Linear Algebra
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Lecture – 09
Row Echelon Form (REF)

Alright. So, let us look at what we have done in the previous class. In the previous class, we had solve the system of equation and we had got the augmented matrix. I have written them here itself alright; to make you understand certain things.

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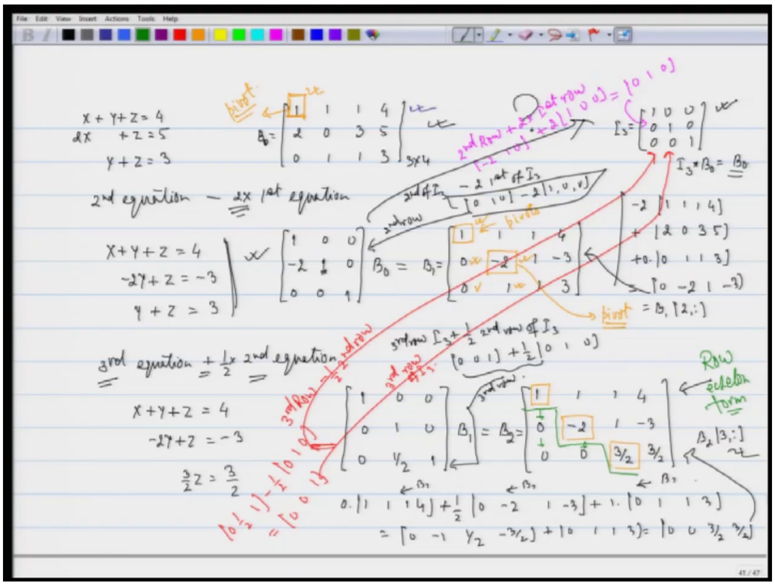

So, here what you see here is in the 1st part, we just wrote the augmented matrix – 2nd part also, 3rd part also. But how did I obtain the 2nd part? The 2nd part was obtained by looking at 2nd equation minus 2 times the 1st equation that was the idea that we subtracted 2 times the 1st equation from the 2nd equation alright.

So, let us look at the identity matrix. This matrix is size 3 cross 4, I want to multiply a matrix on the left alright. So, I want to look at the identity matrix which is 3 cross 3 is that ok? I want to look at this matrix identity matrix which is of size 3. Now, if I multiply this matrix to be 0, so if I want to look at I 3 times B 0, I will get B 0 alright, because this is an identity matrix.

I want to change I three in such a way that I want to get B 1 alright. So, the idea is I want to change I 3 alright, I want to do something to I 3 multiply it to B 2 and get B 1, is that ok? The idea is do something to identity and get B 1 fine. So, what we see is that coming from the 1st equation to the 2nd part, 1st part to the 2nd part, the 1st equation or the 1st row of the matrix did not change. The 1st row of B 0, and the 1st row of B 1 is still the same.

So, since the 1st row is the same I just write 1 0 0 times B 0 because 1 0 0 times B 0 will give me the 1st row alright. I also see that the 3rd row of B 1 and B 0 is also the same. Since they are the same so I can just replace by 0 0 1 nothing to be done. Is that ok? We have replaced 2nd equation by 2 times the 1st equation.

So, if I had just write here 1 and 0, it means that it is 1 times the 2nd row, and 0 times the 3rd row. What we are doing is 2nd equation minus 2 times so minus 2 times the 1st equation. So, I already have the 2nd equation with me, I want to minus 2 times the 1st equation I write minus 2 here. So, I would like you to check here that minus 2 times this B 0 will give me B 1. So, just multiply here.

So, just let us look at this. So, minus 2 times 1 1 1 4 plus 2 0 3 5 plus 0 times 0 1 1 3. This gives you 2 minus 2 is 0, 0 minus 2 is minus 2, 3 minus 2 is 1, and 5 minus 8 is minus 3; which is nothing but the 2nd equation of B 1 – 2nd row of B 1 alright fine.

So, whatever we could do here for the system of equations subtracting one from the other or doing something, I am able to get it by matrix multiplication. Now, let us go from the 2nd part to the 3rd part alright. So, again I want to multiply here something to B 1 alright.

