Linear Algebra
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Lecture - 65
Examples of Conics and Quartics

Alright, so let us take an example in this class. Previous class we gave you the different ideas what happens for two dimension. So, let us take example and proceed.

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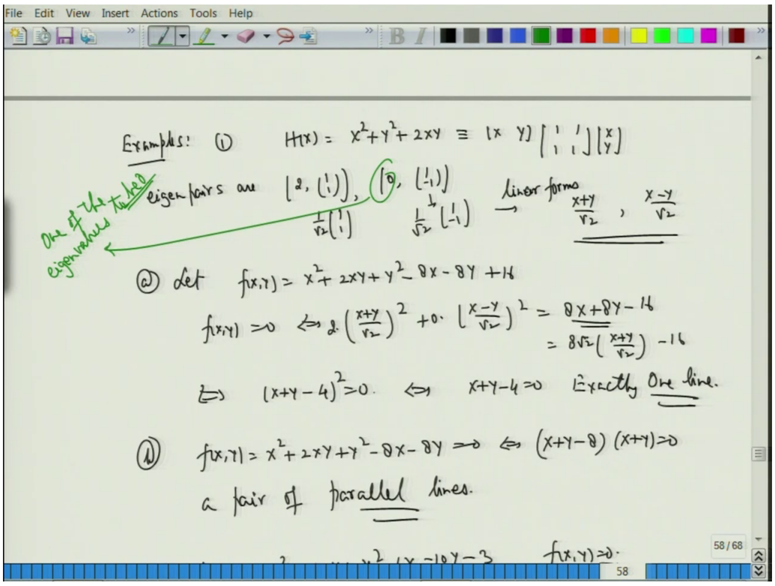

So, actual example with numbers, examples. So, I am going to take H x as the same thing that is the quadratic form is the same and we will see that things changes, alright. So, I will take it as x square plus y square plus 2 x y. So, which corresponds to looking at x y 1 1 1 1 and x y for me, is that ok.

So, the eigenvalues of this matrix, eigen pairs are 2 and 1, 1 and 0 and 1, minus 1. These are the eigen pairs, fine. I want orthogonal; so I will write it as 1 upon root 2 of 1 1 and this as 1 upon root 2 of 1 minus 1 fine, is that ok.

So, those are the linear forms that I am looking at. So, I am looking at the linear forms as multiply by x, y. So, I get x plus y upon root 2 and I get x minus y upon root 2 as my linear forms alright. These are the linear forms that I am looking at, we have suppose to look at the transpose of this and then multiply by x, y. So, if I take the transpose of this, I will get x x, x y; transpose of this will give me x minus y and so on fine.

So, I get this part. So, first example. So, a let f x, y is equal to x square plus 2 x y plus y square minus 8 x minus 8 y plus 16, I want to look at this, fine. So, this is nice form, you can easily see yourself that, it is x square I have taken care of it, x plus y upon root 2.

So, what I have written here is that, f x, y is equal to 0 is same thing as looking at. So, look at this x plus y upon root 2 whole square 2 times this plus 0 times x minus y upon root 2 whole square x square plus y square 2 x y. This cancels out you get this part, is equal to 8 x plus 8 y minus 16.

From here also I can take 8 common; so 8 root 2 common, I can write it as x plus y upon root 2 here fine, minus 16 as it is just like that. So, you can see that this gives me, this expression this what I have written; be careful, I get this part alright fine. So, just, so this implies or it is implied by x plus y minus 4 is 0 exactly one line, fine. So, if you remember we did get exactly one line when alpha 1 was 0, fine that case was there with me, fine.

b, I define, so I take f x, y as x square plus 2 x y plus y square minus 8 x minus 8 y here, fine. So, this tells me that again I have x plus y, but there is no constant here, alright. So, if you look at this part; here I could divide by 2 by 2 and 2 and proceed here, so I could get only complete square here, here I will not get the complete square, fine.

So, look at it nicely. So, what I will get here is it turns out that, this is equal to 0 if and only if x plus y minus 8 into x plus y is 0, fine. So, I get a pair of parallel lines; recall that I had got that set also parallel lines in the previous class, go back and have a look at it yourself, you will get that part. And the next idea was that, in this I got what was called the parabola.

