Linear Algebra
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Lecture – 64
Applications of Quadratic Forms to Analytic Geometry

Alright. So, let us recall what we did yesterday; alright or in the previous class.

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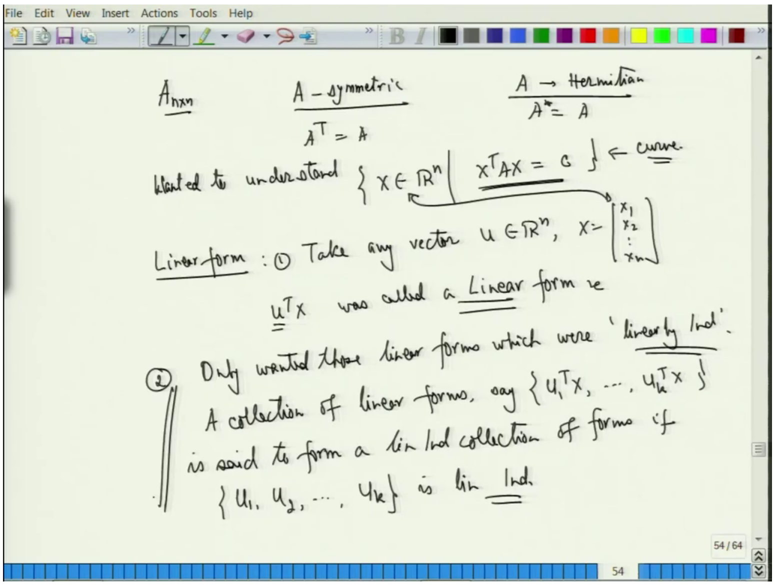

What we did was that we had this matrix A; which was n cross n would to get to be symmetric fine. In general what we wanted was that a as should be Hermitian for complex part we wanted it to be Hermitian, so that A star is A and A it means that A transpose is A this is what we wanted, fine.

Now, after that we defined, we wanted to understand wanted to understand this what is this? At x belonging to all R n such that X transpose AX is equal to say some constant C, alright, fine. We wanted the curve what this curve is? That was the idea for 2 cross 2 we try to understand fine I have not I just gave some examples to make you understand things.

So, to do things what we wanted was there was this notion of what was called a linear form. So, what was a linear form take any vectors. So, take any vector u in R n and X in my vector X 1, X 2, X n alright; which gives me idea from here that I want to look at solve a system in some sense fine or we want to find the solutions of this equation for a fixed C.

So, there take any vector u, then this u transpose X was called a linear form alright; that is one thing. Another thing we talked off was we only wanted those linear forms we wanted only those linear forms which were linearly independent, fine. And by this what we wanted was that, each of the linear forms they come from some vector u.

So, we wanted that so, a collection of; a collection of linear forms a collection of linear forms say u 1 transpose X till u k transpose X this is said to form or said to be linearly form or linearly independent form or to form or linearly independent collection of forms if the corresponding vectors so, if the vectors u 1, u 2, u k is linearly independent alright, fine. So, idea was that I wanted to collect linear forms why we want to collect linear forms? Let us try to understand that part.

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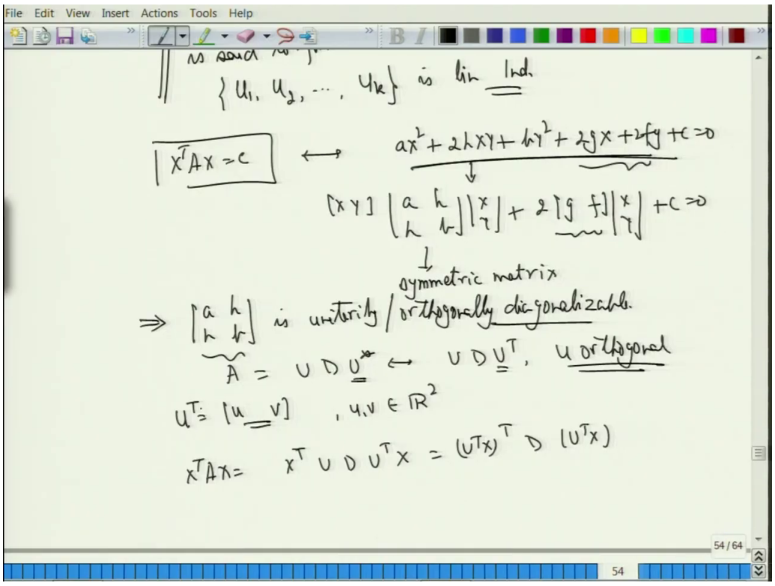

