we do is we write, say ab, you can write it as exponential of log a plus log b and that can be written as exponential of 1 by p log a to the power p plus 1 by q log b to the power q.

So, these can be written as 1 by p exponential of log, so this is log base e this is base e, log a to the power p plus 1 by q exponential log b to the power q. So, this is true because we are using the convexity argument, so this is given by a to the power p by p plus b to the power q by q.

And you see why we are using ab positive because otherwise I cannot define this particular things, log of a plus log of b and this is the place where we are using a convex, so how are we using it? We are using it like this, e power lambda x plus 1 minus lambda y, so f of this is less than equal so lambda times exponential x plus 1 minus lambda times exponential y, this is what we get. So, that is your Cauchy's inequality.

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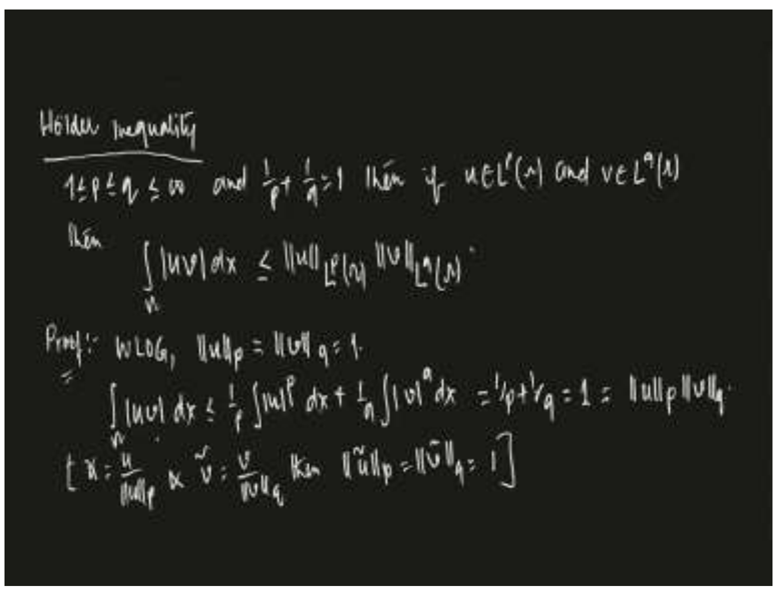

And in the next part what we are going to do is we are going to look at the Cauchy this is the Holder inequality. So, Holder inequality, so what is Holder inequality? What it says is let us say p and q between 1 and infinity and 1 by p plus 1 by q equals to 1, then if u is in Lp omega and v is in Lq omega then integral omega u v dx this can be dominated by norm of u Lp omega norm of v Lq omega.

Now, let us look at the proof of this thing, proof is quite simple, you can use homogeneity and we can assume without loss of generality, let us assume that norm of u p, norm of p q is 1, so let us just prove this thing for this u v. So for the unit u v, norm of u p equal to norm v q equals to 1 and then we can look at the other things. So, let us say this is the pair integral uv dx over omega, this is less than equal 1 by p integral mod u to the power p dx plus 1 by q integral mod v to the power q dx. And this is 1 this is 1 and then it becomes 1 by p plus 1 by q and that will be 1.

And this is norm u, p norm q, so if we this is to, so here I am using the you know Young's inequality and now, why, since can we use this things that norm u p and norm p q is 1, so because, if you take let us say u and v are any arbitrary functions in Lp and Lq, you can actually assume u tilde to be u by norm u p and v tilde to be v by norm vq and then u tilde p, v tilde q is 1. And for this u tilde and v tilde this is true, this inequality is true, so when you put it there, then you will get this inequality. So, that is your Holder inequality.