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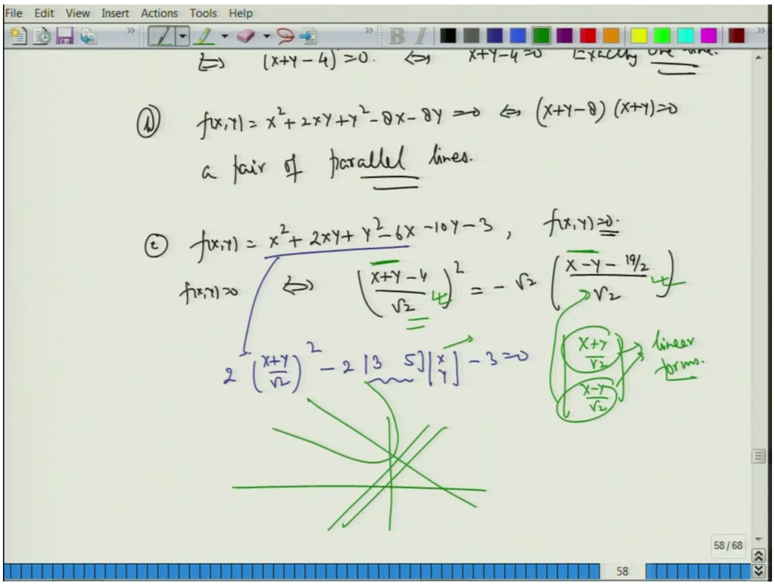

So, let us me take another c, for which I get a parabola. So, f of x, y I define it to be equal to I take it as x square plus 2 x y plus y square minus 6 x minus 10 y minus 3 I take this, alright. And I am looking at f of x, y is equal to 0. I have written it as f x, y equal to 0 is equivalent to looking at x plus y minus 4 upon root 2 whole square is equal to minus root 2 times x minus y minus 19 by 2 upon root 2, this what I have written my expression, alright.

So, this is what I want you to understand that that in general I will get some constants. So, if I want to play with this alright, I am getting here x plus y upon root 2 whole square into 2 fine, minus 6 x or minus of 2 times 3 and 5 x y here minus 3 is equal to 0 I am getting this. So, I should be able to write this alright in terms of first x plus y and this. That is first thing; the second thing I need to do is that, I should be able to get them as x plus y is here, x minus y is here, alright.

So, what I am trying to say here is that, because of the linear forms, I am able to get x plus y and x minus y. See the linear form here was x plus y upon root 2 and x minus y upon root 2, alright. So, these were the linear forms. So, one of the expression x plus y is here, another linear form is here that we are looking at. So, therefore, whenever you have such an expression; because of the linear forms, you can decompose it to get orthogonal lines, is that ok.

So, this line and this line are orthogonal; the slope of this line is minus 1, the slope of this line is 1, is that ok. So, this is just because of the linear force being linearly independent that is the first thing, that you can get a solution of such a type. The second thing is orthogonality comes because of the u being an orthogonal matrix, fine.

So, just look at it, do yourself, you get things fine. So, it turns out that, this is nothing but the parabola which looks like this. So, I do not know I wrote some lines; so it is not passing through origin, where does it pass through I do not know, fine. So, I have something like this and this is a parabola which goes like this, fine. So, verify yourself, get the things yourself and try it out, fine.

So, I have looked at the two dimensional case for you, I want you to understand that. So, this example was only when. So, in this case, we have taken one of the eigen values to be 0. So, one of the eigen values to be 0, fine.

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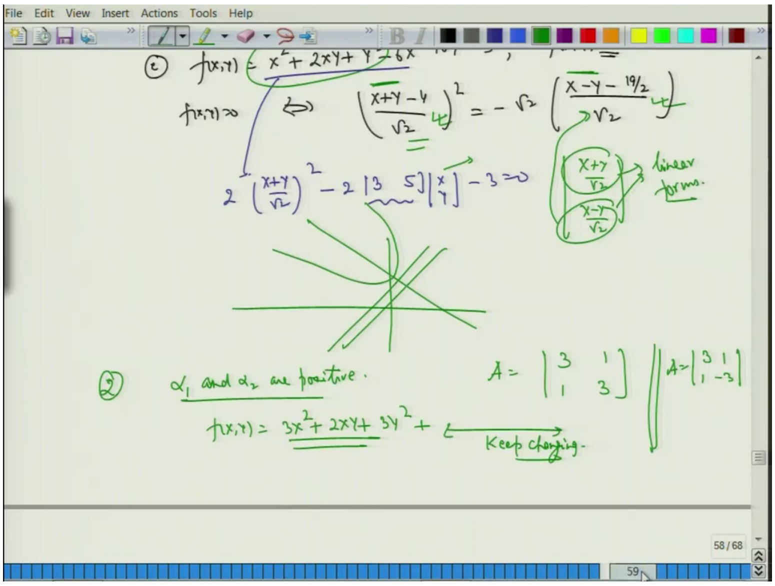

So, now, what are the cases, different cases left out, alright? So, you can look at different cases. So, case 2. So, second case will be alpha 1 and alpha 2 are positive.