So, initially we only wanted to look at X transpose A X is equal to C, but when we look at applications or anywhere else what we basically get is, we get expressions like the following. So, let me just write the second degree; third degree you can think accordingly. So, what we have is AX square plus 2 h XY plus b Y square plus 2 g X plus 2fy plus c equal to 0 this is what we look at generally when we look at second degree.

Similarly, for the third degree we look at something else. So, what we want is that I can write this part alright as a h h b times X Y; times X Y fine, but the this part which needs to be retained. So, this I can write it as g f times X Y plus c equal to 0, alright. So, there is a notion of this g f that comes into play. So, we need to take care of that part also, is that ok?

Now, what does the linear form say? So, the idea was that this is a symmetric matrix and therefore, this will imply that I can unitarily diagonalizable. So, implies this matrix a h h b is unitarily or orthogonally diagonalizable, fine.

So, what we mean is that I can get. So, I can write this matrix A as U D U star or since everything is real here if I assume then I can write it as U D U transpose U orthogonal, fine. That is the first thing that we have and then we had this notion of the linear form. So, the what are the linear forms? Look at this matrix u or this u whatever you want to look at. So, write u transport as u and v, they are two vectors, fine.

So, u and v belong to R 2 for us because I am looking at real orthogonal here, you can go for complex also whatever you want to say fine and this we are multiplying X and Y, is that ok? So, let us recall what we did X transpose AX was X transpose UDU transpose of X which is same as U transpose X whole transpose D times U transpose X fine.

So, I need to multiply this U transpose X look at U transpose X, alright. Now, so, I did not write u transpose like this I needed U transpose X itself like this, alright. So, I did not want this part, alright.

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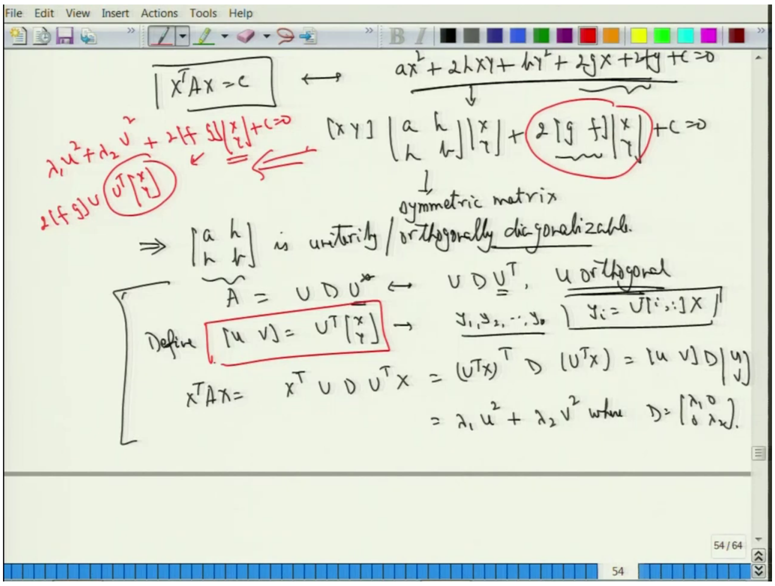

What I wanted was define this vector u v as U transpose of X Y, alright. So, if you recall a linear forms were y 1, y 2, y n and those were nothing, but y i was look at U alright I think U transport whatever it is just. So, it will be U transpose look at the ith row of that times this vector X, fine. This is what we had. So, do that yourself.

So, if I write this as u v alright so, then what I get is here u v here fine, D here and u transport v transpose here or u just a linear form. So, it is just u itself, fine. So, it is u and v itself because they are linear forms the straight lines this is a single expression, fine. So, this u get as lambda 1 times u square plus lambda 2 times v square where D is lambda 1 0 0 lambda 2, fine. So, this part is taken care of nicely. What is more important than this is that, you should be able to so, we should be able to write this also in terms of u v.

