Linear Algebra

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Lecture – 62
Quadratic Forms

Now, let us look at what are called Quadratic Forms, alright.

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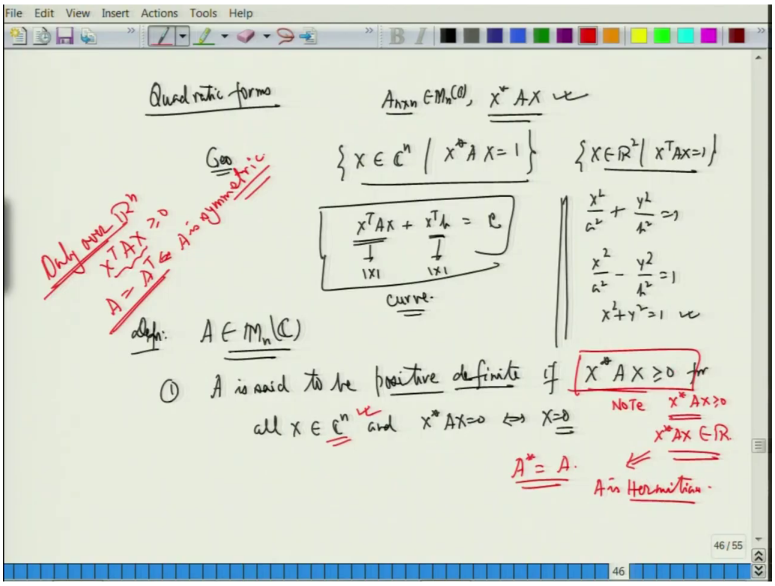

So, there is a notion of what are called Hermitian forms also, but they are similar in nature, alright. So, what we are saying here is that I want to understand given a matrix A which is n cross n, I want to look at X star A X, is that ok, A belongs to the M n of C and I want to look at what is X star A X, is that ok.

The idea appears at many place geometrically if we want to see we want to look at, so geometrically, we want to understand geometrically what we are saying is that I am looking at all X belonging to C n such that A X star A X is equal to 1. So, I want to plot the curve what this curve is or what this object is I want to understand that, fine.

And we have been doing it in our school that we had this X belonging to R 2 such that X transpose A X. So, for certain choices I could have A in certain ways for example, x square by a square plus y square by b square is 1. I could also have x square by a square minus y square by b square is 1, fine x square plus y square is equal to 1 we had such things, alright, fine.

So, you see that you have an ellipse you have a hyperbola, you have a circle, alright. We will see that when we want to generalize these ideas in place of looking at A here and X here, we can have some something more also, so that will come afterwards. So, you can also look at what I am trying to say is X transpose X plus X transpose some b is equal to A constant 1 or something. Because look at this is a 1 cross 1 matrix this is also 1 cross 1, so I can write it as 1 or some C here and then try to look at the graph look at the curve that this represents.

And once I have this then that notion of parabola will also come the notion of (Refer Time: 02:21) straight lines will also come in two dimension. Symbol in higher dimension on what it looks like; and it has got lot of applications to understand this part, alright. So, let us start with our definition, so definition. So, I am going to take A belonging to M n of C, all our examples will be over real numbers.

Once in a while I will look at complex numbers, but the definition is over complex numbers, alright, fine. So, let us also, so definitions. So, matrix 1, A is said to be positive definite, recall id use this id of positive definite means at the time of inner product also. So, A is said to be positive definite if X star A X is greater than equal to 0 for all X belonging to C n and X star A X is 0 if and only if or it implies that X is the 0 vector, is that ok. These are positive definite means.

Note here that, note when I say that X star A X is greater than equal to 0. What we are saying here is that X star A X belongs to real number, alright, because I cannot make any comparison over complex numbers. So, I am already saying here that X star A X is a real number and X is in C n and that implies, so this will imply this condition will imply that A star is A, that is A is Hermitian, alright.

So, when I am looking at C n over C, so when I am looking at C n, I just need to say that this is greater than or equal to 0, alright. I do not had to put any condition because that condition itself implies that A is Hermitian. So, this condition implies A is Hermitian, fine.

But when I want to study only over real numbers, only over real numbers, only over R n, alright, so if I am looking at X transpose A X greater than equal to 0, then whatever A you choose this will always be a real number, alright whatever A you choose. So, when I want to study quadratic forms over real numbers, I will assume that A is equal to A transpose. So, A is symmetric, alright.