So, I would like you to try that out yourselves. So, you can take the matrix A as say 3, 1, 1, 3 fine or let me make it, let it be 3 itself, so that you get symmetrical, you can understand it easily. Look at different cases; so construct the examples yourself. So, what I am saying is that, you take f of x, y as 3 x square plus 2 x y plus 3 y square; fix this part and in this part keep changing, alright.

So, look at this example this is what I did; in all the three example, this part is the same, alright. This part is the same and then I have made changes here to understand the different ideas, is that ok. So, I want to you to do the similar thing that fix them and look at different examples, is that ok. So, this is one example that you can look at, where I am saying that alpha 1 and alpha 2 are both positive.

Similarly, look at the case when alpha 1 is positive, alpha 2 is negative; you can think of that, fine. So, for example, you can take A as 3, 1, 1, minus 3, fine. So, here I can see that one of the eigenvalues will be positive, the other will be negative or I do not know what the expression will be, it may be a bit complicated. So, you can modify it slightly and proceed yourself. So, try that out yourself.

Now, I will go to what is called the three dimension alright, is that ok. So, let me go to three dimension for us now.

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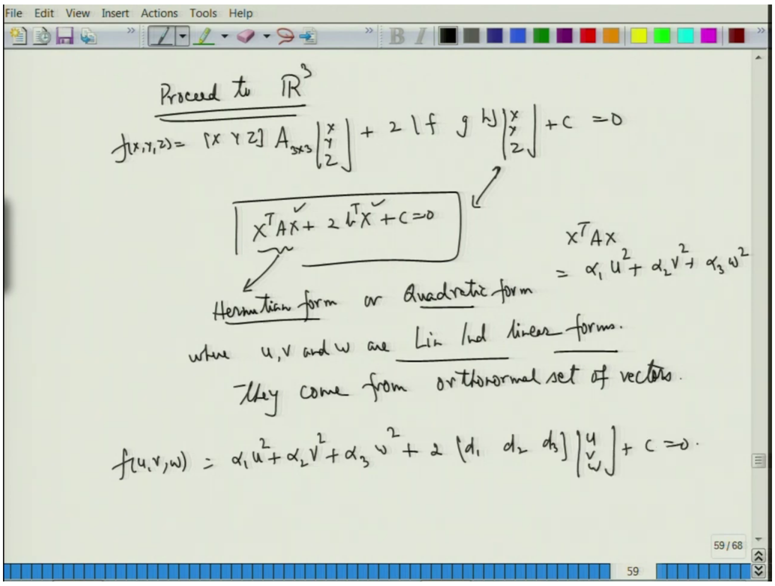

Proceed to R 3, is that ok. So, in R 3 I will have got three things f of x, y, z will be equal to x y z; a matrix A which is 3 by 3, I will have x, y, z. Again I will have 2 times sum f, g, h; x, y, z plus a constant c which is 0. This is what I will get, is that ok. So, the form will remain the same; earlier I had the form was recall the form was X transpose A x plus vector b transpose x, which is b here plus c equal to 0, fine.

I also have the same form here itself, in four dimension also or five dimension whatever dimension you want to understand; the expression will be similar in nature. So, for each one of them you are supposed to look at just this part. The look at the Hermitian form, Hermitian form or the Quadratic form whatever you want to call quadratic form. Hermitian for over complex, quadratic over real numbers and so on.

So, write X transpose A x as alpha 1 u square plus alpha 2 v square plus alpha 3 z square, where u, v and z are linearly independent linear forms, alright. Do that yourself; whatever way you want, you can do that. So, what we are saying is that linear forms are independent means; I have got linearly independent vectors, they are orthogonal also. So, linear forms means, they are they come from orthonormal set of vectors, is that fine.

So, once I have got orthonormal set of vectors, I can replace each x here by u, v and w, fine. So, I will get something nice here. So, I will now write it as f of u, v, z alright; I have written z. So, no this z can be confusing. So, let me write w here; u, v, w as alpha 1 u square plus alpha 2 v square plus alpha 3 w square plus. Now some 2 times d 1, d 2, d 3 times u, v, w plus a constant c is equal to 0, this is what I will get now, fine.

