Linear Algebra
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Lecture – 66
Singular Value Decomposition

Alright. So, this is going to be the last class.

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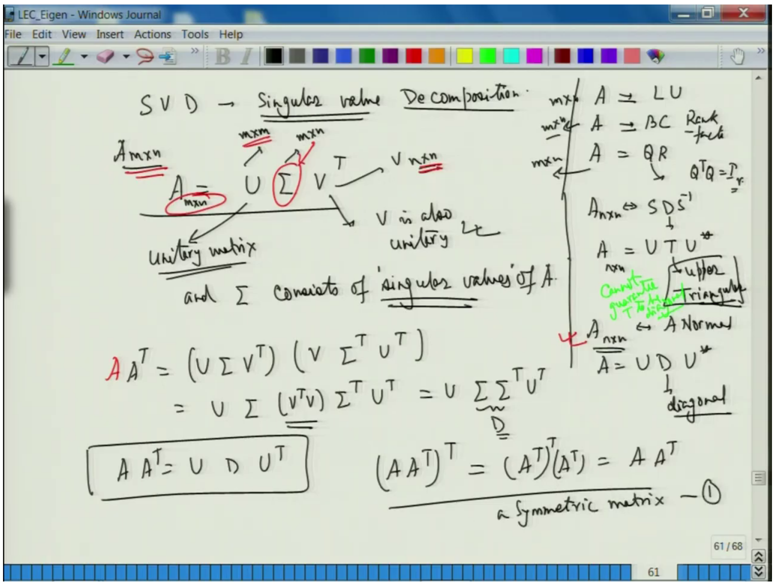

And here I would like to make you understand what is called SVD – Singular Value Decomposition. So, it is a decomposition. But you have to see what are called singular values for us that I will have to explain to you, fine. And this is the most important thing that we have.

In the sense that the starting with a matrix A we had given you LU decomposition that was for a square matrix; and from there we got the linear form also. That LU in some sense upper triangular form should written as m cross n. We also the notion of A is equal to B C rank factorization, alright.

Then A equal to QR was there fine, where Q was such that Q transpose Q was an identity matrix of rank r, whatever the rank of the matrix is, everything was dependent on the rank. Then given a square matrix this we could say, that means, we wanted to understand this as SDS inverse. So, S is already is non-singular for you, and D is the matrix of eigenvalues of A. This is not always possible. Sometimes you could do; sometimes you could not do fine.

Then we went to what is the Schur's theorem which we said that they exist unitary matrix U such that A is U T U transpose. And this T is upper triangular, fine. You could do this. This was possible for everything. From there, at the next stage we went to matrix A which is again this a square matrix. And we assume that A is normal. And in this case, we could show that I can write A as U D U U star, I should have written a star here also in place of transpose, because I am looking at over complex numbers here.

So, A is U D U star for us fine. So, D is diagonal here again. So, when A is not normal, I cannot guarantee that this upper triangular matrix has to be a diagonal alright; so cannot guarantee T to be diagonal, is that, ok that I cannot guarantee. But when it is normal, then I could go do it fine. Now, so everything here that I have done till now in some sense is related with what are called the square matrices.

Here, it was allowed to be m cross n, here also m cross n, m cross n, I could talk of this, but not understand much. And what we have seen is that understanding eigenvalues, eigen vectors are most important as per as our calculations and things are concerned that is the application, because we only look at certain directions in which you want to maximize or minimize.

We do not want full information, because getting full information takes too much of time alright fine. Computationally, it is almost impossible to get the full information. So, whatever minimum information we have would like to get the best information, which is in the direction of maximum or minimum optimizing things, fine.

So, we would like to formulate those idea in terms of now A which is an m cross n matrix, fine. So, what is SVD theorem says that any matrix A can be written as U sigma V transpose I think, yeah. So, this is a note that I have. So, A is this where U is a unitary matrix, V is also a V is also unitary also unitary, and sigma consists of singular values of A. So, what are singular values, I have not defined what are those singular values. I have to define that as well for you. Is that ok?