So, if I look at B 1 and B 2 alright; the 1st row of the 2 is the same. So, I just keep it as it is 1st row is the same to 1 0 0 times B 1 will be the 1st row of B 2. 2nd row of B 2 is also the same as the 2nd row of B 1. So, I do not do anything here remains as it is. Now, I am doing 3rd equation is being replaced by 3rd equation plus half times the 2nd equation alright.

So, the 3rd equation I am not doing anything; I am not playing with the 1st row also. So, the 1st row I am not doing anything. So, it remains 0 here. We are looking at only 3rd equation plus half times the 2nd equation. So, 2nd equation is being multiplied by half. So, I just write half here, because it was half and I am adding it, so there is a plus sign here.

So, I would like you to see here that it is 0 times 1 1 1 4 plus half times 0 minus 2 1 minus 3 plus 1 times sorry 1 1 3 alright. So, these are the entries of B 1, they are coming from B 1 alright. So, I would like you to see that this is 0; here it is nothing but minus half so this is equal to 0 minus 1 half and this is minus 3 by 2, so it is minus 3 by 2 plus 0 1 1 3 which is 0 0 1 plus 1 by 3 is 3 by 2, 3 minus 3 by 2 is 3 by 2 which is nothing but the 3rd row alright.

So, the question is it just a chance factor, or it does really always hold? Alright. So, if you see what exactly we have done is that we have the identity matrix, the 1st part was replace or look at 2nd equation minus 2 times the 1st equation alright. I had done that 1st system of equation. Look at I 3 if I look at the 2nd equation of I 3; 2nd equation of I 3 minus 2 times 1st row of I 3, I do get here 2nd row of I 3 is 0 1 0 and this is minus 2 times 1 0 0. And I do get back this part. Look at that minus 2 1 0, this is nothing but the 2nd row here alright.

Similarly, here also I would like you to see that I am looking at 3rd equation. So, 3rd row of identity plus half of 2nd row of identity. So, it is 0 0 1 plus half of 0 1 0, which gives me here the 3rd row alright. So, whatever I am doing here to the equation, I am doing a similar thing to the rows of identity matrix, and then just multiplying to get my things. Is that ok? So, this is what you have to be careful about that you are able to get them through matrices alright.

Now, one thing extra that you need to know that will come to afterwards is that if from identity I can get back this equation 1 0 0 minus 2 1 0 0 0 1, can I go back from here to identity, can I go back, alright? The answer is yes, you can go back.

And if I want to go back to this matrix, I would like to just look at 2nd row plus 2 times 1st row alright. So, look at the 2nd row. 2nd row is minus 2 1 0 plus 2 times the 1st row, and this is nothing but this gives me minus 2 plus 2 is 0 alright, 1 minus 0 is 1 and this. So, I get back the 2nd row as it is. Is that ok?

Similarly, if I look at the 3rd part here, can I from this matrix, can I go back and get identity? So, the 1st two are the same, I am just change the 2nd the 3rd row. So, again I can do the 3rd row. So, look at 3rd row minus half times the 2nd row. And this is nothing but so 3rd row is here 0 half 1 alright minus half times the 2nd row 0 1 0, check that this is indeed equal to 0 half minus half is 0 and this is 1. So, you get back the 3rd row of identity. Is that ok?

So, you can see that in some sense I am able to get back the original part that is the identity. We will look at them in matrix multiplication in the next class, alright. Now, what I would like you to stress here is what are called pivots here. So, if you look at the equation that we had here augmented matrix, the 1st row of this augmented matrix alright, 1st row of the augmented matrix had a 1 here.

This 1 we say it is a pivot because we are using this 1 to make other entries in that column to be 0. Is that ok? So, this 1 in the 1st row is a pivot. So, this is called a pivot. And this is the coefficient of x alright. Is that ok? In B 1 this is a pivot alright, this is a pivot fine.

So, what are pivots look at the rows that you are looking at in the 1st row look at the 1st nonzero element that is the pivot alright. There is no other pivot here as far as 1 is concerned for me for the time being. Now, when I come to the 2nd equation it is minus 2 which is a pivot and 1 was also a pivot coming from the previous part fine. So, we have two pivots here.

This is also a pivot, so two pivots, 1st nonzero entry alright. And there is something also which is important is look at the part which is below I am those things can be made 0 using this pivot I can use minus 2 to make this entry 1 as 0 alright. So, therefore, that is a pivot. So, I have a pivot here; I have a pivot here; I have a pivot. So, I got three pivots. Is that ok? So, I would like to understand these pivots, and what are called row reduced echelon form alright.