So, the complicated part which was X transpose A x is nicely written like this, fine.

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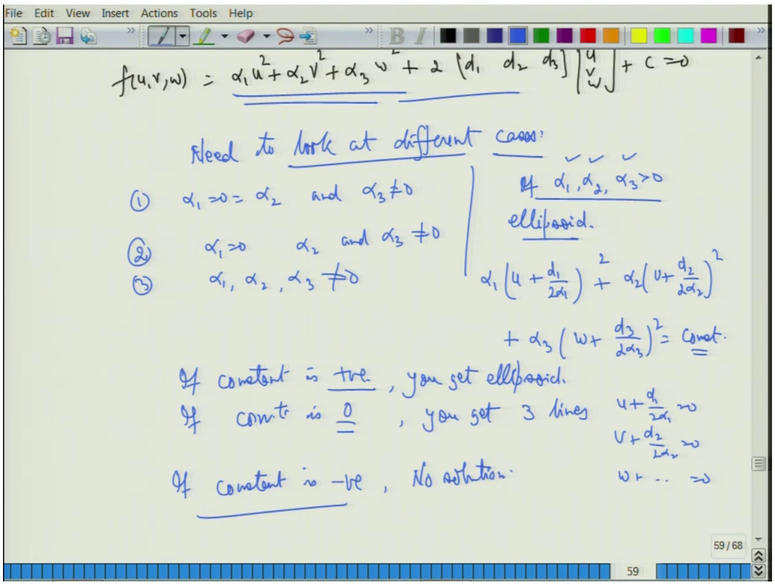

And now I can again do the same thing, I can look at d 1 times u is here; so I can write it as. So, if I am assuming that alpha 1, alpha 2, alpha 3 are all non-zero; I can do nice things, if it is 0 I will have issues. So, need to look at different cases, need to look at different cases.

What are the cases that we need to look at? One will be that alpha 1 is equal to 0 is equal to alpha 2 and alpha 3 is not 0 fine; you have to look at this case. You have to look at the case alpha 1 is 0, alpha 2 and alpha 3 are not 0 and the third case none of them is alpha 1, alpha 2, alpha 3 not 0. So, let me look at all the three cases. In each of the cases again I have to look at positive, negative and so on, we will have to be careful and proceed, fine.

So, you need to understand each of the cases separately. So, if alpha 1, alpha 2, alpha 3 are positive; then you can get what is called an ellipsoid. Why you will get an ellipsoid? Again understand that nicely; all these are the axes the length of the axes in some sense they correspond to that part. And you can look at this here, this expression in this case will turn out to be alpha 1 is outside. You will get u plus d 1 upon 2 alpha 1 whole square plus alpha 2 v plus d 2 upon 2 alpha 2 whole square, unless I have done a mistake.

So, be careful, because I am very good at mistakes. So, you will have to rewrite yourself; w plus d 3 by 2 alpha 3 whole square is equal to some constant, fine. So, if this constant is positive, if constant is positive, you get ellipsoid. If constant is 0 fine, you get 3 lines u plus d 1 upon 2 alpha 1 is 0, v plus d 2 upon 2 alpha 2 is 0 and third one. So, you get a point of intersection of those lines; third one write yourself d 2 plus something is 0, you will get it.

If constant is negative, no solution fine. Which I will assume that alpha 1, alpha 2, alpha 3 are positive. Or you can just say that alpha 1, alpha 2, alpha 3 are of the same sign and then relate the two ideas directly yourself, alright fine.

You can also from here say that, if alpha 1 equal to alpha 2 equal to alpha 3 all three are constant; then you will get an sphere, fine. So, you can change this idea according to circle and ellipse; similarly a sphere and ellipsoid that you have, fine.