So, what we have done is that I have rewritten this as if you look at this part lambda 1 u 1 square plus lambda 2 u 2 square plus I still have f g and X Y with me fine. So, I still have X Y as well as u 1 and u 2, sorry u and v, u and v square alright.

So, I have got four variables, now I have got u, v as well as X and Y. So, that makes my problem more difficult in some sense alright. So, what we do is that, we want to replace X, Y also by u and v itself. So, we write this part as 2 of f g alright multiply by u; multiply by u transpose X Y I can do this, fine.

Now, what is this? This is nothing but look at our definition of u v, this is what it is, fine. I should have written it correctly, alright.

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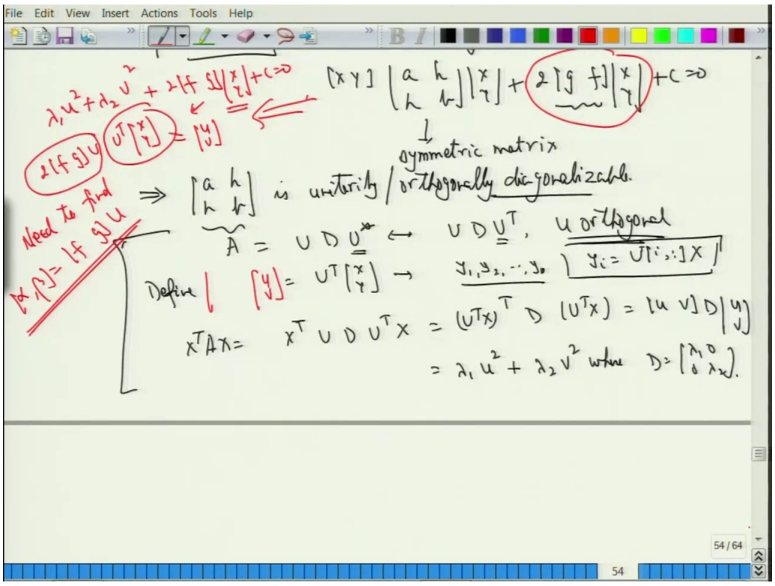

So, let me rewrite that size I think; this size is wrong it should be u v here, u and v it should be alright, fine. So, that you get the transpose and then this, alright. I should have written like this, fine.

So, now I can replace this by u v, is that ok? Once I replace this u v this part will give me another set of null values. So, I have to look at f g times u and so, need to look at need to find alpha beta so that alpha beta is equal to f g times u, is that ok? I need to find this, fine. So, I need to find alpha beta such that u times f g is there.

So, f and g are known, u is known. So, I want to find alpha beta, fine; that is the first thing that we need to do.

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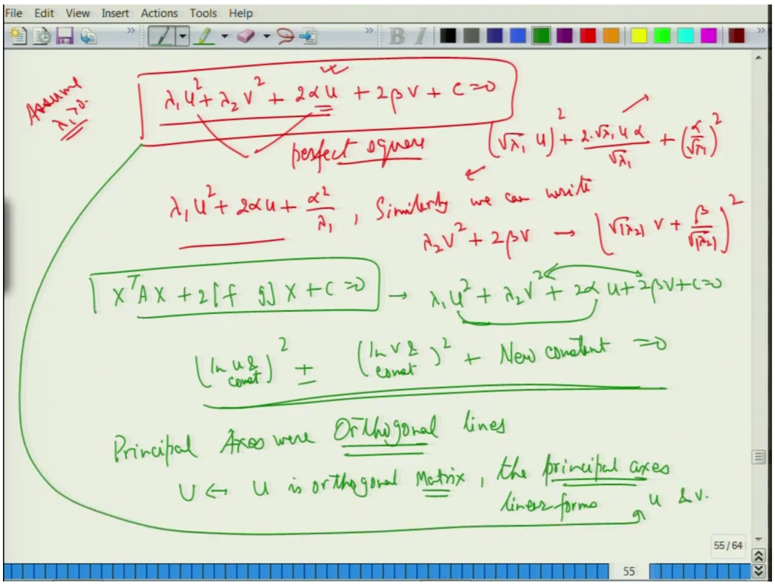

So, that you can write this expression the previous expression as lambda 1 u square plus lambda 2 v square plus I can right now 2 alpha u plus 2 beta v plus sum whatever it is. So, there is nothing else. So, plus c is equal to 0, I can write like this, fine? That is the first thing I need to do to replace X Y by u v and then now I can do this I can look at now. So, I can divide by whatever it is. So, alright.