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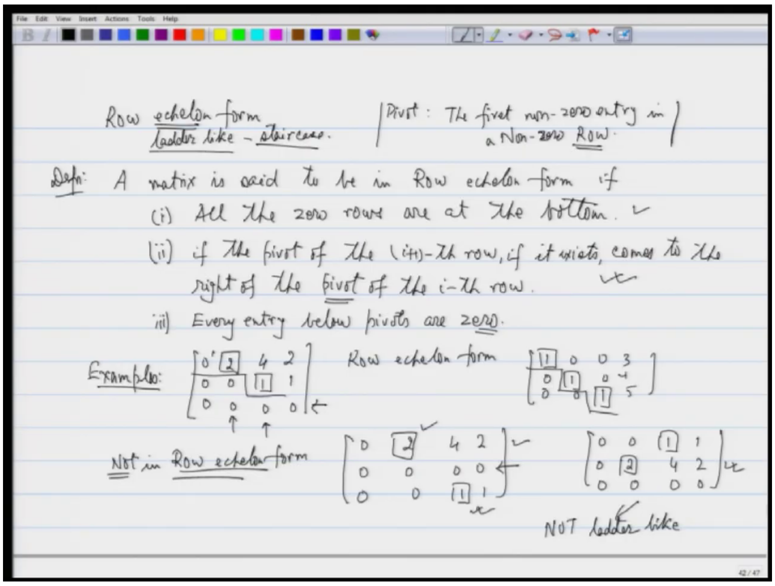

So, first let us look at what is a row echelon form; row echelon form. So, echelon basically means ladder like for us, in general it is called a staircase, alright, fine. So, we are saying it is a ladder like. So, look at the pivots. So, a matrix is, so it is a definition that I am writing here definition, a matrix is said to be in row echelon form if all the 0 rows are at the bottom.

So, in the previous example, I did not have any 0 row everything was non zero; ii – if the pivot of the i plus 1th row if it exists comes to the right of the pivot of the ith row; iii – once you have got the pivots alright in certain rows every entry below that pivot every entry below that pivot below the pivots below pivots are 0 alright.

So, let us go back the previous example and see what we have. So, in the previous example, what we have is that this matrix is in row echelon form. Why, because look at this there is no 0 row. So, there is no question of bottom here. We have pivots here alright. And pivot at the 2nd row is on the right of the first, pivot on the 3rd is again on the right of the 2nd and so on fine. 3rd part is every entry below the pivot should be 0 alright. So, they had 0 for us.

Let us have a better understanding, also one thing more I wanted to say is. So, why it is called echelon or ladder like basically because what you are looking at you have these as your pivots these are the pivots. So, there is a ladder that is coming into play here alright. So, there is a ladder that is playing the role here. It is not necessary that it has to be at each step.

So, let us look at some examples where they are echelon form or not I would like you to give the answers, so examples. So, let us look at this matrix. Now, if I look at this, look at the definition all the 0 rows are at the bottom, yes, it is at the bottom, there is no problem. If the pivot of the I plus 1th row comes to the right of the pivot of the ith row. Now, what are the pivots here?

Pivots are as I said it is the first nonzero entry in each row alright. So, these are the pivots that I have. So, I have pivot here, I have a pivot here. So, these pivots if I look at they go to the right here. So, this is the stair that we are looking at here fine. Every entry below pivots are 0; so this entry is 0 and below everything is 0. So, this is a row echelon form.

Another example of row echelon form, the identity matrix itself. So, you can put here 3, 4, 5 whatever you want to say. So, these are the pivots that you are looking at everything below that is 0, and this is the way it appears fine.

Now, example why it is not a, so not in row echelon form. So, let us take an example previous example is 0 2 4 2, 0 0 0 0, 0 0 1 1. So, these are the pivots I can see here; these are the nonzero rows alright. So, I think I forgot to say what is a pivot. So, pivot is, so pivot the first nonzero entry in a nonzero row. This was the one that I was writing, all that I forgot it. So, that was a mistake fine.

