Linear Algebra
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Lecture – 63
Sylvester's Law of Inertia

So, we will in the previous class we learnt about given a matrix A which was Hermitian valued at X star A X; what was called the quadratic form or the Hermitian form and we gave conditions for that to be positive semi definite or positive definite and so on; fine. I do not know when we will be able to use that idea, but let us try to understand the crux of the thing, what is called the Sylvester's law of inertia.

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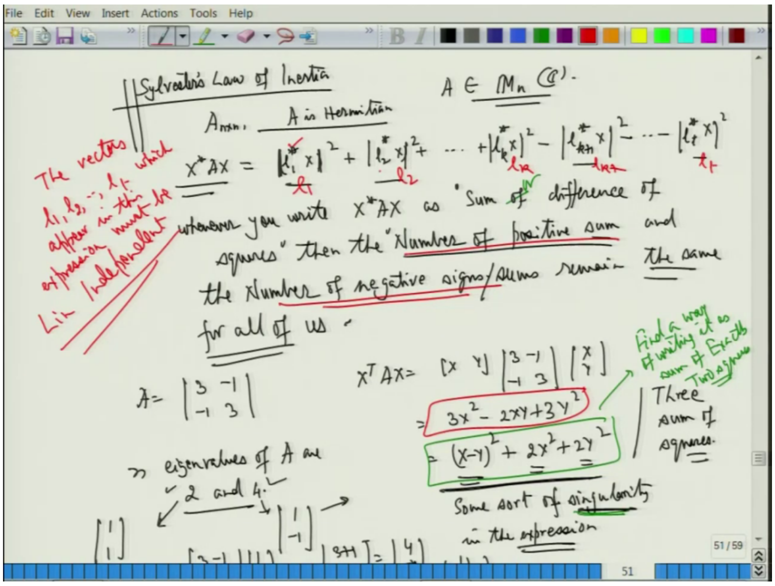

Everything follows because of this idea Sylvester's law of inertia; Sylvester's law of inertia. What it says is that I have a matrix A which is n cross n; I am assuming that A is Hermitian alright; fine. I want to compute X star A X fine and if when I write it, I will get some expressions here of the type something like say; some vector l 1; transpose; X whole square plus l 2 transpose X whole square things like this plus l k transpose X whole square.

And then some minus will be there some l k plus 1 transpose X; l t transpose X whole square. So, what do I mean by saying this? So, let us try to understand that by an example and then I will build upon this idea. So, it says that whenever you write X star A X as; whenever you write X star A X, as sum or difference of a squares; then the number of positive signs, positive sums and the number of negative signs positive sums or negative sums or signs whatever sums or signs remain the same for all of us; is that ok.

So, let me try to understand this. So, there is a quote that I should write here sum of difference of squares alright, that is one thing; I need to understand what is that. When the number of positive sums and the number of negative sum remain the same for all of us; this is very important to understand here. Now, I forgot to say that I am looking over A over complex numbers. So, A is over M n of C (Refer Time: 03:34) over M n of C, in place of this; I would had to put this and a star here; is that.

So, I will put a star here square; so it was l something l k star X whole square, again this star and l t star X whole alright. So, I have take absolute values of every term here fine. So, let me try to make you understand this by an example; let us take A as 3, minus 1, minus 1, 4. Suppose, I take this fine; I compute X star A X for me. So, this will be equal to X Y; 3 minus 1 minus 4 X Y.

So, this is equal to 3 X square minus 2 X Y plus 4 Y square. So, I can write it as X minus Y whole square plus 2 X square plus 3 Y square; I can write like this fine. I can also write it differently; so here when I am writing here; please understand here that it is X minus Y whole square; 2 X square, 3 Y square. There are three sum of squares here fine. What I would like to do is; I would like to write it only in two terms.

So, in some sense there is a notion of what is called here there are three terms; so there is some sort of singularity that I am writing here. So, there is some sort of; some sort of singularity in the expression. Determinant is not 0; alright, it is similarity in some in the expression I am saying; I am talking in terms of the expression alright. So, let me try to make you understand things here what I am trying to do.

So, let us look at the eigenvalues of this matrix fine. So, if I look at the eigenvalues of this matrix p of lambda or p of X will be equal to X minus 3 or let it make it easy so that; I should make it 3 here itself alright; so that I can do my calculations very easy. 3 times Y square, 2 times Y square; so then I can see that this will imply that eigenvalues of A are 3; that is 2 and 4, these are the eigenvalues. Corresponding to 2; the eigen vector is 1, 1 and corresponding to 4; alright the eigen vector is 1 minus 1; I would like you to check that.