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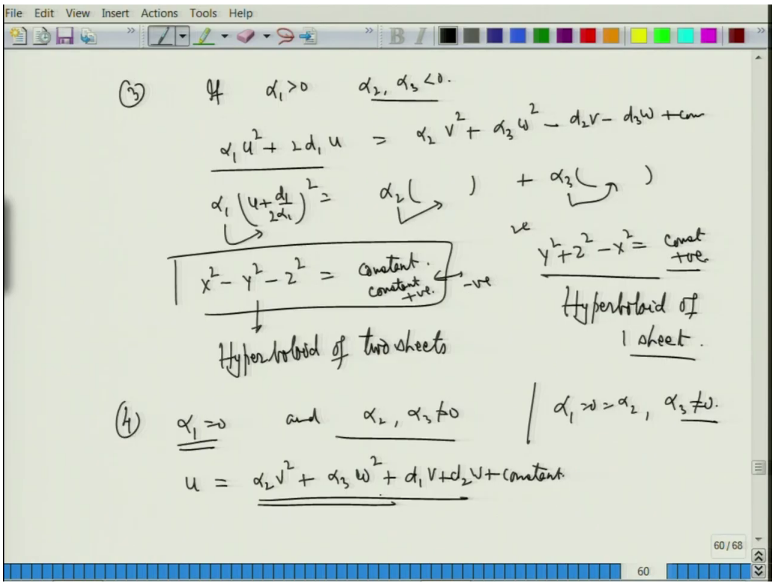

Now, if here in the third case, so third case, if alpha 1 is positive alpha 2 and alpha 3 are negative fine. Then I will get it as alpha 1 times u square, I will just keep d also itself. So, d times u here, so 2 d I think I wrote; I do not know what I wrote. So, I think I wrote 2 d. So, whatever I write 2 d 1 d u. So, I can put this alpha 2 out the other side, alpha 2 times v square plus alpha 3 v, alpha 3 w square or minus whatever it is, minus d 2 v minus d 3 w plus constant here, fine.

Again I can write it as alpha 1 u plus d 1 upon 2 alpha 1 whole square is equal to alpha 2 times something plus alpha 3 times something, fine. So, I will get something like this part fine. What will this object be, fine? So, you have to again careful. So, what you are writing that you can write like this; but you need to understand that I, so also I can put these alpha 1s inside, alpha 2 inside, alpha 3 inside I can do it.

So, in some sense I am going to get an expression like something like x square minus y square minus z square is equal to constant fine, is that. So, I can assume that constant is positive; I can assume constant is positive, constant positive.

But if it is negative, negative will imply that I can look at y square plus z square minus x square as constant, which is positive, is that and I just multiply minus 1 throughout, fine. So, they are going to give me what are called; do you remember something, hyperboloid of one sheet and two sheets, alright. So, there is a notion of sheets here that comes into play; I think I have written it in my notes or not let me just see, I have written, alright fine.

So, if I am take this as positive, this will be hyperboloid of two sheets. This will give me hyperboloid of 1 sheet, fine. So, what I am trying to say is that, you can just build up your ideas from here, fine. So, positive negative I have taken care of all of them here in some sense; this also tells me that two eigen values are positive, the other is negative. So, that case is also taken care of. My only case now left out in some sense is one of them is 0 and the other two are not 0; alpha 2 and alpha 3 are not 0, fine.

I also have a case where alpha 1 is 0, alpha 2 is 0, alpha 3 is not 0; you can look at that part also. So, alpha 1 0 is equal to alpha 2, alpha 3 not 0. So, you can look at that part also. So, when alpha 1 is 0; what does it mean? I get something like u is equal to alright; something like alpha 2 v square plus alpha 3 w square plus d 1 v plus d 2 w plus constant, alright.

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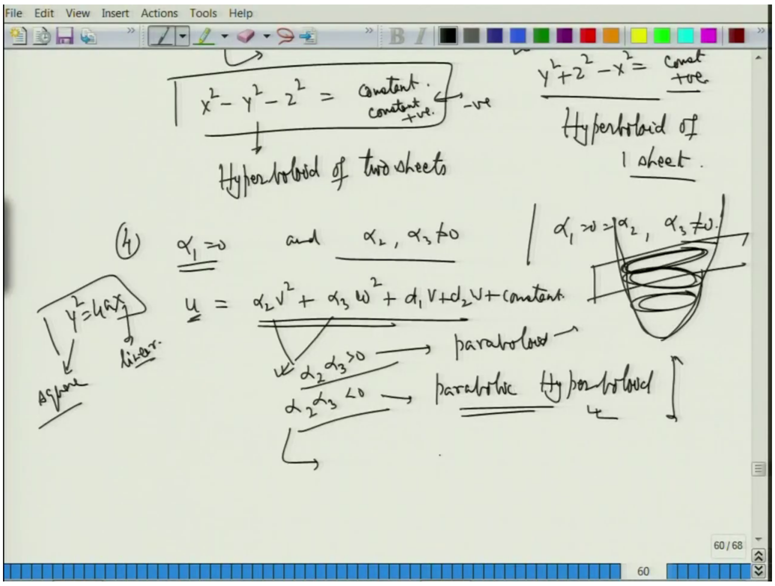

So, you can see that you are getting in some sense of parabola because in parabola the question was y square is equal to 4 a x. So, there was one square term and there was a linear term here. So, a square, this was a linear term, fine. So, I need to understand both of them separately. Further if I look at these two, if alpha 2 and alpha 3 are positive that is one case, or alpha 2 alpha 3 is negative.