So, let us try to understand first what do I mean by saying that A is U sigma V transpose. So, A is an m cross n matrix. So, this U which is an unitary matrix, it has to be m cross m fine. Sigma has to be an m cross n matrix. And this V has to be n cross n fine understand them nicely.

I want A to be an m cross n matrix. So, U has to be m cross m; V has to be n cross n, I do not have a much choice. So, the middle thing that I have has to be m cross n. So, that matrix multiplication makes sense, that is the first thing that we need to understand that we cannot get away with that part.

The next idea that we have here is let us just multiply and see, because we know about n cross n matrix how do you use this idea of n cross n matrix to get to an m cross n matrix alright, fine. From where do I get U and V that also has to be understood, and from where do I get sigma, what do I mean by singular values, so we need to understand that also, fine. So, let us do that one by one.

So, let us compute what is A times A transpose. A times A transpose is nothing but we are we want to write U sigma V transpose this is your A. A transpose will be transpose of this it will be V sigma transpose, alright. Look at nicely U transpose for us, which will be equal to U sigma V transpose V sigma transpose U transpose.

Now, V is an unitary matrix orthogonal matrix whatever you want to say let us say it for the time being unitary, because we would like to then build up on those ideas fine. But what we know is that over complex numbers everything is nice, we have eigenvalues eigenvectors and therefore, everything is nice. So, V is unitary for us.

So, if I look at this here for the time being, I get here is it is U sigma transpose U transpose, alright. This is more important for me, understand this. That if I had this write this matrix as D, fine, if I write this matrix as D, I have got that A A transpose as U D U transpose, fine.

Now, what is A A transpose? Look at this matrix A A transpose, take the whole transpose of this, you get A transpose A transpose whole transpose and this which is same as A A transpose, so a symmetric matrix, a symmetric matrix that is one thing. I observed that this is a symmetric matrix that the first thing we observed.

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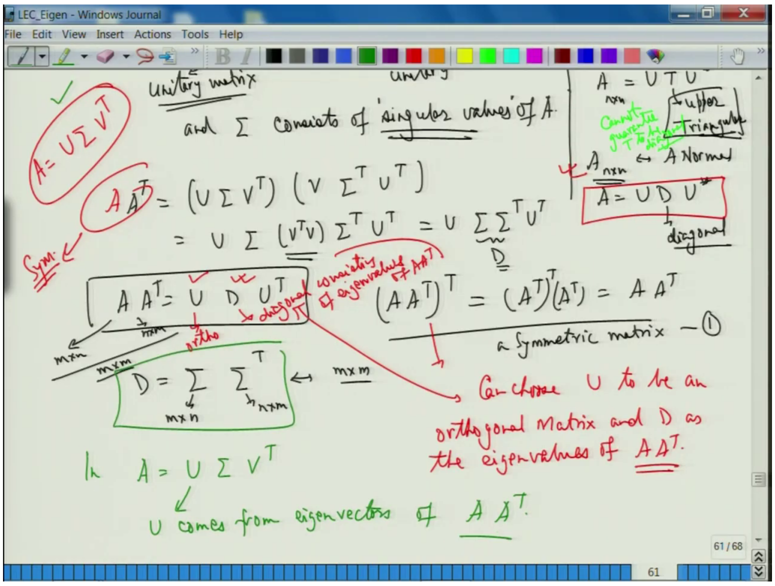

Now, what is D? D is sigma times sigma transpose. And what is this sigma is of the size m cross n, this is n cross m. So, I am getting a matrix of size m cross m, fine. So, keep track of things. A is also m cross n; this is n cross m. So, my everything is m cross m. So, what we are doing here is that in some sense I have got this matrix which is symmetric matrix.

I already wrote it is a symmetric matrix. And then I am looking at the decomposition of A A transpose in terms of eigenvalues eigenvectors, alright that we have already done, this what we said that for normal matrices we already have this. And for Hermitian matrices, we had something special. And what was that? That the eigenvalues are always real and you can choose the eigen vectors to be in R n itself.

So, we can choose. So, can choose U to be an orthogonal matrix, and D as the eigenvalues of A times A transpose, is that ok, that is more important understand it nicely. What I am trying to say, what we are saying is that I want my A to be U sigma V transpose. I want this A, fine.