So, let me just verify for you; 3, minus 1, minus 1, 3 times; 1 minus 1 is 3 plus 1, that is 4 and minus 1; minus 3 which is 4 minus 4 which is 4 times 1 minus 1 alright. So, you can check that things are fine.

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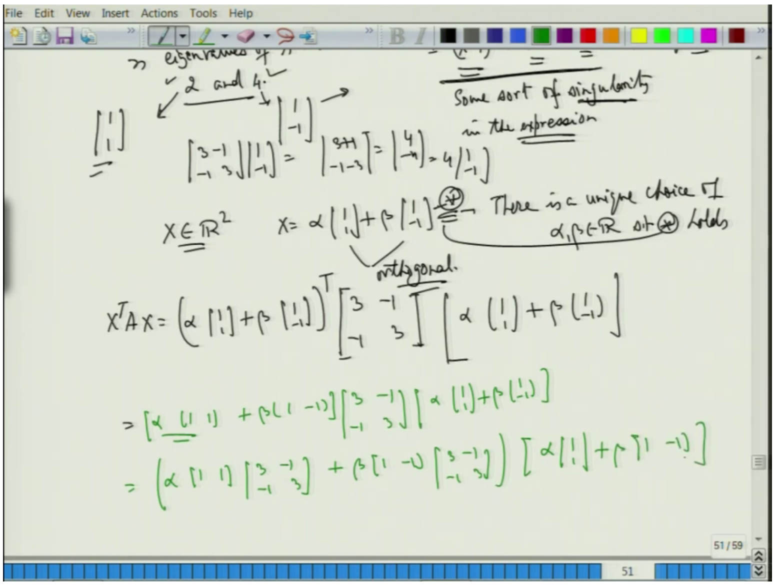

So, I have got two eigenvalues 2 and 4 fine; this is an eigenvector, this is an eigenvector; in place of eigen vector 1 comma 1; so what I can do is, if I look at any X. So, I can write any X belonging to R 2 as X is equal to X; so, as alpha times 1, 1 plus beta times 1 minus 1 alright.

So, there is a unique choice of alpha beta belonging to R such that a star holds alright. So, this I am writing it as a star; is that fine. What does the unique solution, now which is same thing as saying whatever you want to say that I am not bothered about; fine. So, understand this and 1 and minus 1; they are, these two vectors are orthogonal.

Note, I am looking at orthogonal basically because for us I am taking it a symmetric matrix. So, everything is nice; I can do orthogonality and so on fine. So, now let us rewrite X transpose A X with respect to this part.

So, X transpose A X; for me will be equal to X is alpha times this vector 1, 1 plus beta times 1 minus 1; this whole transpose 3, minus 1, minus 1, 3; fine, this is the matrix that I have. Again, here alpha times 1, 1 plus beta times 1 minus 1; is that ok, this is the vector that I have.

So, let me write it nicely fine. So, just look at nicely yourself there is nothing special about it, but you have to be careful in writing it because I always do lot of mistakes; so be careful in doing it fine. So, this matrix has to be multiplied with this; how do I multiply it? So, alpha transpose here with this transpose, I will have to write alpha here 1, 1 plus beta times 1 minus 1; this times 3 minus 1, minus 1; 3 times again alpha 1, 1 plus beta times 1 minus 1.

So, this is same I just multiplied this with this alright. So, it is alpha 1, 1; 3 minus 1, minus 1, 3 fine plus beta; 1, minus 1, 3, minus 1, minus 1, 3; this whole thing has to be multiplied with alpha 1, 1 plus beta 1 minus 1 alright.

