Linear Algebra

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Lecture – 07
Introduction to System of Linear Equations

So, let us just start the class today. So, we had learned matrix multiplication and the inverse of a matrix. In this class we will try to look at system of equations and how does the matrix multiplication and invertibility help us to understand things. So, this whole chapter is going to depend on that we will slowly build up the ideas. So, let us start with the basic ideas that we do. So, we start with a system of one equation only. So, let us look at this example.

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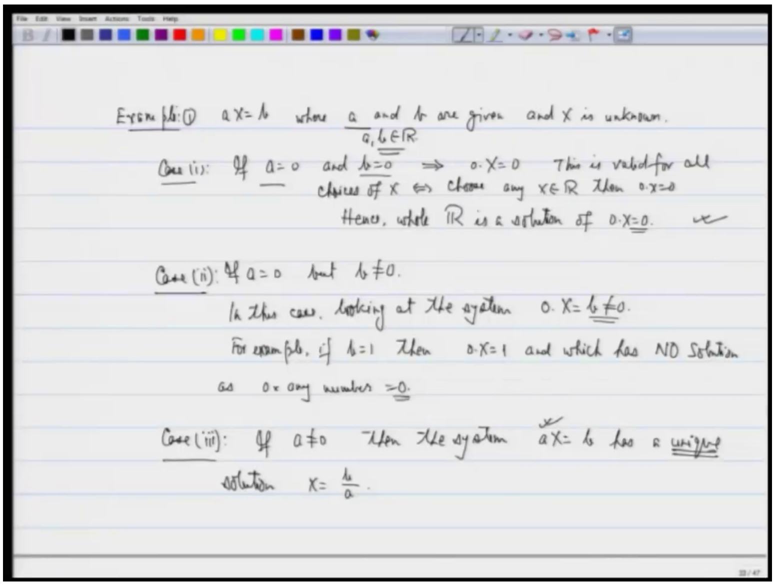

So, consider the system a X is equal to b, where a and b are given and X is unknown alright. So, I have a system here which is a X equal to b, a is a scalar b is a scalar. So, they are real numbers or complex numbers or rational numbers whatever we want to say alright and X is unknown I want to find X fine. So, generally we said that the solution is X is equal to b upon a without putting any condition. So, now we just like to look at things carefully.

So, let us look at certain cases. Case i, if a is 0 and b is also 0 fine then we are looking at the equation 0 times X is equal to 0 now this is true. So, this is valid; this is valid for all choices of X. So, which I am trying to say that or equivalently what I am saying is that if you take.

So, choose any X belonging to R then 0 times X is 0 and the hence whole R is a solution of 0 times X is equal to 0 alright. So, we generally do not worry about this part when we are studied it, but we need to be careful because we want to generalize ideas. So, therefore, we need to understand what happens at the basic level and then proceed alright. So, that the basic level is starting with a equal to 0 what happens to b equal to 0.

Now, case ii a is 0, so, if a is 0, but b is not equal to 0 alright. So, this is a very important case a is 0, but b is not 0 in this case what happens is that in this case just looking at the system 0 times X is equal to b and b is not equal to 0 alright.

So, for example, if b is 1 then we are looking at 0 times X is equal to 1 and which has no solution and which has no solution; no solution as 0 times any number is 0 alright. So, this case is also taken care of that the first case we see that whole of R is a solution or there are infinite number of solutions.

In the second case I do not have a solution unless look at the third case when a is not 0. So, case 3 if a is not 0 then the system a X is equal to b has a unique solution X which is given by b upon a now whether b is 0 or not that does not matter to us because since a is not 0 I can divide by a and do all the things.

So in this case I have a unique solution So, this is the case when I have got only one equation in one unknown now let us proceed with the second example in which you are going to look at two equations in two unknowns.