So, I may not use the word symmetric again and again. I may not use the word that A is Hermitian again and again. But in this chapter for us it is going to be symmetric, A is symmetric or A is Hermitian, fine. I am not going to check this part, I would like you to do it yourself that if X star A X is greater than equal to 0, then A has to be Hermitian try that out yourself, fine. Let us go to the next, definition 2.

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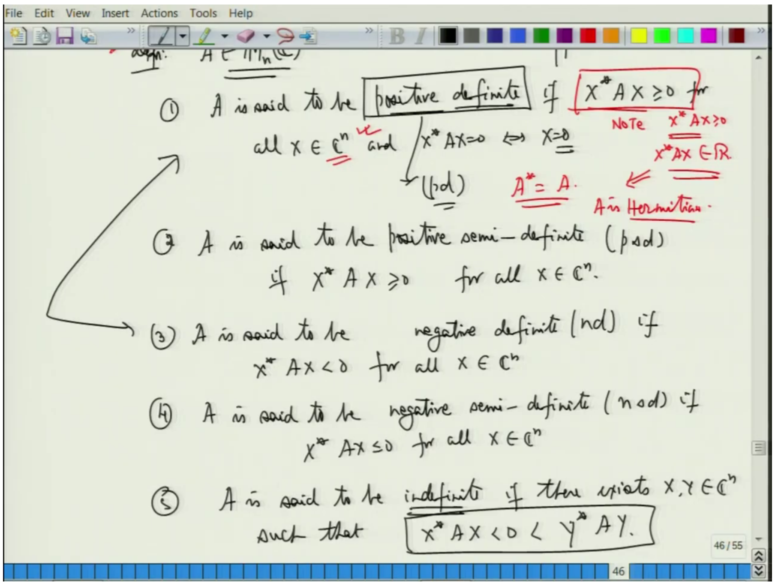

So, here I looked at what is called positive definite. Now, there is notion of positive semi-definite. A is said to be positive semi-definite if X star A X is greater than equal to 0 for all X belongs to C n, alright. In short we write it as psd, this in short is written as pd, fine.

Third definition, let me write all the definitions here. A is said to be non-negative definite, rewrite non, I have said no I have said negative definite negative semi-definite. So, I have not used a word I have used negative definite, negative definite nd, if X to the star A X is less than 0 for all X belonging to C n; so, just the reciprocal of this part or not reciprocal the negative of this part.

Fourth, A is said to be negative semi-definite, so nsd if X star A X is less than or equal to 0 for all X belongs to C n. And the last one fifth, A is said to be semi-definite or indefinite X is said to be I think I will tell indefinite if it is neither of these, alright. So, if there exists x and y belonging to C n, such that X star A X is less than 0. So, there are some portions where it is negative and there are some places where it is positive, alright. So, both the things happen, negative as well as positive, is that ok.

So, indefinite means, we are not sure whether it is on the positive side or we are in the negative side. So, positive definite means that everything is on the right hand side beyond 0, in some sense semi-definite means 0 and something negative definite means you are on the left in the real line and so on, is that ok. So, let us look at a examples.

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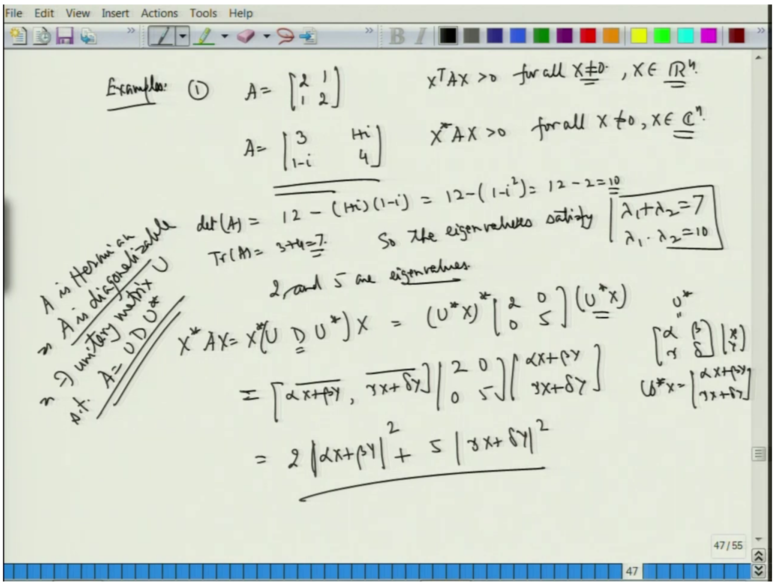

Examples, 1. So, I define A as 2, 1, 1, 2. So, it is symmetric matrix, fine. You are, I would like you to check that X transpose A X is greater than or equal to 0 for all X not equal to 0, I would like you to check that. Try that out yourself. I take complex example, 3, 1 plus i, 1 minus i and 4. So, now, looks look at X star A X. So, this will be greater than 0 for all X not equal to 0 and X in C n. Here I can take X in C n, but I can take in R n also whatever you want to say you can do it, fine.