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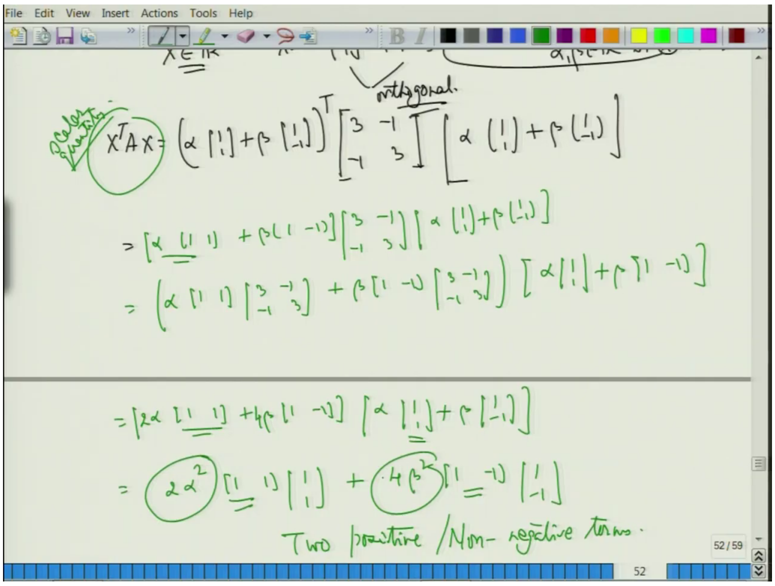

So, this is same as multiply this; it is alpha 1, 1 into this is 3, 1; that will give you 2; look at again here 1 and 3; that is 2 alright. So, 3 minus 1; minus 1 plus 3; so it will be 2 alpha times 1, 1; this will give you plus beta times; look at this 3 plus 1 is 4, so I will get 4 beta times 1, minus 1; times alpha 1, 1 plus beta 1; minus 1.

So, now, I can write it as 2 alpha square; this vector times this vector, 1, 1; 1, 1; here plus 4 beta square; 1 minus 1, 1 minus 1 fine. So, I am able to write see; this here scalar quantity, the scalar quantity. So, you can see that I have been able to write it as sum of alpha square, beta square; there are only two terms here; two positive oblique non negative terms.

So, I want you to understand that they have come from eigenvalues eigenvector, they are come from eigenvalues and eigenvectors fine. Eigenvectors were 1, 1 was the eigenvector, 1 minus 1 was an eigenvector, 2 and 4 were the corresponding eigenvalues alright. So, I want you to understand there was that expression and there is this expression; there are two different expression.

There the number of positive terms was 4; sorry 2, here the number positive terms is 3 fine. What this theorem tells me; statement says that the sum whenever you write X star A X as sum of difference; sum or difference, sum or difference of squares; then the number of positive sums and the number of negative sums or signs remain the same for all of us.

So, it means that there is something wrong in writing this way; this is what I am trying to say; so I will have to rewrite it in a different way. So, can you think of rewriting it in different ways so that there are exactly two expressions.

So, find a way of writing it as sum of exactly two squares alright. I would like you to think it yourself how do you do it and now I will explain you the ambiguity part which is important for me that what is the ambiguity here that I would like you to explain fine.

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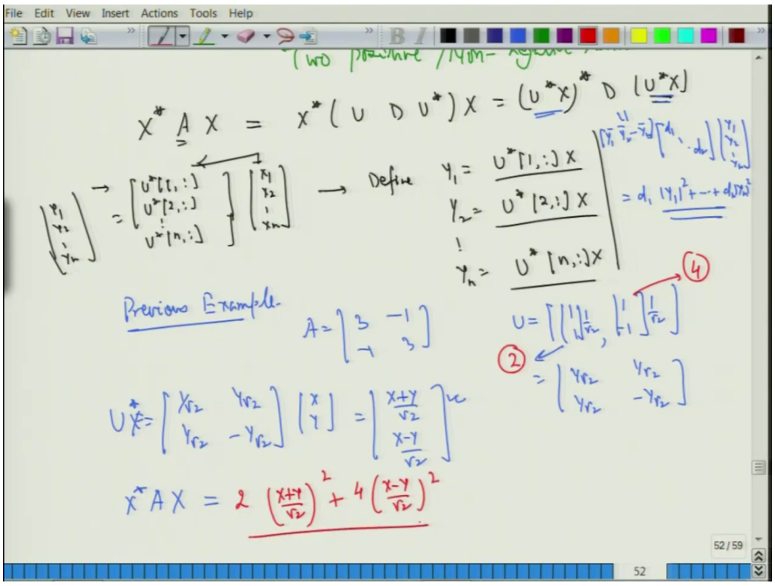

So, let me go to that part fine. So, what we have? You have A; you are looking at X star; A X fine. So, as we said earlier also we are writing A here; so A is U D; U star fine, X here fine. So, which is U star X; whole star D times, U star X fine. Now, what is U star X? Let us understand is; U is a matrix fine. I have X which is X 1, X 2, X n fine.