So, mode of lambda 1 square root u for u alright whatever it is. So, assume let us assume lambda 1 is positive, I do not know about lambda 2. So, what I am trying to do is, that I want to use this to make it a complete square perfect square perfect square. So, in the sense that I want to write it as a square root of lambda 1 u 1 plus square of this, so that I can take care of this.

So, if I want to do it is 2 times the square root of lambda 1 u into alpha u divided by square root of lambda 1, alright, fine. So, if I just multiply by alpha here it is not u 1 u itself u and alpha. So, let us look at square of lambda 1 square root of lambda 1 cancels out I get 2 alpha u I get 2 alpha u. So, this square if I look at this gives me this gives me lambda 1 u square plus 2 alpha u cancels out, fine plus a square of this.

So, what is this expression if you see nicely? It is so, I want this. So, let me write this. So, this cancels out this cancels 2 alpha u plus. So, this part comes from here. So, alpha square root of lambda 1 whole square, this what I am looking at, fine.

Let us see this square 2 times the square root of lambda 1 u upon this and alpha up on this. So, 2 alpha 1 this plus cancels out this plus alpha square by lambda 1, I can make this. Similarly, we can write this part lambda 2 v square plus 2 beta v in terms of.

So, if lambda 2 is positive I can still do the same thing if lambda 2 is negative I can replace lambda 2 by minus of something and I still do it. So, again I will get similar expression here which will be square root of mod of lambda 2 v plus alpha. So, it is beta upon mod of lambda 2. So, square root of lambda 2 something like this if I do. Let us see square of this. So, I will get lambda 2 v square plus 2 v v lambda 2 will cancel out 2 beta v and something will come out, alright.

So, you can see that I can write this expression in terms of this alright, fine. So, let us understand again. We wanted to look at X transpose AX plus 2 f g times X plus c equal to 0, I want to look at this first we write it in terms of lambda 1. So, we write it as lambda 1 u square plus lambda 2 v square plus 2 alpha u plus 2 beta v plus c equal to 0. Now, I subsumed these two together and these two together and there will be some constant.

So, I will now get something which will look like. So, lambda 1 or forget about lambda 1 it will look like some expression fine in u and constant fine plus or minus another square which will be in v and constant fine plus a new constant, is that ok? So, this is what the expression that I am going to get, fine and from here I want to say that I get all the results. So, let us take examples to understand it one by one nicely and proceed, is that ok?

So, what was the idea of linear form? The linear form idea was basically that you get this u which has all the dimensions. So, if I am looking at R 2 u has two columns if I have got R 3 I will get three of them and four of them if it is R 4 and so on, fine. So, the linear form helped us to get this part and talk of the inverts and things like that so that I can get answers, fine. So, let us look at it.

Principal axes are orthogonal. Principal axes were orthogonal lines, is that ok? they were orthogonal lines. So, what we have here is that here the matrix U which is U is orthogonal matrix orthogonal matrix and what were the lines? The lines are the principal axes are nothing, but principal axes are what are principal axes? The linear forms linear forms. So, in this example if I look at in this example the linear forms are u and v, is that ok, fine?

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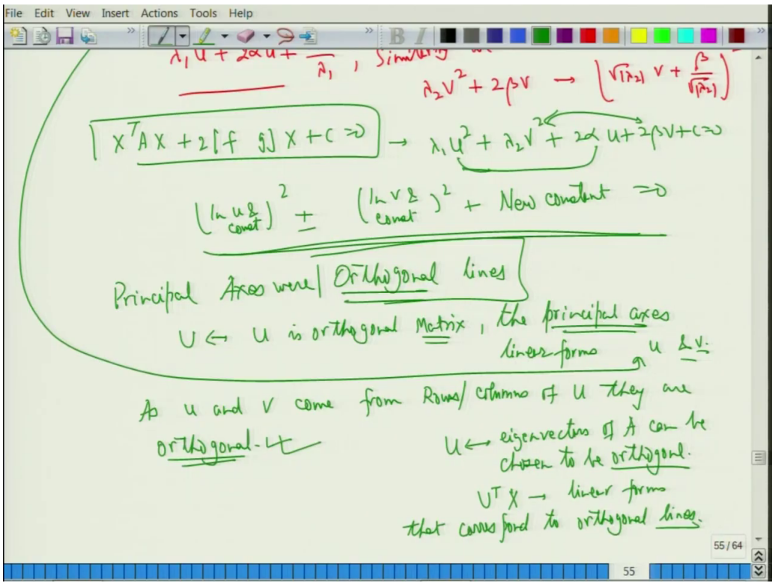