So, I should not write alpha 2, alpha 3 positive; alpha 2 into alpha 3 is positive, accordingly the direction will change. So, there are two cases; this is positive, the other is negative. This will give me hyperboloid of a different type, so this is called parabolic hyperboloid. This will give me paraboloid itself and this is positive; basically it means that, I have got some sort of ellipse. So, at each place if I put u to be something; fix a constant for this u, I will get this.

So, in this case I will get an expressions. So, the figure will be something like this that I will be looking at. So, this is the parabola that I am looking at that, each of these are nothing, but these are ellipse. Because alpha 2 and alpha 3 are positive, so I will get an ellipse here, fine. If u is negative, these are some high numbers; this part will become 0. So, I am going to get only one part, is that ok.

This basically means that, each of the intersection, each of the say what should I say the cut off that you look at alright. If you cut it something, then alpha 2 is positive, alpha 3 is negative or vice versa will give you each of them should be hyperbola, alright.

So, if I look at this cutting here by a plane, I am getting a circle here alright, an ellipse in some sense, alright. What I want here is now in this case, all the intersection that I am going to look at to the plane; that is this u equal to 0 is going to give me a hyperbola and therefore I got a hyperbola, is that ok.

So, I find it difficult to plot these graphs, alright.

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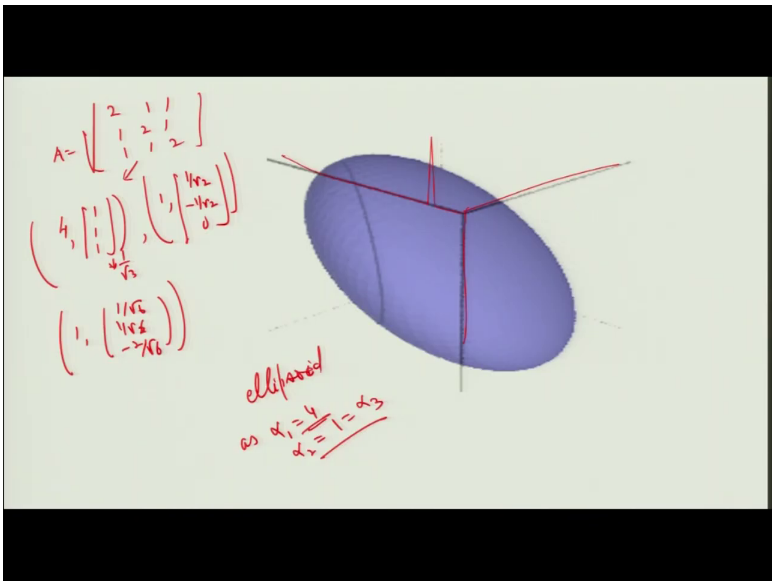

So, this is the example which I wanted you to see here that, what happens in case of three dimension. So, I have got some of the figures for you. So, this corresponds to, this figure corresponds to looking at A as this matrix 2. So, the matrix is 2 1 1 1 2 1 1 1 2 which corresponds to.

So, eigen values are 4; eigen vector is 1 1 1. So, this will give me 1 upon root 3 times this as my unit vector. I have an eigen value which is 1, which corresponds to 1 upon root 2 minus 1 upon root 2, 0 as an eigen vector fine, this is an eigen pair. This is another eigen pair and the third pair is again 1; but here it is 1 upon root 6, 1 upon root 6, minus 2 upon root 6 alright, this is the eigen pair that I am looking at. So, my orthogonal vectors are these three orthogonal vectors

And therefore, I get this; if you can see here that this is a ellipsoid that I have got, ellipsoid as alpha 1 is 4, one eigenvalue is 4, the other eigen value are the same. So, I get an ellipsoid, is that ok.

So, all three are positive and I have got this for you, is that ok. So, this gives me the ellipsoid and these are the three. Look at here, these are the three which are orthogonal alright fine, principal axes that you are looking at and that is what the requirement was, fine.