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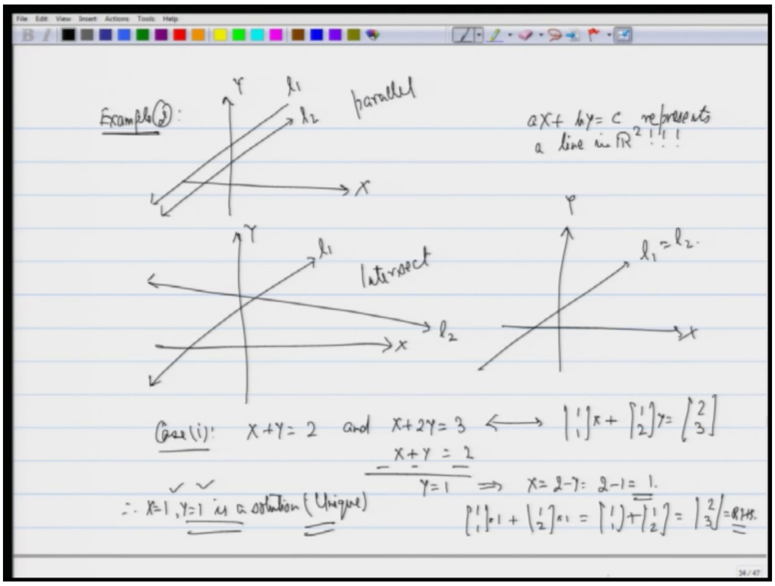

So, example 2, fine. So, what we have learnt is that, this case if I look at geometrically then this corresponds to looking at. So, generally we look at lines here. So, any linear system of the type a X plus b Y is equal to c represents a line in R 2 fine.

I will put a exclamation mark here to for you to understand that when such a thing happens if I put say for example, a is 0 b is 0 do I still make sense alright. So, it make sense when either a or b is not 0 and then we proceed and get lines fine. So, we have to be careful when we say that ax plus b Y equal to c represents the line. So, there is a pinch of salt that is important for us and we have to be careful

So, now I have got suppose two lines, these two lines could be parallel alright or I could have I have one line here I have another line here. So, this is line l 1 this is l 2; l 1 l 2 fine. So, here the lines are parallel here they intersect, it may happen that the two lines are same. So, l 1 and l 2 are same alright. So, these are the cases that we need to look at.

So, I will try to see things according to that, but be careful there is one case which you are not writing here where a is 0 b is 0 alright. If a is 0 b is 0 fine, then we do not have lines is that ok. So, let us look at examples to understand things. So, numerical examples for us. So, case i; so I am going to look at the system X plus Y is equal to 2 and X plus 2 Y is equal to 3 we know how to solve it we have already done it.

So, if I solve it what do I get I can write here X plus Y is equal to 2 subtract here cancel out this X I will get Y is equal to 1 and therefore, this will imply that X is 2 minus Y which is 2 minus 1 which is 1. So, I see that. So, therefore, X is equal to 1 and Y is equal to 1 is a solution alright. So, there is the point of intersection and it is a unique solution and the solution is unique fine.

As I pointed out in the first class or I think second class, we also said that we would like to look at not just a line, but vector here. So, if I look at the vector corresponding to this two equation these two equations, the X part comes from 1 and 1 times X plus 1 comma 2 of Y is equal to 2 comma 3 alright. So, once you have already do matrix multiplication. So, you can see that X plus Y is 2 and X plus 2 Y is 3 these are the two equations

So, I would like you to see that the solution X is equal to 1 and Y equal to 1 tells me that 1 1 times 1 plus 1 2 times 1 is same as 1 1 plus 1 2 which is same as 2 3 which is the right hand side of the system of equations alright. So, system of equation it was looking at intersection of lines has given us something about vectors that I can get the vector that I require which is on the right hand side fine.

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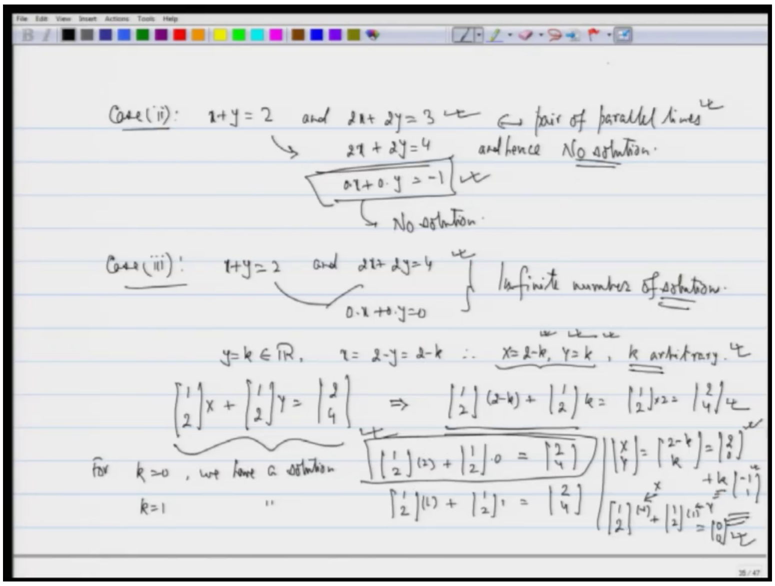