They will be the first column here which will be; so I think I should write let it be U start itself alright. So, I can look at U star; the first column, the first row sorry; the first row, the second row of U star and the nth row of U star; is that ok?

So, I want to look at from here; I want to define, Y 1 is equal to U star of X; multiply this whole thing with this whole thing. So, I am writing this as Y 1, Y 2, Y n; that is all I am doing here. Y 2 is equal to U star; this time X and Y n is U star fine; n of this. So, I am writing like this; is that ok?

So, I am just writing this part myself; nothing else, I am not doing anything I am just writing it out. So, from here now if I want to look at this part; now gives me as look at this part, I have a diagonal here d 1 to d n; the right hand side is Y 1, Y 2, Y n. The left hand side here is the star of that; so the left hand side is Y 1 bar, Y 2 bar, Y n bar fine. So, this is equal to d 1 times mod of Y 1 square plus so on plus d n, Y n square; is that ok? Fine.

So, you can see here that there are n terms; d 1, d 2, d n are eigenvalues and nothing else; this is one way of writing it. So, let us go back and write it for the previous example. So, the previous example; so I had A as 3, minus 1, minus 1, 3 fine. So, you can see that I can write U as one eigenvector was 1, 1; the other was 1, minus 1; fine these were the two eigenvectors , but this is not a unitary matrix; I will have to multiply by; so, this is 1 upon, root 2; I will have to write here; 1 upon root 2.

So, these are the two vectors which will give me my things; is that fine. So, my first column; so I can I am writing it as 1 upon root 2, 1 upon root 2, 1 upon root 2, minus 1 upon root 2; this is my U for us. So, U star is equal to; so in this example, it turns out to be the same fine and therefore, U star X will be equal to multiply by X, Y.

So, write this as X plus Y upon root 2, X minus Y upon root 2; write like this fine. Then X star A X; I would like you to check that this is equal to corresponding to X, Y or corresponding to this U alright; 1, 1 gave you the eigenvalue 2, this gave you eigenvalue 4 fine. So, therefore, I will get 2 times; X plus Y upon root 2 whole square plus 4 times; X minus Y upon root 2 whole square, this is what we will get fine.

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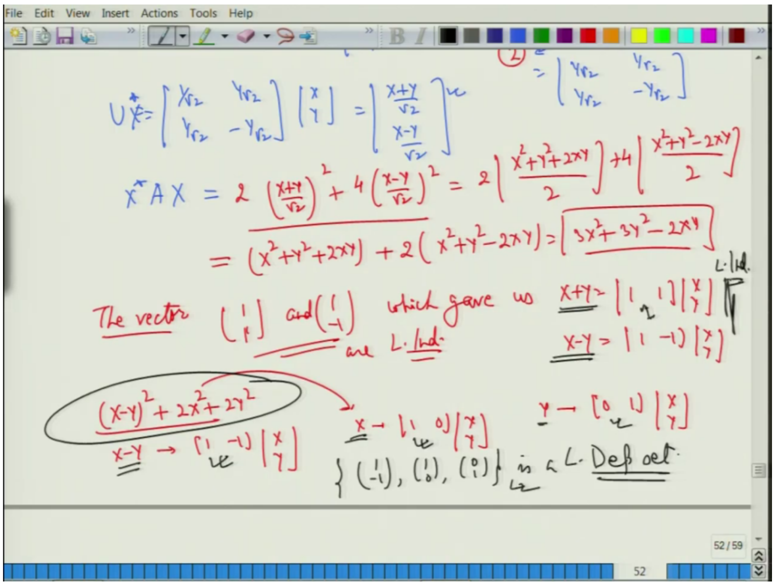

Let us verify this out. So, this is expression is nothing, but 2 times X square plus Y square plus 2 X Y over 2 plus 4 times X square plus Y square minus 2 X Y over 2. So, which is same as 2, 2 cancels out; X square plus Y square plus 2 X Y plus 4 here; so it is 2 and 2 cancels out 2 times; X square plus Y square minus 2 X Y which is equal to 3 X square plus 3 Y square minus 2 X Y and this is what our expression was.

If you go back; this is the expression 3 X square plus 3 Y square minus 2 X Y alright. So, when I want something to be written; I want to be written in terms of things which are nonzero alright. So, there in some sense; there was linear dependence coming into play alright. So, here look at; so what we are trying to say here is that the vectors 1, 1; 1 minus 1 which gave us X plus Y was 1, 1 times X, Y.