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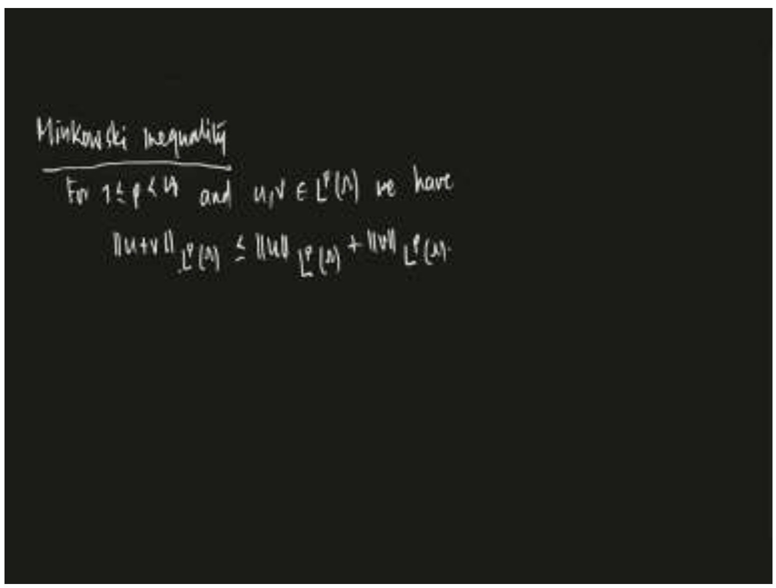

Another very important inequality, which we are going to use is called the Minkowski inequality. This I am not proving but this is also very simple inequality to prove, it says that for 1 less than p less than infinity and uv in Lp omega we have norm u plus v Lp omega less than equal norm u Lp omega plus norm p Lp omega.

So, this is basically the final inequality in terms of Minkowski when think of it like that. This can proved using this norms inequality, but I am not going to prove it right now. So, this you can just remember. Now, what we want to do is we want to (())(39:34) calculate.

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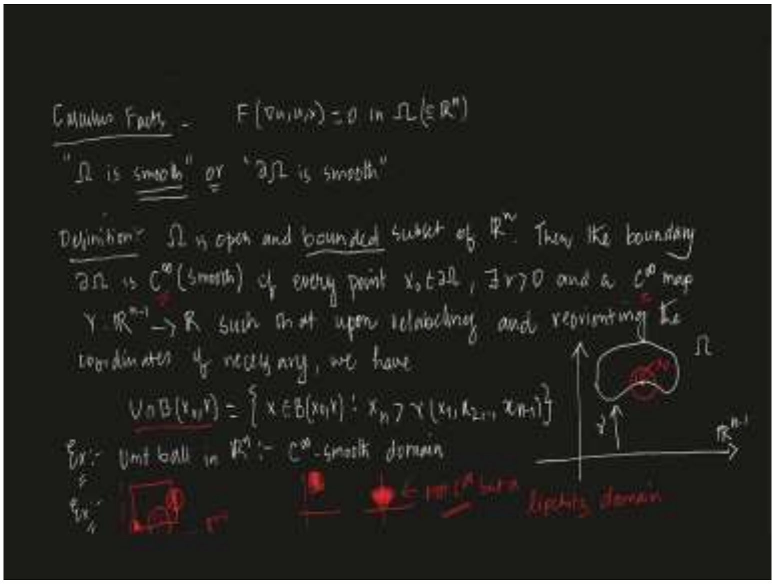

So, what I mean by this you see generally whenever we are talking about this PDE's, let us say some PDE, I do not know maybe let us write it like f of gradient u, u x this is some relation that is equal to 0 in let us say, we say omega some domain, domain omega which is the let us say Rn and as usual we, I mean that omega is open and connected and we generally assume the omega is smooth or also in some books they say Del omega, the boundary of omega is smooth.

Now, the question is this, what is smooth? What do you mean when you say omega is smooth? So, we want to treat properly in a definition, so we say, so obviously with this assume that omega, now we will assume this omega is open and bounded subset of Rn. So, we are giving this definition for open and bounded subset of Rn. So, what we are saying is, let us say omega is open and bounded, sorry, and unbounded subset of Rn.

Then the boundary Del omega, is Ck, is C infinity, let us start with C infinity, C Infinity is this is what we say smooth, C infinity if for every point x naught on the boundary there exists r greater than 0 and for C infinity map let us say comma from Rn minus 1 to R such that upon relabeling and re-orienting the coordinate if necessary we have, so u intersection B x naught r, this part of the domain is x in B x naught r such that an x is the x1 x2 xn, so xn should be greater than gamma of x1 x2 xn minus 1.