Another example, case ii. So, let us look at x plus y is equal to 2 alright and the second equation is and 2 x plus 2 y is equal to 3, now you can see here that I have got. So, x plus y is 2 and again I can rewrite this second equation as x plus y equal to 3 by 2. So, therefore, I get a parallel of pair of parallel lines and hence there is no solution; and hence no solution fine.

But in general I cannot talk in terms of parallel lines. So, what we do is that again we multiply by 2 and subtract. So, I can write use this first equation to write it as 2 x plus 2 y is equal to 4 subtracted you get 0 times x plus 0 times y is minus 1 and therefore, this has no solution that is there is no x and y no real number x and y such that 0 times x plus 0 times y will be equal to minus 1 fine.

So, that is the idea that we would like to follow that we would like to get a equation in which finally, we have something from which you can directly conclude whether the system has a solution or not that is the idea alright. The next case is infinite number of solutions you already know it I can just rewrite the above equation as x plus y is 2 and 2 x plus 2 y is 4 and therefore, you can see that 0 times x plus 0 times y will be equal to 0 and there are infinite number of solutions alright.

Now, here I would like you to observe something which is important for us which is the two equations are same alright. So, the solution is going to be I can write y as some k a real number and then x is equal to 2 minus y which is 2 minus k therefore, the solution is x is equal to 2 minus k y is k and k is arbitrary alright.

So, let us rewrite it this part again in terms of a matrix notation. So, what we see is that these two equation they give us 1 2 corresponding to X plus 1 2 corresponding to Y is equal to 2 comma 4. So, if I want to replace X and Y here for example. So, we can see that 1 2 times 2 minus k plus 1 2 times k is nothing but 1 2 times 2 which is 2 4 fine.

But also I would like you to understand something which is more important for me for the time being is that I have got the solution here for the system of equations, there infinite number of solutions, but let us look at the two solutions separately in the sense that I just look at corresponding to what happens to k alright. So, I would like to look at k as a separate part is that ok.

So, for k equal to; so for k equal to for; k equal to 0 we have a solution 1 2 times 2 k is 0. So, X is 2 and Y is 0 plus 1 2 times 0 is equal to 2 4. Similarly, if I put k equal to 1 alright we will have a solution which will again be of the type 1 2 times 1 plus 1 2 times 1 which is 2 4 fine. So, here you are not able to see the much difference here what I am trying to say here is that, if I write X and Y as 2 minus k comma k which is what we had I can write it as 2 comma 0 plus k times minus 1 comma 1 alright. So, this is important part.

So, 2 comma 0 was already used here to get a solution what is the need of using 2 comma 0 again and again alright. So, if I want to look at the solution space as such set of all solutions what I see is that any linear any combination for value of k does give me a solution, but if I just look at the part corresponding to k which is minus 1 comma 1, then it has some nice property that 1 2 times. So, let it 1 2.

Look at this minus 1 plus 1 2 times 1. So, this comes from the X component this comes from the Y component and they give you 0 alright. So, what you are saying is that in some sense this vector these two vectors 1 2 and 1 2 they are not invertible because I have got a system of ax is equal to 0 or solution of ax is equal to 0.

So, let us proceed further with these ideas. So, in this case now we are going to solve a system of equation 3 equations in 3 unknowns fine and you already know how to solve it so, but let me look at this example again and we will come back to it again and again.