Another thing I got X minus Y was 1 minus 1 times X Y; so these two vectors are linearly independent fine. In the previous expression, what I had written was I wrote it as X minus Y whole square plus 2 X square plus 2 Y square fine. If I want to get X minus Y; X minus Y corresponds to 1 minus 1; times X Y fine. This 2 X square; so X is coming from 1, 0 times X, Y and Y is coming from 0, 1 times X, Y fine.

So, I have got three vectors here 1, minus 1, 1, 0 and 0, 1. So, there are three vectors 1, minus 1; 1, 0 and 0, 1 is a linearly dependent set alright fine. So, when I when I want to write things; I do not want linear dependence to come into play; is that ok? That is a very important idea that I am trying to say that you can write a sum of three non negative terms, you can write it fine.

But in this expression, whatever you have done; these vectors that you are getting here which gives which is giving you X and Y or the terms here I gave X plus Y, X minus Y fine. Here also I am getting X minus Y, X and Y; they are coming from these three vectors. So, these three vectors are linearly dependent here, but the vectors that I have got here in this expression; they are linearly independent alright.

So, what the theorem says, what the statement of Sylvester's law of inertia says that when you are looking at Sylvester law of inertia; it says that you have to write the number of positive sums and number of negative sums when you write, the vectors that are coming to appear there alright; these vectors l 1; so there should be a star here; the vector l 1 here, l 1, the vector l 2 here, l k; l k plus 1; l t; all of them, they have to be linearly independent.

So, the vectors l 1, l 2, l t which appear in that expression; which appear in this expression must be linearly independent; is that ok? And then only I can talk of Sylvester's law of inertia; if they are not independent I cannot talk of Sylvester's law of inertia is that. So, understand them nicely; I am not able to prove it at this stage basically because it is out of syllabus.

Idea is similar; it is a bit complicated, but the important thing is that whenever you are writing expressing X star A X in terms of some linear terms, these are linear forms; I have not used the word linear form. So, these forms are called; so these terms what are called l i star X; these are called linear forms; there are special name they are called linear forms. So, why what we are trying to do is that we are saying that write X star A X as sum or difference of linear forms fine.

Write X star A X as sum and difference of linear forms; you have to keep track that those linear forms are linearly independent. When I say linear forms are linearly independent, I mean that the vectors alpha i's are linearly independent.

Once you have written it, then the positives and negatives that you get in the expression; they are same for all of us fine. So, let me now clarify you; what do I mean by sayings that alright. So, again let us look at it nicely, so we have already done some work.

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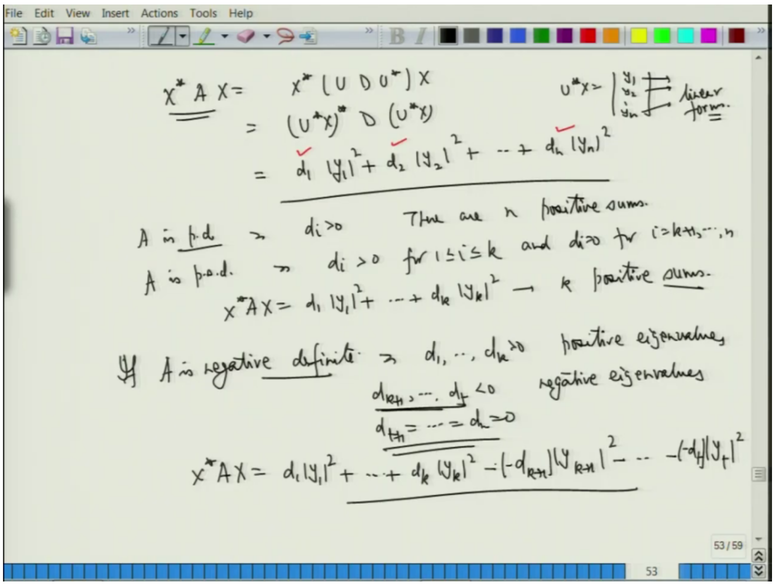

So, again let me write X star A X as fine; we are going to write it as X star U D U star; I am writing it again and again, it is very important. So, this is U star X; whole star D; U star X. Then we wrote U star X as y 1, y 2, y n; they were the linear forms each of them were the linear forms fine; we got them as linear forms. So, from here what we see is that you can write this as d 1 times, we wrote it as this plus d 2, y 2 square plus d n; y n square, I can write like this fine.