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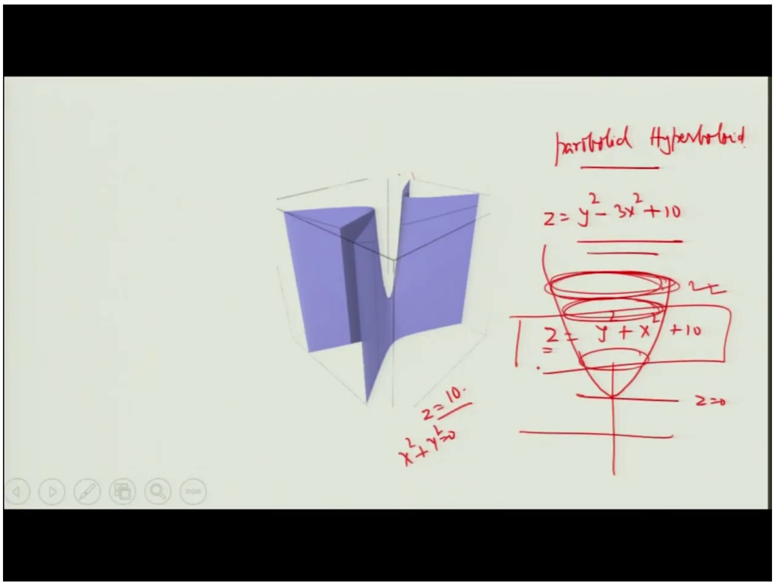

Look at the next one; the next one is this is the parabolic hyperboloid, bolic hyperboloid.

So, you can see that, if I cut anywhere. So, if I want to cut like this here from here, I will get something like this, fine. So, I am going to get here and the again something here I am going to get like this, is that ok. So, each of these sheets are separate in some sense, fine.

If I just cut it, I will get two parts and things like that. So, I would like you to see this nicely and the equation of this is I have written as z is equal to y square minus 3 x square plus 10, this is the one that I am looking at, fine. So, depending on z is positive or negative, you can keep changing here and you can keep getting it here, fine. So, this was parabolic hyperboloid.

The other one I forgot to write I think, you could have taken here as y square plus x square plus 10 if I take, I will get z is increasing with this height. So, as z increases these things the size of these things will also increase, fine. Just look at it; if z is increasing, y x square the value y x square can also increase. So, I will get an, if I take z has to be minus 10, z is equal to minus 10; I will get x square or z as 10, sorry z as 10.

Then x square plus y square is 0. So, x and y will be 0. So, at this point at z is equal to 10, it will start and it will go like this, fine. And each of them will be hyper, will be an ellipse that will come into play, is that ok.

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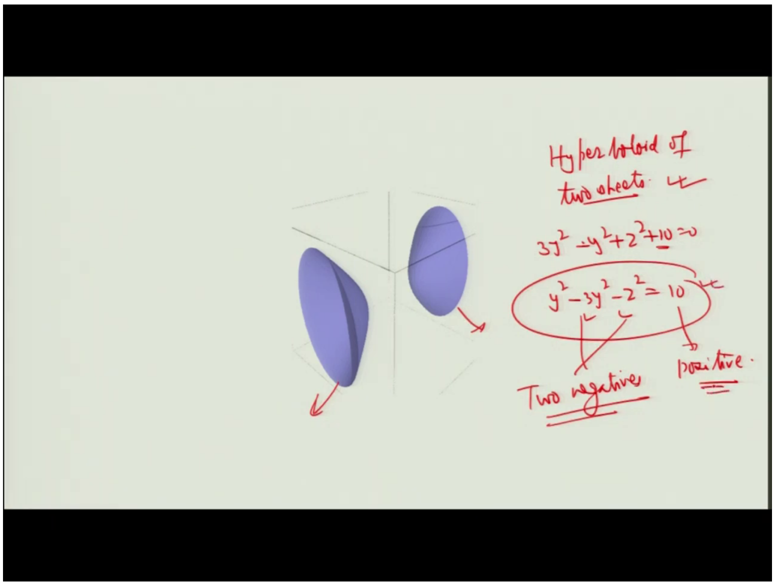

The next is this is the hyperboloid of two sheets and the equation that I have written here is hyperboloid of two sheet is 3 x square minus y square plus z square plus 10 is 0. So, what you are writing here is, just see here two sheets basically; because plus 10, I have to put it that side. So, I will get negative, fine. So, what I will get is y square minus 3 y square minus z square is 10, this is what I had written; I had taken everything with respect to this being positive, alright.