So, as u and v, v come from rows or the columns of u they are orthogonal, alright. So, the idea was to get the principal axes to be orthogonal lines were indeed getting orthogonal lines basically because I can orthogonality or can use orthogonal matrices to diagonalize the symmetric matrices, is that ok?

So, everything we are able to do because of the matrix A being symmetric. Matrix A being symmetric means I have got eigenvectors of A can be chosen to be orthogonal. So, once I choose them to be orthogonal, I get U transpose X linear forms that correspond to orthogonal lines.

And, therefore, I will get all the principle axes, is that ok, fine? So, let us now take examples one by one and proceed to get our answers, is that ok? So, let me look at the examples that we have examples.

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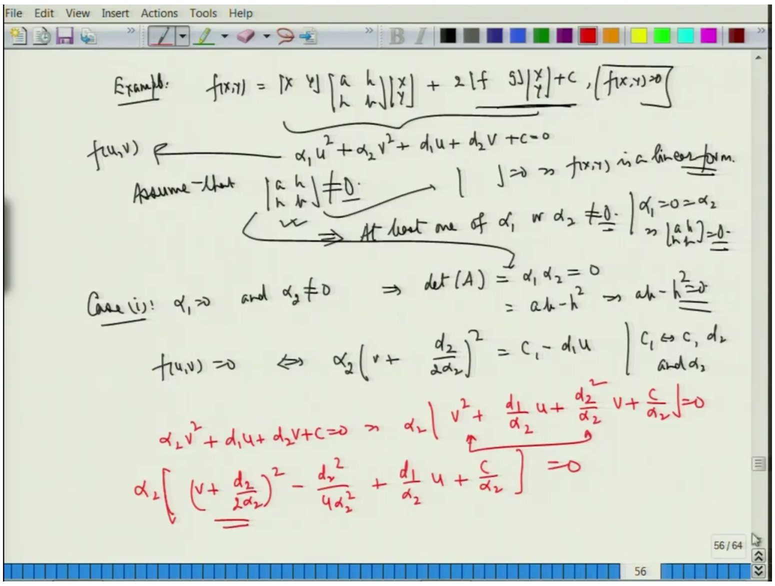

So, example so, I am starting at two dimension now, fine? So, I wrote. So, from f of X, Y which was our X Y a h h b X, Y, fine plus 2 f g X Y plus c equal to 0. So, we are looking at f X Y as that and we are looking at f X, Y is equal to 0. This is what we are looking at to get all the points of the curve that we need to understand, fine.

So, from here we are able to write it as some. So, I have written in my notes as alpha. So, let me write alpha itself in place of lambda. So, I wrote alpha 1 u square plus alpha 2 v square plus d 1 of u plus d 2 of v plus c is equal to 0, I wrote it like this. Now, I will get different cases. So, case 1, alright.

So, before I write this I also have to assume that this vector this matrix a h h b is not the 0 one, alright. If it is 0, then I have just one linear equation. So, this is equal to 0 implies f X, Y is a linear form, alright. I do not want to look at the linear form, I want to look at the quadratic form, alright.

So, let us look at quadratic form. So, I get a matrix like this it is a symmetric matrix. So, I will get u and v, is that. I will get two linear forms which are linearly independent, fine. That is more important that I have in indeed got two linear forms u and v alpha 1 alpha 2 could be 0. One of them could be 0, the other may not be or the other cannot be because diagnosable and I am looking at this not 0, fine.

So, therefore, this part also implies prove it yourself that at least one of alpha 1 or alpha 2 is not equal to 0; because if both of them are 0, then this matrix will turn out to be 0 matrix fine. Alpha 1 is equal to 0 is equal to alpha 2 implies this matrix a h h b is a 0 matrix. Try that out yourself prove it yourself, fine. So, that assuming this part.