So, if I compute A times A transpose, this is a symmetric matrix. And therefore, by the theory that we have developed till now, we know that I can decompose it into U D U transpose, where U is unitary. Since, it is Hermitian, so I can take U to be orthogonal. So, I have this is as orthogonal matrix, D is diagonal matrix diagonal consisting of eigenvalues of A times A transpose, is that ok. So, I can do this.

So, the idea is to compute A A transpose and get the vector U get the matrix U and D is that, ok. So, in the expansion when I want my A to be U sigma V transpose, I know what U is. I can compute U as your eigenvectors coming from U, alright.

So, in writing A as U sigma V transpose, U comes from eigenvectors of A times A transpose, fine. Sigma also comes from this part that we have also seen, fine. Now, let us compute from where do I get V or V transpose whatever we want to talk off, from where do I get it?

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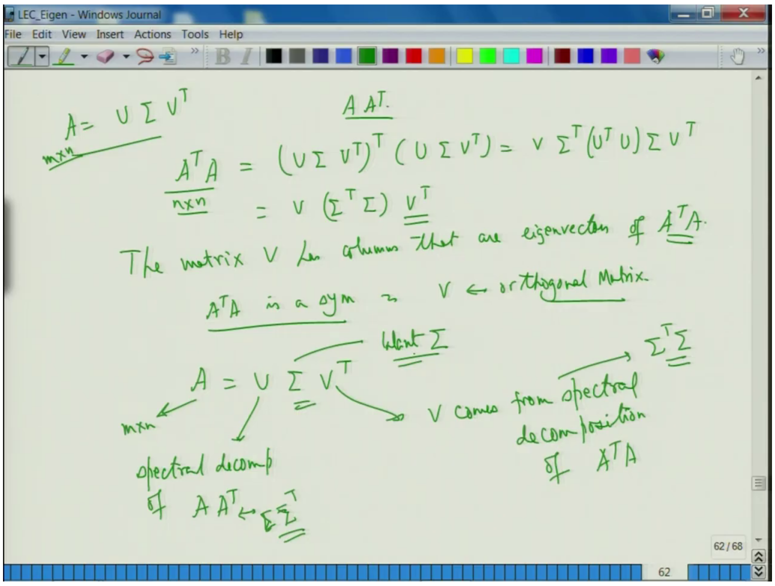

So, I want to now look at, so I wrote A as U d V transpose, and we got A times A transpose. Now, let us compute A transpose A. So, what is A transpose A? A transpose A is now a was m cross n. So, this is now my n cross n matrix for me, fine. So, let us do that. So, it is U sigma V transpose whole transpose U sigma V transpose.

This is same as V sigma transpose U transpose U sigma V transpose, which is same as V sigma transpose sigma V transpose alright. So, you can see that now the matrix V by the same argument the matrix V appears as the columns of the matrix V. The matrix V has columns that are eigenvectors of A transpose A, alright.

Again note here very very important that A transpose A is a symmetric matrix or Hermitian matrix whatever you want to say. So, if A has real entries A transpose A is a symmetric matrix. And this implies V can be chosen to be orthogonal matrix, orthogonal matrix. And therefore, V transpose makes sense, I do not have to compute the inverse I just add V transpose because V transpose is same as V inverse is that ok.

So, when I want to write A as U sigma V transpose, A is m cross n. U comes from the spectral decomposition of A times A transpose. This V comes from a spectral decomposition of A transpose A, fine. Sigma, understand eigenvalues here they come from sigma sigma transpose; eigenvalues here they come from sigma transpose sigma fine, is that ok. So, these are the two things that have to be careful and I want the sigma for us, want sigma.

I do not want sigma sigma transpose or sigma transpose sigma; I want sigma that is more important for me, fine. So, what we say is that the sigma. So, look at this. So, what we are seeing here is that note.