So, what we are saying is that whenever I have A is positive definite; it means that A is positive definite, imply each d i is positive. So, I have all; so there are n positive sums; if A is positive semi definite, this will imply that there will be some di’s which will be positive and so d i will be positive for say 1 less than equal to i less than equal to k and d i is equal to 0 for i is equal to k plus 1 till n. In this case, I will get X star A X, as d 1, y 1 square plus d k, y k square.

So, there will be k positive sums fine. Similarly, you can express them for negative semi definite and negative definite; for indefinite; so if A is negative definite, this will imply that there will be say some d 1 to d k will be positive eigenvalues, d k plus 1 till; d t will be negative eigenvalues and then d t plus 1, till d n will always 0, alright multiplicity of 0.

So, in this case you can write X star A X as d 1, y 1 square plus d k, y k square minus d k plus 1, y k plus 1 whole square minus so on minus d t into y t square; is that ok; so that is the way we can write. In many books, they do not write like this; they improvise on this alright. What they do is that since d 1, d 2, d n are positive numbers alright was I should have done here there is a small error; here I should written here minus of this, minus of minus d t fine because you have taken that dk's are negative; so, alright.

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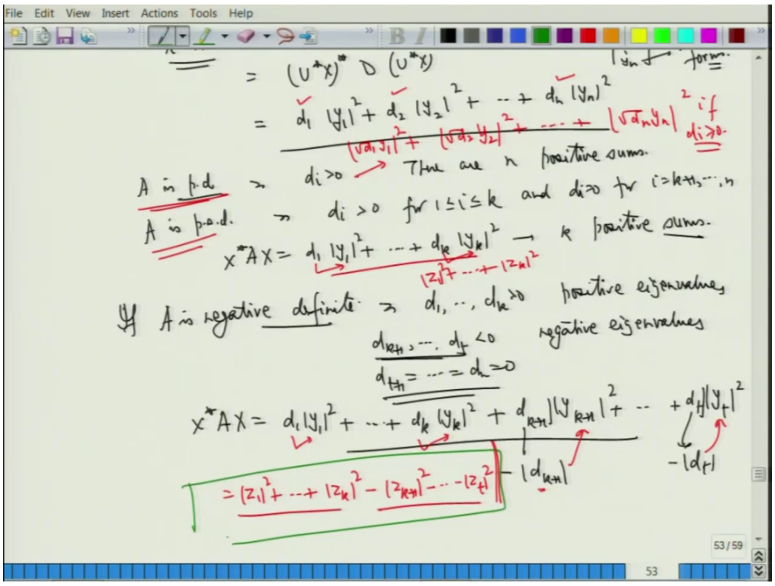

So, let me just write plus itself d k plus 1; so you have to be careful to understand this part, otherwise there will be error plus d t here , but d t's are negative; so minus say mod of d t alright. So, minus mod of d k plus 1; is that fine.

So, books they use that since d 1, d 2, d n are positive numbers; I can just push this d inside. So, I can just right here a square root of d 1 times Y 1; this square plus the square root of d 2, y 2 whole square plus so on; a square root of d n fine. So, if they are positive then; so if positive. So, if; so this time positive if di's are positive; alright fine, I can write like this.

I should not have written here but I should have done it here; positive definite, so for this case I will get this part. Positive semi definite also; I will get the same thing, but these di's will go inside only for these parts alright, for the rest it will be 0; so I will get only things here.

Similarly, if I look at this part; I can push this absolute value inside, inside here. So, what I will get here is; I can push this inside this also inside. So, I will get here as some z 1 square till z k square and then I will get minus z k plus 1 square minus so on minus z t square alright.

So, I will get this part; so there will be some positive term, there will be some negative terms alright. Here, it is going to be all positive terms; z 1 square till z k square alright. So, many books they follow this notation, when they write things; so they follow this notation. They do not put di's outside, but the ideas of di's they come from eigenvalues.

So, I want you to keep track of that and understand that the linear form that you are talking about which is the most important, they need to come from a columns of something. For us, we are going to get it from the eigenvalues, but in general you can get it from whatever way you want, but you want them to be linearly independent.

You want them here these things should be linearly independent; if they are not linearly independent, then I am not following what is required of Sylvester's law of inertia; is that ok? So, that is all for now.

Thank you.