So, now, let us look at different cases case 1 alpha 1 is 0 and alpha 2 is not 0, fine. I have to be careful otherwise I will have problem. So, this implies what? I am assuming that alpha 1 is 0, recall what is the determinant of the matrix A? Determinant of the matrix A is nothing, but product of eigenvalues.

So, therefore, this has to be 0, is that ok? And, what is the determinant of this matrix? Determinant of this matrix is also equal to a b minus h square. So, we need that. So, these two together will imply that ab minus h square is 0, fine. So, you get this condition. So, let us rewrite this now.

So, f of u v which was this for me f of u v is equal to 0 if and only if alpha 2 times v plus d 2 upon 2 alpha 2 whole square is equal to c 1 minus d 1 of u. Now, what is c 1? c 1 is a function of c and c d 2 and alpha 2. So, basically what we have done is so, I will give you for this example for the next I will not do it. So, I will just explain you how I got this, for the rest you have to see it yourself, alright.

So, let us look at this alpha 1 is 0. So, what I get is alpha 2 v square plus d 1 u plus d 2 of v plus c is 0. This implies I can divide throughout by alpha 2 or I can just take alpha 2 common, I get v square plus d 1 upon alpha 2 u plus d 2 upon alpha 2 v plus c upon alpha 2 is 0. So, alpha 2 is common, fine. So, I can just look at this part v and this here together.

So, I get v plus d 2 upon 2 alpha 2 common this whole square I have added this I have to subtract minus d 2 square by 4 alpha 2 square. I have to do that plus d 1 upon alpha 2 u, alright plus c upon alpha 2 this whole thing is 0 for me just look at this nicely, fine.

So, this gives me v square which is v square is there d 2 square 4 alpha 2 which is cancels out, I get the second term which is 2, 2 cancels out; d 2 upon alpha 2 v which is also there, this part is taken care of. d 1 upon alpha 2 u is here and c upon alpha 2 is here. So, you are writing this part fine, alpha 2 was here also is alpha 2.

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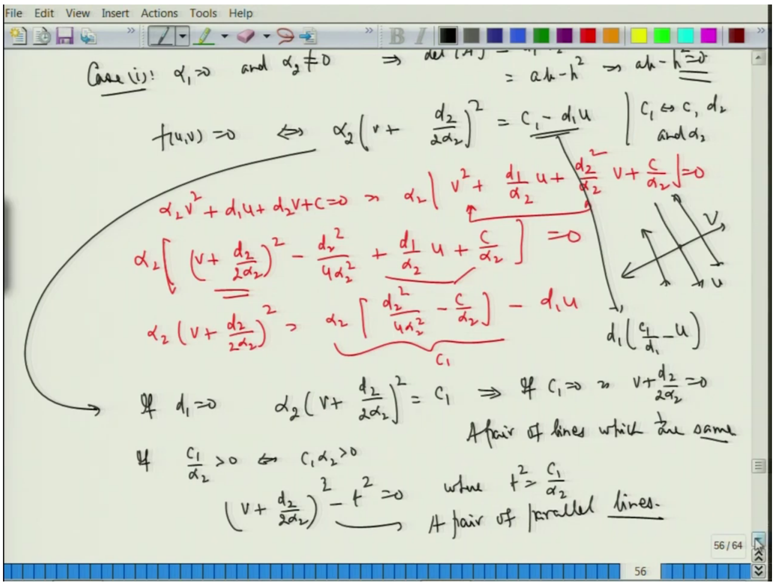

So, I can take out alpha 2 one side v plus d 2 upon 2 alpha 2 whole square one side and the rest the other side. So, I can take the everything the other side. So, it is alpha 2 common. Again, if you see here d 2 square by 4 alpha 2 square minus c upon alpha 2, they will come together; alpha 2 alpha 2 cancels out, fine. Look at this minus d 1 of u is this what you are going to get, fine.

So, this I took as my c 1 in my example, fine. So, if my calculations are correct then c 1 is this and minus d 1 of u as come up as, it is fine? So, you can see that everything is nice I have got my expressions here. So, I can now proceed further, fine. That is one thing.