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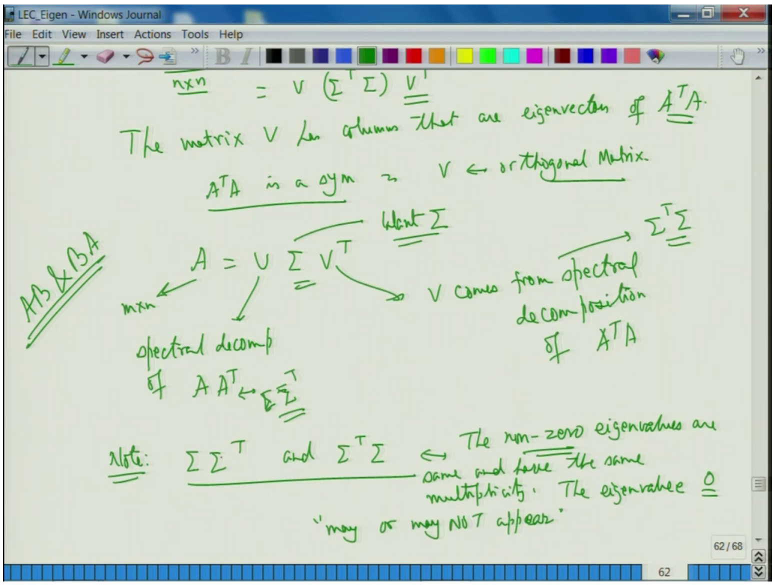

Fine, that sigma sigma transpose and sigma transpose sigma, they have same eigenvalues, same non-zero eigenvalues I will say, same the non-zero eigenvalues the non-zero eigenvalues, eigen values are same. And have the same multiplicity, fine. The non-zero have the same fine. 0 maybe there, 0 may not be there. The eigenvalue, 0 may or may not appear.

And this I will put it under quote mark, because it is m cross n if m is not equal to n, then at least one of them will have 0 eigenvalue coming into play because what we had seen was that the matrices A B and B A, they have same non-zero eigenvalue same multiplicity, but there is a problem with adding extra zeros this is what we had seen for our matrices A B and B A, alright; A B and B A we had looked at that part, fine.

So, this is the same thing here A is sigma B is sigma transpose AB and BA we are looking at fine. So, let us try to understand this by an example. Let me compute this with an example, fine.

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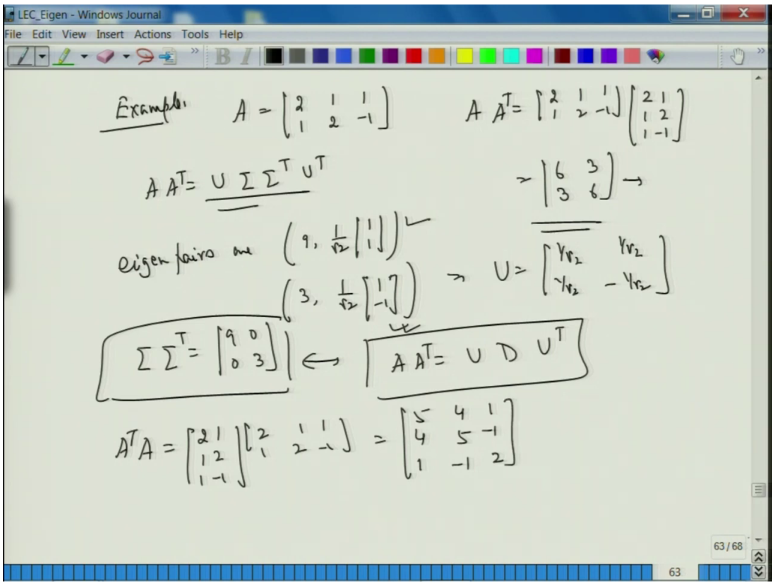

So, example, so I will take only one example. Let me define A as something easy which is 2 1 1 2 1 minus 1. Let us compute A times A transpose. So, A times A transpose is 2 1 1 2 1 minus 1 times 2 1 1 2 1 minus 1. I hope I am correct. So, I do a lot of mistakes alright. So, 4 plus 5 plus 1 is 6, 2 plus 1 is 4, 4 and minus 1, no, 2 plus 2 4, 4 minus 1 is 3 yes indeed 3, it has to be symmetric. So, it is 3. 1 4 and then 6, this is what I get, fine.