So, again what is a pivot? Pivot is the first nonzero entry in the nonzero row. So, I have those here. So, this is a pivot 2 is a pivot, 1 is a pivot here. But if we look at the row echelon form, we have a problem here in the sense that look at these the first says that all the 0 rows are at the bottom; here the 0 row is at the middle place, it is not at the bottom. So, it is not in the row, it is echelon form. I can also have for example, 0 0 1 1 this is a pivot; I can have 0 2 4 2 again this a pivot because is the first nonzero row.

And then I can have this fine. This is not in the row it is echelon form basically because it is not ladder like. So, not ladder like, or the pivot at the 2nd row comes on the left not on the right of the pivot of the first. Is that ok? So, this part is getting contradicted is that fine not only that even 3rd part is contradicted. So, I would like you to look at these examples and get something nice for yourself alright.

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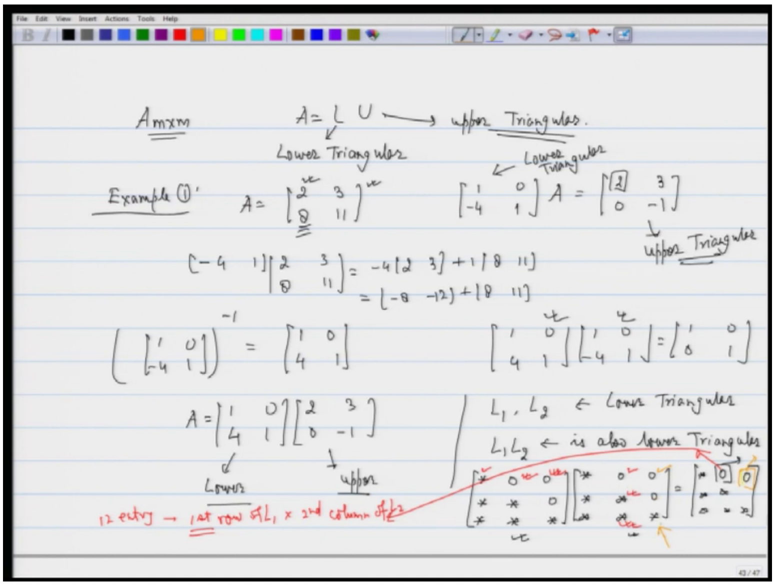

So, the last part for this lecture is what is called given a matrix A which is m cross m would like to write it as a equal to LU if you remember that was the idea that we want to decompose a into certain things.

So, L is lower triangular and U is upper triangular fine. So, let us go back to our examples alright. So, example 1. So, let me write A as 2 3 8 and 11, and simple example 2 by 2, and then we will go back to the 3rd example. So, if I look at this matrix, I would like to make it times A, I want to have it as in the row echelon form alright.

So, there is a 2 here. So, I can think of 2 itself as my pivot, and have 3 as it is there is no change as such. I can multiply, so the 1st row of this remains the same for me alright. I want to make this entry 0. If I want to make this entry 0, I need to multiply the 1st row by 4 and subtract. So, I need to multiply the 1st row by 4 and subtract. Subtract means minus sign and 1 here.

So, if you look at minus 4 1 times 2 3 8 11 is nothing but minus 4 times 2 3 plus 1 times 8 11 which is nothing but minus 8 minus 12 plus 8 11 which is nothing but 0 and minus 12 and this is minus 1. Is that ok? So, what we see here is that I have this matrix is lower triangular, and this matrix is upper triangular fine.

But I did not want it as lower triangular times A, but I wanted A to be terms of lower and upper triangular. So, I would like to take the inverse of this matrix. So, I want to compute 1 0 minus 4 1 inverse of this matrix. So, inverse of this matrix will be 1 and 1 itself fine. It will be 4 here, 0 here.

So, let us check it out. So, 1 0 4 1 times 1 0 minus 4 1, this is nothing but 1st row remains as it is it says that 4 times the 1st row plus 1 times the 2nd row 4 times the 1st row is 4 0 minus 4. So, I get back this here fine. So, this is indeed the inverse of this.

So, I can write A as 1 0 4 1 times 2 3 0 minus 1. So, I have been able to write A as a lower triangular and an upper triangular fine. So, before I move to the next example, I would like you to see that suppose I have a lower triangular matrix L 1. So, L 1 and L 2 are both lower triangular. Then what can you say about the matrix product L 1, L 2? Can I say that L 1, L 2 is also lower triangular?