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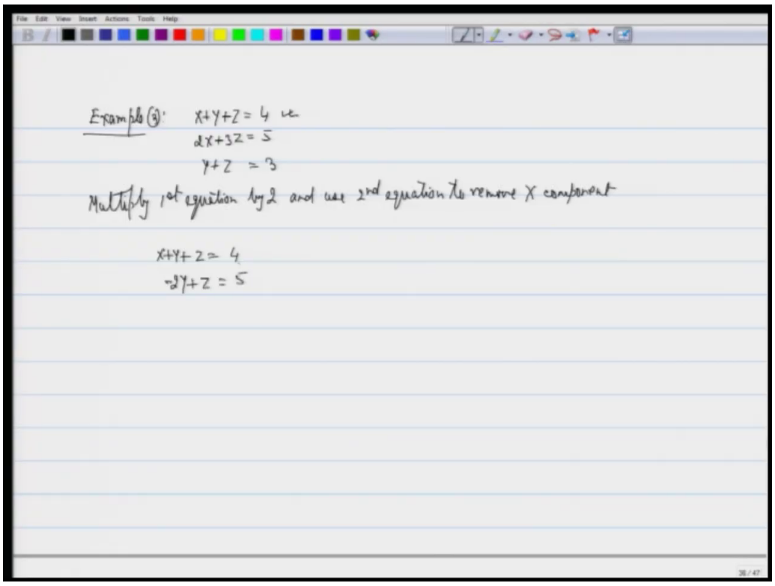

So, I have the system X plus Y plus Z is 4 2 X plus 3 Z is 5 and Y plus Z is equal to 3 this is the system I have alright.

Next stage what I can do is that I can multiply the first equation. So, multiply first equation by 4 by 2 and use second equation to remove X component alright fine. So, if I do that the first equation remains as it is it is X plus Y plus Z is 4, I multiplying by 2 and subtracting. So, 2 X minus 2 X is going to be 0 times X, the next stage will be there is no Y here. So, it will be minus 2 Y and 3 Z minus 2 Z is alright 3 Z minus 2 Z is Z. So, I am going to get it as 5 is that ok?

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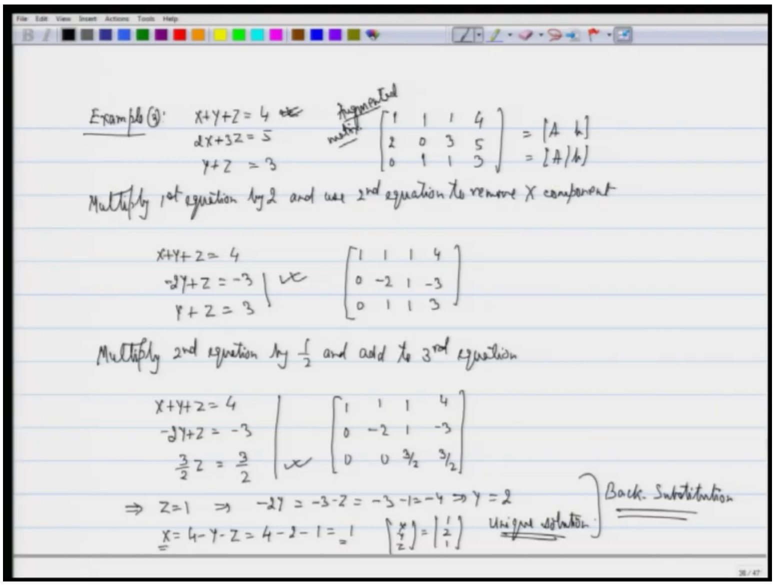

So, 5 minus 8 is minus 3 and this I leave it as it is because I there is no component of X here. So, I leave it as it is; at the second stage I use these two to make 0 here. So, I multiply this by half and add. So, multiply 2nd equation by half and add to 3rd equation; 3rd equation.

So, what I get is X plus Y plus Z is 4 minus 2 Y plus Z is minus 3 alright and here I get is 3 upon 2 Z is equal to 3 upon 2 and from here I have the system of equation for me as. So, this will imply that Z is 1 alright. So, I have done elimination. So, I have done it now I am looking at Z is equal to 1. Once Z is 1 this will imply that minus 2 Y is equal to minus 3 minus Z which is same as minus 3 minus 1 which is minus 4 and this gives me Y is equal to 2.

Now, can you use the first equation to get X is equal to 4 minus Y minus Z which is 4 minus Y is 2 and Z is 1. So, I get 1 alright. So, I have got the equation as X Y Z is equal to 1 2 1 a unique solution alright. So, I would like you to understand here is that what exactly we have done.

So, what exactly we have done is that I have my first equation, in that there is a coefficient of X which is non zero, I have used that coefficient to make the first component that is the corresponding to X every coefficient of X to be 0 alright. So, here if I look at the first equation coefficient of X was 1, I have used it to make coefficient of X 0 in the 2nd stage fine.