So, what I know is that I am supposed to write A A transpose as. So, let me write; otherwise, I will always get confused as usual. So, A transpose for me is U, is it [FL], yeah A transpose is U I think I wrote here. So, A A transpose is, no, A transpose A is V. So, A A transpose is U. So, A A transpose was U sigma, sigma transpose U transpose alright, fine.

So, what I have here is U I can take. So, eigenvalues of these, so eigen pairs here are, eigen pairs pairs are 6 plus 3 is 9 with 1 upon root 2 1 1, this is 1, the other is 6 minus 3 that is 3 and 1 upon root 2 1 minus 1. So, these are the two eigen pairs. And therefore, this implies that U is equal to 1 upon root 2, 1 upon root 2 corresponding from this part; and then 1 upon root 2, minus 1 upon root 2 coming from this part is that, ok. So, I have got the matrix U.

And sigma sigma transpose is nothing but 9 0 0 3 alright, because A transpose this is there because A A transpose is U D U transpose, fine. So, therefore, D is this that we are got here. Let us compute A transpose A A transpose A is 2 1 1 2 1 minus 1 times A 2 1 1 1 2 minus 1, which is equal to multiply this by 2 4 plus 1 is 5, 2 plus 2 is 4, 2 minus 1 is 1.

I hope I am correct, I do not know, 1 2 plus 2 is 4, 1 plus 4 is 5, minus 1 multiply this by 1. So, just add. So, it is 1 here, minus 1 here, and 1 plus 1 is 2, I hope I am correct. It is symmetrical, I hope it is correct, alright fine. So, what we know is that the eigenvalues of A B and B A are same as per the non-zero part is concerned.

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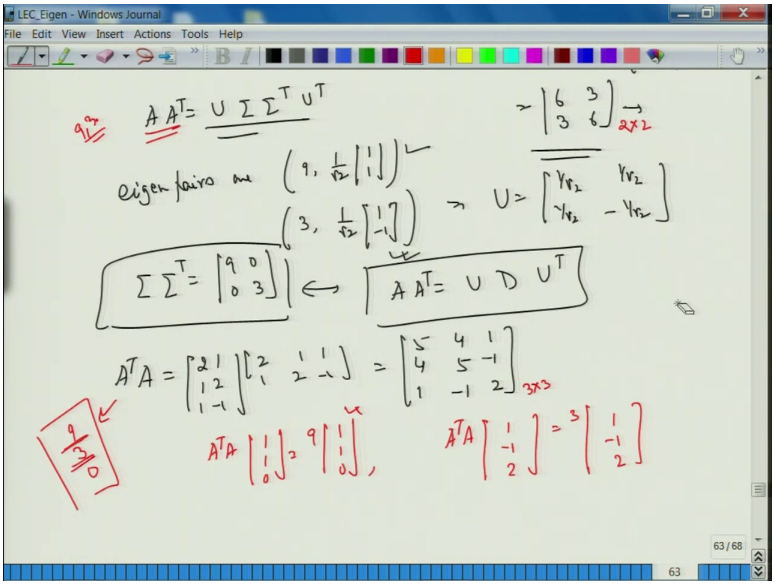

So, I already know that the eigenvalues of this matrix are 9 and 3. So, the eigen values here have to be 9 and 3, does the first thing that we understand, fine. What was the third eigenvalue this is a 2 cross 2 matrix this is a 3 cross 3 matrix. So, 0 has to be here. So, these are the 3 eigenvalues that I need to look at. Let us try to get those eigenvectors.

So, if I see the eigenvectors, yes, I get it. So, you can check that A transpose A of 1 1 0 is 5 plus 4 is 9. It is 9 times 1 1 0 you can check that yourself, fine. Then I need to go the other way around. So, I want 3 to be then eigenvalue, 3 as an eigenvalue. So, 5 plus 4 is 3. [FL] So, 1 minus 1 has to come somehow. So, A transpose A times I want something to be equal to 3 times the same number. So, I should get here.