Now, what this means is you are basically saying that, so let us draw a small diagram, let us say that your domain omega and this is Rn minus 1, this is minus 1, but it essentially citizen now what is gamma, gamma is this. So, it is saying that the graph I mean if this set is C infinity, so basically the boundary of the set is infinity simply it means that the boundary is smooth, it means that you take any point on the boundary let us say this is the point on the boundary.

Let me put it in a different colour. So, this is the point in the boundary and you have a ball, you can have a ball around, let us say this is the point x naught in the boundary, x naught and there is a ball around that such that the part, this part of the ball, this this part through the intersection part of the ball with the domain is always above the graph of a C infinity function. So, this is always above the graph of the C infinity function.

So, for example let us just assume, so for example, let us say example if you take a unit ball, so unit ball, ball in Rn definitely you can see that you unit ball in Rn, I mean you can obviously always find a function gamma such that this thing happens every point on the unit ball so this is a C infinity smooth domain or C infinity domain whatever you want to call it. Of course here I am assuming open and boundaries.

So, another example let us see, so let us say a domain which looks something like this, let me call it like this rectangle, so is it a C infinity domain? Of course, if you take these points and if you look at ball around this point of course, it is a C infinity domain, because I mean you can be orient it and maybe look at it in this way and your domain your axis is like this and it is always above the intersection, this part will always be above the graph of a C infinity function.

What is the C infinity function in this case? This one. So, the line, so in this, if you reorient this line like this here, then you can understand that the C infinity function, so if you reorient this line, the axis, if you re-orient the axis you can think of this line as like this. And you have a axis like this, this is your Rn minus 1 and the above part, this part, in this case, in this part is always above the graph of C infinity function.

What is the C Infinity function in this case? This line, so this is a linear map and obviously this is C infinity and hence this is a smooth domain, but it is smooth almost everywhere not in any point, because if you let us say take this point or this point, any corner points, if you take a corner point like this, the intersection, so this is the intersection part and so it look like this no, intersection part, intersection part it will look like this.

So, this is the axis. Now let us say if you re-orient it, you may think of this looking like this no, so the graph is not, I mean the intersection part, this part u intersection b, this is definitely not the graph of a C infinity function, not C infinity, so this kind of domain is not a C infinity domain. You can understand that you can have the same definition by replacing C infinity with C1 and then you just have to replace here with C1.

This is, then we call it as C1 domain, so basically the boundary C1 if there is a C1 map with there all of this. So, if you do that then maybe, this is also not a Ck, C1 function, this is also not a C1 function. So this is mod x kind of things. So this is (())(49:01) continuous function and we say this kind of domains, the rectangular domain is a Lipchitz domains as far this definition.

Lipchitz is a set to replace the boundary C infinity with a Lipchitz, so the boundary is Lipchitz and Lipchitz is for every point x naught, there exists R such that there exists and are Lipchitz mapping just as a. So, if he just replace this definition with Lipchitz definition then this particular domain, this particular domain is a Lipchitz domain part, is this clear?

Because the graph is above, the intersection is above Lipchitz graph, graph of a Lipchitz continuous function, but not a C1 continuous function. The intersection, this intersection point, the intersection region this is above the graph of a Lipchitz continuous function, but not as C infinity continuous function, not C1 in that case. So, this is not even a C1 domain, but it is of course a Lipchitz domain. So, that is how we deal with domain, now so this is what we mean, when we say it is a domain, smooth domain.