The other thing is that u and v are already perpendicular to me, so, I can even replace we can even write this as d 1 common, alright. c 1 upon d 1 minus u, fine? Look at this, I can write this as this also. So, what I see here is that if u is perpendicular to v, then a shift of u by the constant alright I am just shifting the origin because u was a linear form. So, it went through 0 and it is perpendicular to v.

So, v is this line suppose u was this line which are passing through the center from the 0 origin. Now, I am shifting it by some scalar. So, I will get here or here, but they will still be perpendicular because the slope remains the same, is that ok? So, that is important that we have maintained the slope and hence orthogonality is there hence I have got orthogonal axes with me, fine.

So, now I would like to use this to get my answers. So, if d 1 is 0 I get alpha 2 times v plus d 2 upon 2 alpha 2 whole square is equal to c 1, fine. So, therefore, if. So, this will implies. So, if d 1 is 0 I get this equation. And, this implies what? This implies if c 1 and alpha 2 both are positive then I have a solution; if c 1 is 0 then I get only one line, alright.

So, if c 1 is 0 implies v plus d 2 upon 2 alpha 2 is 0 I get same lines with appear of lines which are same appear of lines which are same, fine, is that ok? I am just looking at this line. If c 1 is not 0, if c 1 upon alpha 2 is positive, alright. So, if this is positive so, just divide here take it minus here.

So, if this is positive or which is same thing as saying that c 1 times alpha 2 is positive this will imply that I can write this as v plus d 2 upon 2 alpha 2 whole square minus sum t square is 0 where t square is c 1 upon alpha 2 I can write like this. So, here I get a pair of lines that are parallel, alright. The slope does not change it is just a distance a pair of parallel lines pair of parallel lines, is that fine?

So, you can see that I am able to make a statements one after the other using these ideas fine.

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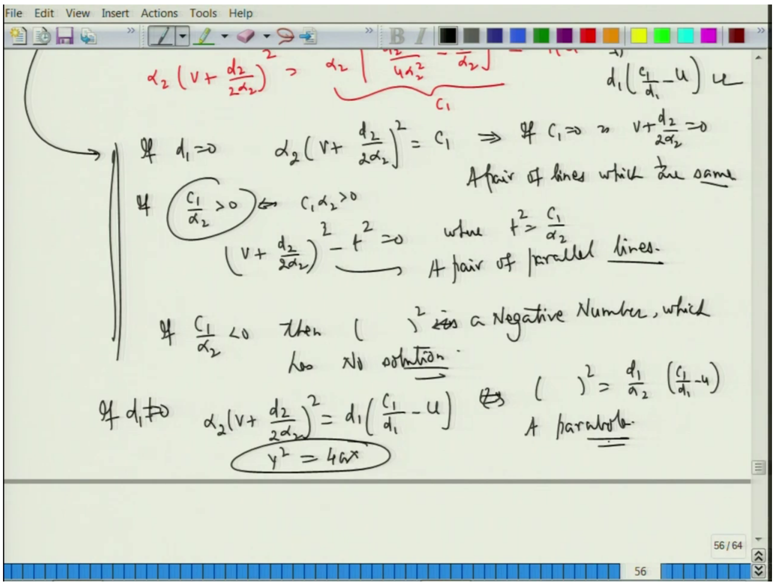

Similarly, now if c if d 1 is; so, this was with d 1 is 0 we could do things. If c 1, so, this I assumed. So, if c 1 upon alpha 2 is negative, then we are saying that this square is a negative number negative number which has no solution which has no solution, alright. So, I cannot get any point in R 2 any line or any point in R 2 which will satisfy that condition and hence I do not have any curve, is that ok?

So, if d 1 is 0, then you go what I have if d 1 is not 0; then I can divide by d 1. So, then I can write my things as alpha 2 times v plus d 2 upon 2 alpha 2 whole square is equal to d 1 times c 1 upon d 1 minus u. So, this represents nothing, but yeah do you remember y square is equal to 4ax.

So, you have some y square here I can. So, I can write here equivalent to looking at this square is equal to d 1 upon alpha 2 c 1 upon d 1 minus u. So, I get y square is equal to 4ax coming into play in some sense. I do not have the axes as x axis and y axis. My axes now are v plus d 2 upon 2 alpha 2 and c 1 upon d 1 minus u. Those are the other two axes which are orthogonal. So, I get in this case I get alright a parabola, fine.