See the idea is that this is a symmetric matrix A transpose is symmetric. So, the eigenvectors have to be orthogonal alright. So, this one eigenvector is already 1 1. So, I have to look at 1 minus 1 and then something alright. So, I have to have 1 here, minus 1 here, 1 and minus 1 here, and something that you need to put. So, what should that be, fine?

So, since it is this, I would like to look at here, so 5 minus 4 is, 5 minus 4 is 1; I want 3. So, maybe I could put 2 here. So, let check whether it is or not. So, 5 5 4 1 minus 4 minus 5 and 1 plus 1 and with 2 plus 2 minus 2 plus 4 2, so it gives me 6, it gives me 4 minus that is minus 3 and this is correct, yeah. So, this is the one that I am looking at. So, I have this part. And then there will be one which will corresponding to 0.

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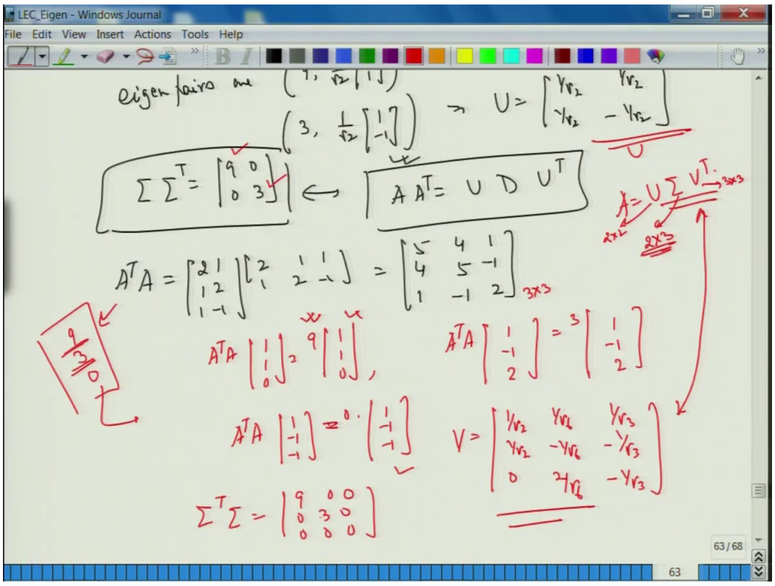

So, A transpose A times something should be 0 times something. So, this seems I think 5 and 5 cancels out, 4 and this cancels out. So, it will be 1, minus 1, minus 1, alright. So, you can see that everything is nice. I hope this correct know, 1 minus 1 will give me 2, minus 1, yeah. So, this is correct yeah. So, the eigenvectors for me. So, what you are looking at is V, V will be coming from these of length one unit vectors because I want V to be orthogonal. So, it will be 1 upon root 2 1 upon root 2 0. This will consist of the second one. So, this was 9.

See, I have to keep the same order 9 is here 3 is here. So, I have to keep the same order. So, first I am looking at 9. The second is 3 3 is nothing but 1 upon root 6, minus 1 upon root 6, 2 upon root 6 unit vector. And this is the last one which is 1 upon root 3, minus 1 upon root 3, minus 1 upon root 3, alright.

So, I have got your as far as we are concerned, and here our sigma transpose sigma is 9 0 0 0 3 0 0 0 0, alright, this is what I have. So, I have got everything here nicely. Let us try to understand what we want. We wanted A to be equal to U sigma V transpose, alright; this is what we wanted, fine. U is already there with us. Where was U? I wrote somewhere U; I wrote U here. This is my U fine. V is there. So, I can just compute V transpose from here. So, I can get this also fine.

Now, sigma is supposed to be a matrix of size U is 2 cross 2. So, this V sigma has to be 2 cross 3, because this is 3 cross 3, alright. So, matrix multiplication tells me that sigma has to be 2 cross 3. So, just write this part, use these two ideas to write yourself. And it is sigma not sigma square or sigma sigma transpose, it is just sigma.