So, this is positive, there are two of them. So, there is a two sheets; as one sheet here and one sheet here, there are two sheets, fine.

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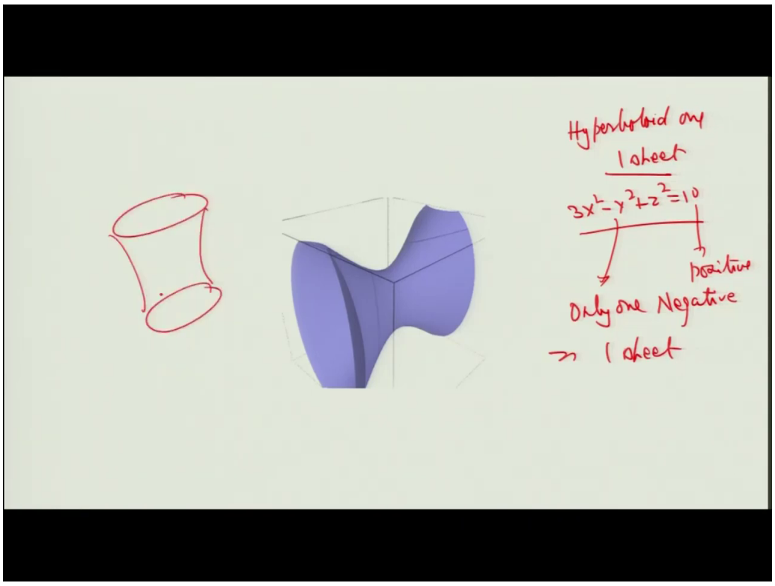

And the last one is hyperbola of 1 sheet. So, let me write that equation. So, hyperboloid of 1 sheet here; the equation is 3 x square minus y square plus z square is 10, that the same thing I wrote, fine. So, this is again I have taken as positive and there is only one negative alright; only one negative implies 1 sheet, alright.

In the previous one if you see, I had two negatives here and therefore, there were two sheets, is that ok. So, they get related with these ideas that you have.

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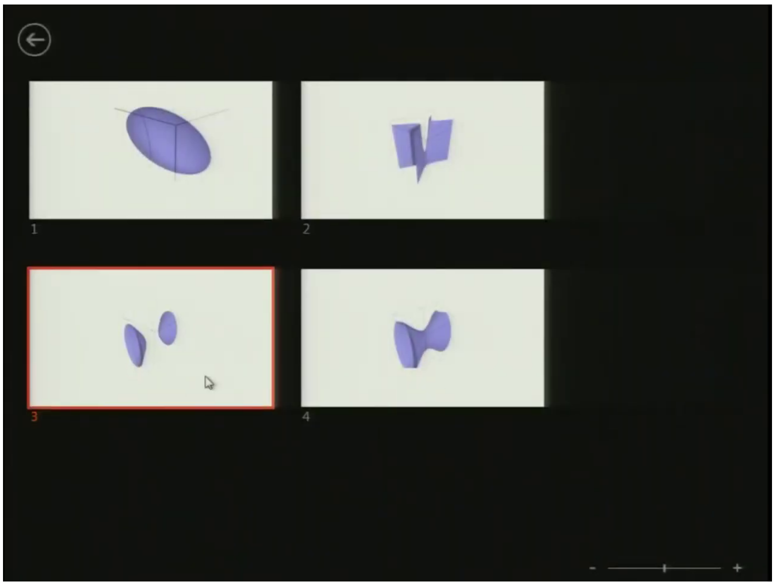

So, these are the four diagrams that I have with for you; one is ellipsoid, so one was ellipsoid, the other is parabolic hyperboloid. For the main paraboloid, where, my each of these things are ellipse that I have not plotted; but I have given you this equation here, that you can check for yourself. This is two sheets, because they are two separate parts.

And there are two negatives, so when I write the constant as positive constant have to be always written as positive, so that you can make a statements. And the last one is this, only one sheet here and this is the one that we generally have what is called a mudda at our place, at our home. It is sold in the market, you have these mudda’s that you have this size. So, these are the mudda’s that you are looking at, fine.

So, that is all for this course as far as geometry is concerned. As the last lecture, I will talk about alright; what is called the singular value decomposition, which is not in the syllabus as such. But I will just give you some brief idea about it, so that you can use it in your applications; that is all.