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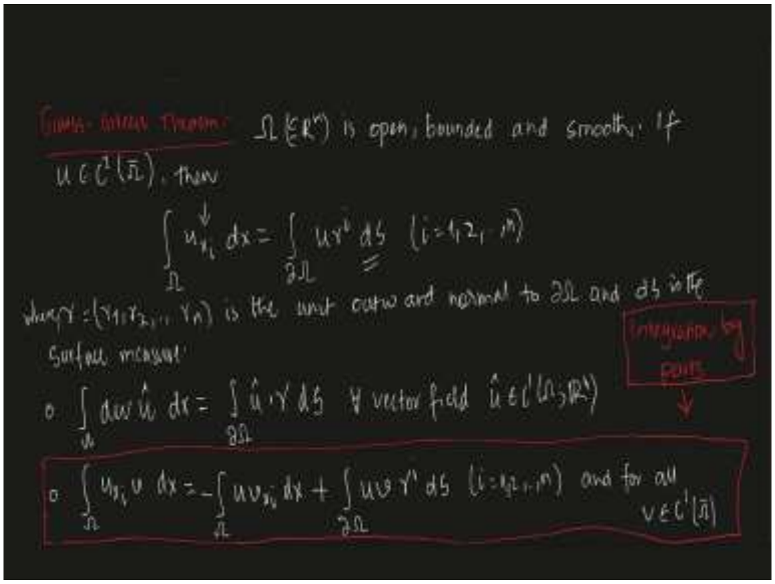

Now, let us come to a very important thing which we call as the Gauss Green theorem and let me change the colour, so Gauss Green theorem, what does that theorem say? So, first of all omega obviously it assume to be open and bounded, please keep that in mind omega subset of Rn is open and bounded and smooth, if u C1 of omega bar, so essentially if omega is bounded C1 of omega bar consists of all uniformly continuous function.

So basically the function is uniformly continuous, the derivative is it has a derivative omega bar, the derivative continuous and not only continuous the derivative is also uniformly continuous. So, basically one type continuously differentiable not only continuous differentiable the derivative function is also uniformly continuous.

So, this is the case then this happens very-very important this is the one thing which you should know, I mean you do not have to know anything else in this course, you just have to know this thing, once you know this this is everything is done. So, this is through u xi over the domain omega dx if Del omega u gamma i ds, this whole for i equals to 1, 2, n.

So, what does that mean? What is gamma, gamma where gamma is gamma 1, gamma 2, gamma n is the unit outward normal to Del omega, see what is saying this, it if you take the derivative of u, the partial derivative of u and if you are integrating it on omega, then that will give you so you do not have to look at the whole domain to do that, just look at the boundary of Del omega and you just compute u dot gamma i ds, that is done, I mean once you do that you calculate this thing and this will give you your, I mean evaluate this particular thing u xi over omega.

And this ds, what is s? s is the surface measure as we already did. So maybe I can just write it as Del omega normal to omega and ds is the surface measure, so basically I am integrating it with respect to omega, so I need to use ds. Now, so this is Green Gauss theorem, I mean this is essentially Greens theorem, but in a more useful form and of course you can just use this thing to write this integral omega divergence of u dx equal to integral over the boundary u dot comma ds.

So, this holds for all vector field u in C1 of omega Rn, so this source and a very important property, this property is this, if you replace u with uv listed, so u and v both of this C1 listed so the product is in C1 of omega bar, now if you replace u with uv then what happens you have the integration by pass formula, so that gives you u xi of v dx equals to integral over omega with a negative sign u v xi dx, how do you getting this?

See if you replace C with uv, uv of xy by Lipchitz rule u xi of v plus v xi of u and you have boundary term, what is the boundary term? Plus integral the boundary uv u is replace by uv, uv gamma i ds, gamma i ds and this is for of course i equals to 1, 2, n and for all v in C1 of omega bar. So, basically what I am doing is I am replacing u with uv.

See, if you have two arbitrary function u and v in C1 of omega bar then the product of u and v is also in C1 of omega bar, I am just replacing u with uv, so once you replace it is u xi of v plus v xi of u, this is what I wrote integral over omega that is equal to integral over the boundary uv gamma ds, that is what I wrote. And this is the most important formula which we are going to use in PDE’s.

You do not know much about PDE’s does not matter, if you want to study about PDE’s or the one thing which I want you to understand in this course is only the formula, this is the most important formula, integration, this is called the integration by parts formula, integration by parts. This is a region integration by parts in one dimension you know right and this is just the in dimensional version and you can obtain this thing using Green Gauss theorem. With this I think we can end this lecture.