It means that I am looking at a square root of that in some sense fine. So, recall that we had so those ideas with us. So, what we want is they have to be a square root of 9, a square root of 3. And singular values we only allow what are called positive, alright. So, let me write that part also. So, what are singular values now? I will define what are singular value definition.

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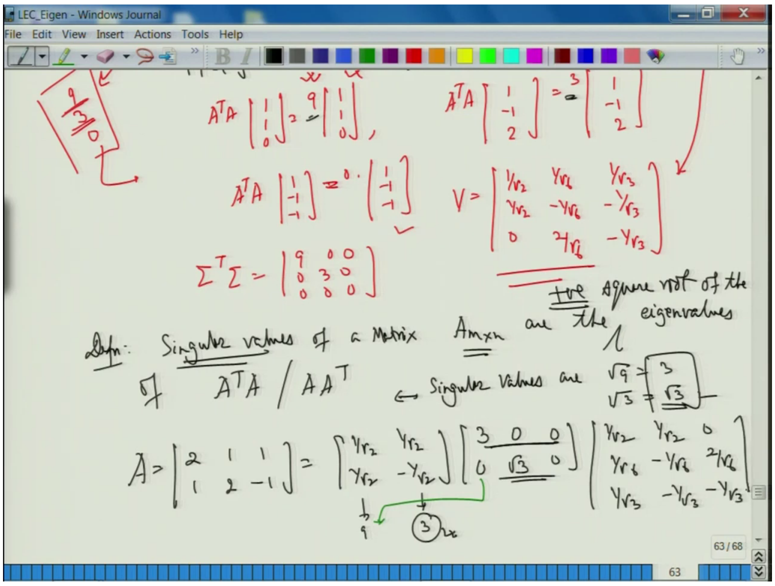

Singular values of a matrix A which is m cross n are the eigenvalues of A transpose A oblique A A transpose are the square root positive square roots alright, no negative, only positive square roots, positive square root of the eigenvalues of this, is that ok, fine.

So, I can have 0 sometimes, I may not have 0, it will depend on the rank alright. So, here in this matrix A, the rank is 2. So, I have got exactly 2 of them which are non-zero that is 9 and 3. If the rank was less, I would have got 0 as an eigenvalue because of some result that for any symmetric matrix, the rank is same as something number of non-zero eigenvalues and things like that, fine.

So, let us do what is singular values. Singular values of a matrix A which is a rectangular matrix are the positive square root of the eigenvalues of these. So, here in this example, my singular values are the square root of 9 which is 3, the square root of 3 which is same as the square root of 3 itself, alright. So, these are the two things that I have. So, therefore, my matrix A, now I can write A which was 2 1 1 1 2 minus 1 is nothing but U; U was this matrix. Now, the singular values, singular value will be 3.

So, be careful do not change the order. This corresponded to 9; this was corresponding to 3. So, I have to take care of 3 0 coming from this part. This is coming from 3. So, this corresponds to root 3 that is this. so, it is root 3 here for U fine. So, be careful. So, first column, first column comes from this matrix multiplication, alright. This is what you have to be careful about.

The last is 0 0 from the row point of view also you can see here that 3 corresponds to the first one that is 1 upon root 2 1 upon root 2 0. But now I have to look at the row, alright, see I am looking at the row transformations now because it is on the left I am writing V. So, I have to write V transpose to make sense here.

So, let us write 1 upon root 2 1 upon root 2 0; root 3 comes from 1 upon root 2 fine, sorry 1 upon root 6 minus 1 upon root 6 2 upon root 6; and 0 eigenvalue comes from 0, sorry [FL] 1 upon root 3 minus 1 upon root 3 minus 1 upon root 3, is that ok. So, I have computed for you the matrix U.