So, let us understand this. So, I have some entry here 0 here, 0 here fine, star here, star here, 0 here. So, I have these entries with me. This is L 1. I have L 2 which is again of similar type. Can I say that everything is nice here, fine? So, I want to show that this entry I am not bothered about. I want to show that this is 0.

Now, what is this entry? This entry is 1st row of this times the 2nd here alright. So, this entry, so this entry here is nothing but or let us let me write it here I think that entry is the 1 2 entry. So, it corresponds to 1st row of L 1 times 2nd column of L 2. So, if you look at the 1st row of

L 1, there is a star here, you have to multiply with 0 here, then 0 here has to be multiply with this star, and this 0 has to be multiplied with this star. And therefore, it is 0 fine.

Similarly, the 1 3 entry is also 0, this entry is also 0 alright, because I have to multiply 1st row of L 1 with the 3rd here fine. So, it is a star time is 0, 0 times 0, and 0 times a star which is 0 alright. So, just check that L 1 into L 2 is also 0 alright. And the last example for now is the previous example that we had done.

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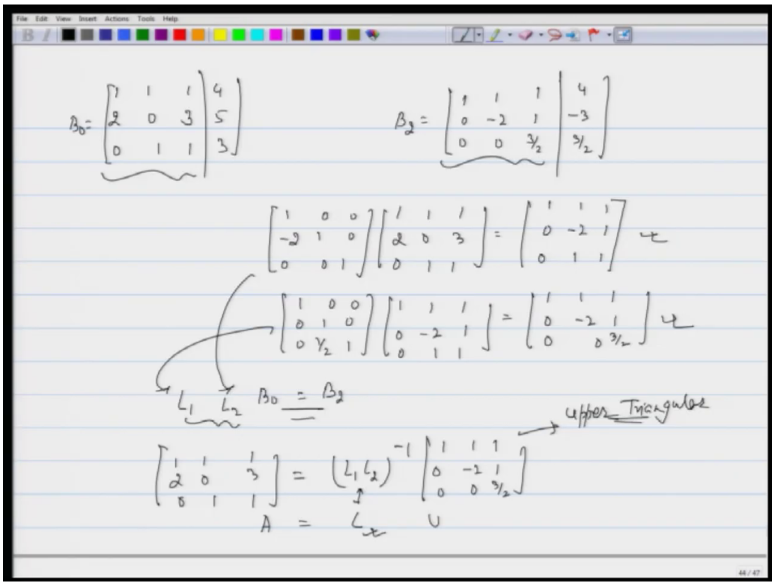

So, in the previous example we had our augmented matrix was 1 1 1 4, 2 0 3 5, 0 1 1 3, this was my B 0. And my B 2 was 1 0 0 1 minus 2 0 1 1 3 by 2 and 4 minus 3 3 by 2. So, here it is not a square matrix. So, it is not a square matrix. If you recall I had said that would like to look at the form upper or the lower triangular form, but I can still do this I can look at these two together forget about this part.

And therefore, if you see what we had done was let us go back. So, the 1st thing we had done was to remove this. So, I had this matrix with me 1 0 0 alright, 0 0 1 times 1 1 1 2 0 3 0 1 1 as something alright. So, this was minus 2 here and 1 here 0 here and that gave me 1 1 1 0 1 1 0 minus 2 1, this is what I got fine. In this, so given this matrix 1 1 1 0 minus 2 1 0 1 1, we have 1 1 1 0 minus 2 1 0 0 3 by 2.

So, here I had looked at the 3rd one. So, it has 1 0 0 0 1 0. So, you have to multiply half here had half here this is what I had this. So, if you see on B 0, 1st we multiply this matrix which is L 2 for me and then L 1 here to get me B 2 is that ok? So, what I see here is that if I write this as my 1 1 1 2 0 3 0 1 1, then the inverse of this L 1 L 2 whole inverse times this part 1 1 1 0 minus 2 1 0 0 3 by 2 is the one that I am looking at A equal to LU.

You can see that this is upper triangular alright. And L is nothing but inverse of L 1 L 2. L 1 is lower triangular, L 2 is lower triangular, so their product is lower triangular, and hence L is lower triangular for us. So, even write able to write a as LU. Is that ok?

So, in general there is a way of doing these things what are the conditions and so on that will come in the next class.

Thank you.