So, I would like you to look at different cases and then see that in different cases I am going to give get different things, fine. I will just give one more.

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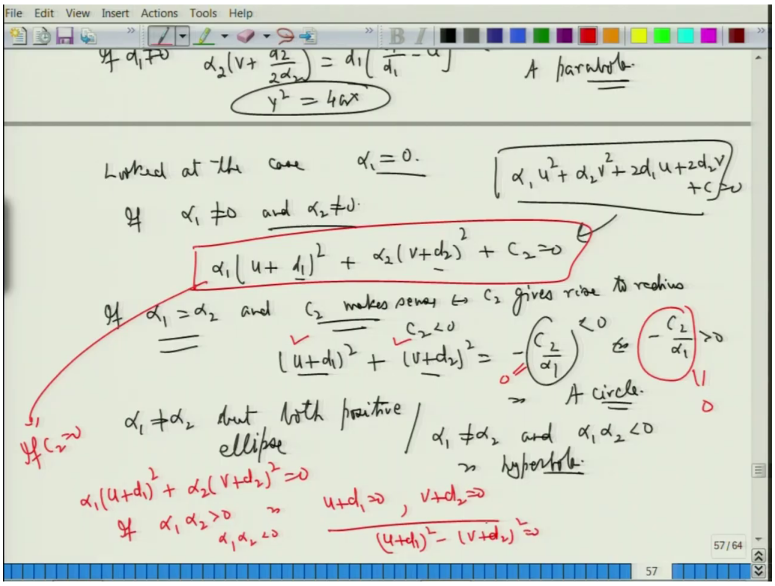

So, what we have seen is that. So, we looked at the case looked at the case alpha 1 was 0. So, if alpha 1 is not 0 and alpha 2 is not 0, then what you get is you will get alpha 1 times let me write I think I have written it somewhere I hope. No, I have not written it here, alright.

So, I will get something like u plus some d 1 square plus alpha 2 times v plus d 2 square plus c 2 equal to 0. I do not know those numbers, but the idea is that our thing was alpha 1 times u square plus alpha 2 times v square plus 2. So, let me write as 2 d 1 u plus 2 d 2 of v plus c 2 is equal to plus c equal to 0, fine. Then from here I will get to this part.

And, hence if alpha 1 is equal to alpha 2 and c 2 make sense make sense means we get c 2. So, that I get a radius c 2 make sense means c 2 gives rise to radius or if c 2 is negative, fine. If c 2 is negative I can go that side. So, I will get here alpha 1 times. So, let me write I think let me do that u plus d 1 square plus v plus d 2 square is equal to minus c 2 upon alpha 1 I am assuming this part.

So, if this is positive so, this is negative implies that minus c 2 upon alpha 1 is positive and therefore, I will get a circle, alright and a circle with this and this as my point of intersection will give me the center and things like that, fine. So, something I will get, think about it, fine. If they are not same and both are positive then I will get an ellipse.

So, if alpha 1 is not equal to alpha 2, but both positive will give me ellipse and, if alpha 1 is not equal to alpha 2 and alpha 1 alpha 2 is negative will imply I will get a hyperbola, that is one thing. Also you have to be careful this number this coefficient that I am getting this maybe 0, alright. If c 2 is 0 then I will not get a circle or an ellipse, that is very important.

Whenever this is 0, I will get that this term is 0 and this term is 0 and therefore, or in the previous term also if c 2 is 0 here. So, if c 2 is 0 I will get that alpha 1 times u plus d 1 whole square plus alpha 2 times v plus d 2 square is 0, fine. So, if alpha 1 into alpha 2 is positive, fine. I will get that u plus d 1 is 0 and v plus d 2 is 0 and then it will give me a intersection. I will get 2 lines and at point of intersection I will get things like that.

So, you have to be careful when you make a statements; when alpha 1 alpha 2 is negative fine, then I will get u plus d 1 square minus v plus d 2 square is 0. You can look at you will get a pair of a straight lines and things like that. So, you have to be careful when you do these arguments, alright. Each case has to be taken off carefully and then appropriately understood, alright.

So, I will just end the class here in the next class we will look at actual numbers and then do all the work is that ok. So, that is all.