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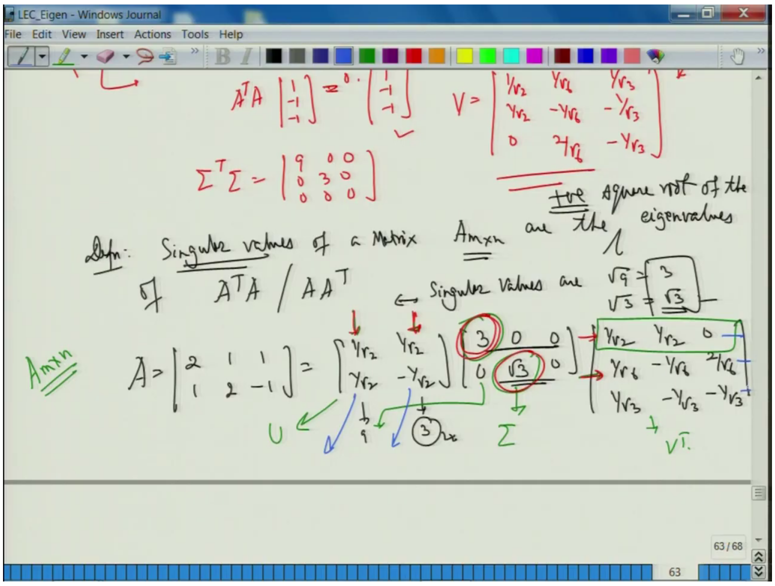

This is your U, this is your sigma, and this is your V transpose. Look at V and V transpose, this is what I have, is that ok. So, you need to understand this and understand them in a very nice way. In the sense that, when we do actual computations, I have the matrix of size A is m cross n, alright. The n could be very large as compared to n or vice versa whatever it is. So, what we are interested in trying to understand, the largest eigenvalue in some sense, alright.

So, we would like to say that, which is the one which is more crucial alright which is the more important direction that I want to go to, alright. So, here this 3 comes from this part, and it also comes from something which is this part, is that ok. So, these are the two parts in which I need to play around is that if I want to look at the minimum, 0 is not allowed for me. So, I need to go through this direction. So, maybe this direction sorry this direction and this direction that I need to look at, is that ok.

So, for each of them, you need to understand where do I need to play with, is that ok, fine. So, in your examples in your theory part everything looks ok, but when you look at applications you have to be careful which rows and which columns need to be looked at. So, again as I said matrix multiplication is the one which comes to play.

If I want the largest, this is the largest among the two singular values; 3 is larger than root 3. So, and this comes from the first one, and the corresponding entries this from this side. So, these are the two things that I need to play around with, fine, similarly minus and so on.

So, these ideas have lots of application when we look at what are called application in biosciences or biology and so on, where we are looking at that there is some drug that has come you want to experiment and then find out things, or means how they behave on different people what are the positives and what are negatives, also who is infected with what and so on, and percentage of infection.

So, these vectors will give you which are the most important factors when which things are going to be localized. Again, I want you to be very careful in the sense that understand that these two vectors that I am looking at fine, so the vectors that I am looking at here this vector sorry this vector and this vector they are perpendicular, this vector this vector this vector are perpendicular.

So, what you are trying to say is that you are trying to get what are called decomposition alright, orthogonal decompositions in some sense. And orthogonal decomposition what we saw was that they will always give you linearly independent things fine, because any set of linearly independent vectors is linearly independent.

So, we are able to decompose whatever issue you have in terms of these are small, small ideas and then build up is that ok. So, what we will say is that I would like to end this course with the following idea that, we started with very simple ideas of solving system of linear equations. From there we saw that this idea can be understood from the point of view of vectors. So, we went to vector spaces. From vector spaces, we could try to find out what happens for finite dimensional cases, what do I mean by bases and things like that.

Once you have understood it, then we went to what are called linear transformation functions. To say that every finite dimensional vector space is basically R n over R or C n over C, and then build on our those ideas, understood matrix multiplication back again. We also showed that every linear transformation from one finite dimensional vector space to another finite dimensional vector space is basically comes from matrix T of B comma B or T of A comma B whatever you wanted to say.

Then we went to eigenvalues eigenvectors and got all the decomposition that was required, alright. So, I would like you to keep track of all the things, alright because they will come in the exam. Singular value decomposition will not come in the exam, but we will have questions about A transpose A, A and A A transpose, because they also come in the chapter on inner product spaces where we looked at the not the actual solution, but the nearest solution alright, a solution which is nearest to the given vector B, is that ok, so that is all.

Thank you.