In the 3rd equation there is no coefficient of X. So, I did not worry about it fine. Now, from second stage to third stage I already see that there is an equation which as X now the other two equations have only Y and Z fine. Since the other two equations have only Y and Z I can use my 2nd equation which has a coefficient of Y to be non-zero and then use it to make the 3rd equation being independent of X and Y both. It was already independent of X now I am making it independent of Y also.

I have done that. So, the 3rd equation now reduces to only coefficient of Z and therefore, I can solve for Z and get my answer. Once I have got the value of Z I back substitute the value of Z in the 2nd equation I get the value of Y and I use the 2 to substitute it in the 1st equation where I can get the value of ax alright.

So, this part that getting z is equal to one and then substituting the values back or the 2nd equation and 1st equation this is called back substitution alright. So, I will come back to this equation again. So, before I come back to that let us make a general statement about all these things. So, that I can come back whenever I feel like alright. So, definition.

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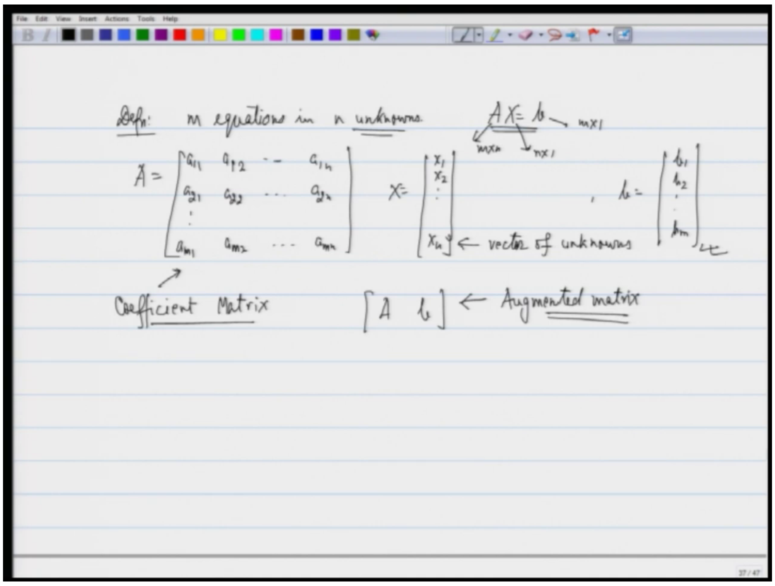

So, we have been solving a system of equations which has been always a square matrix. One equation in one unknown two equations in two unknowns three equations in three unknowns, but in general we would like to look at m equations in n unknown. So, we are looking at m equations in n unknowns alright and our system is going to look like AX is equal to b.

So, A is m cross n matrix X is n cross 1 and b is m cross 1 A is we write it as a 11 a 12 a 1 n a 21 a 22 a 2 n a m 1 a m 2 a mn, X is x 1 x 2 x n alright this is what it is called vector of unknowns or vector of unknown variables or vector of variables whatever you want to say and b is going to look like b 1 b 2 b m and generally we are given that this is known, but in general this may not be given also fine.

So, we are going to look at the setup. So, this matrix a has a special name what is called coefficient matrix fine and the matrix this in which A and b come together, this is called an augmented matrix fine. So, in the previous example let us construct back these coefficient matrix and augmented matrix.

So, I am going back to the previous slide alright. So, here my coefficient matrix is going to look like 1 2 0 there is this is the coefficient of x that you are looking at. Now, let us look at coefficient of y coefficient of y is 1 0 1 and coefficient of z is 1 3 1 alright I do lot of mistakes in calculations. So, and all I need to be careful 1 1 1 2 0 3 alright. So, its correct now.

So, this is my coefficient matrix this part, but now if I put b also here, so b is 4 5 3. So, I get it as sometimes you will just write it as A b or sometimes as A and b to differentiate the 2. So, I have got the augmented matrix here augmented matrix alright.

I would like to get the augmented matrix for this as well as the third one. So, the 2nd equation for the second one it is again 1 0 0 1 minus 2 1 1 1 1 and then 4 minus 3 3 and the third one it is 1 0 0 1 minus 2 0 1 1 3 upon 2 4 minus 3 3 upon 2 alright.

So, for all the three equation I have given you the augmented matrix. We will see how to use them and the idea is coming from there fine. So, for the time being I have just got to you the augmented matrix for the three parts and we will come back to it afterwards again.

Thank you.