And how do I check this part? So, let me check this part for you, fine. So, let us look at the eigen values of this matrix. So, let me look at eigen values of A. So, I want to look at determinant of A. Determinant of A is 12 minus 1 plus i into 1 minus i which is 12 minus 1 minus i square which is 12 minus i think 2 which is 10, fine and trace of A is 3 plus 4 which is 7. So, the eigen values of A are; so, the eigen values satisfy lambda 1 plus lambda 2 is 7, and lambda 1 into lambda 2 is 10.

So, I can see that the solution for this is going to be 5 and 2. So, 2 and 5 are the eigen values; 2 and 5 are eigen values. So, let us go back and try to understand. So, I had A here, A was UD U star, fine. So, when I want to compute X star A X I am looking at X star UD U star of X.

So, I can write it as U star X whole star D here for me is 2 and (Refer Time: 10:52) is 2, 0, 0, 5 and U star X, fine. I write like this. So, this is same as; so, I do not know what U is, but what we know is that every Hermitian matrix is diagonalizable.

A is Hermitian, implies A is diagonalizable, implies they exist unitary U, U, such that A is equal to UD U star, alright. This is what we are using, fine. So, I can take U star X. So, if I look at U star U star will be sum U 11 or say let me write it as alpha beta gamma delta whatever it is X is your x and y, fine. So, U star, so this is my U star I am writing here. So, U star X is equal to Alpha X plus beta y this is gamma x plus delta y, fine.

So, I would like you to see that this is same as you are writing it star of this. So, alpha x plus beta y this gamma x plus delta y bar 2, 0, 0, 5 times alpha x plus beta y, this is gamma x plus delta y. So, this is nothing, but 2 times absolute value of alpha x plus beta y whole square plus 5 times gamma x plus delta y whole square, alright.

So, you can see that this is positive for every choice of x and y and is 0 only when x and y is 0, is that ok. I would like you to try that out yourself for the first example also. But the idea is that they come from eigen values.

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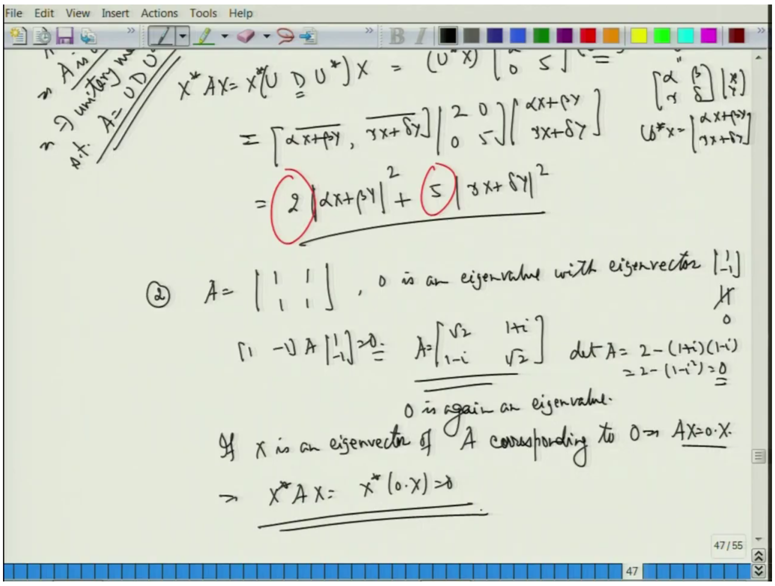

This is more important for me that, if you look at the calculation here finally, these eigen values are coming into play, is that ok. If the eigen value was negative, I would have got a negative number here, fine. This is what is important.

Second example. So, I have given you two examples. Now, let us look at, so this is about positive definite. So, both the examples were positive definite here, fine. Now, let us look at positive semi-definite. So, I define A as, take A as 1, 1, 1, 1 and so you can see here that 0 is an eigen value, 0 is eigen value with eigen vector 1 minus 1. So, check that 1 minus 1 times A times 1 minus 1 is 0, alright.

So, I have a non-zero vector for which things are 0 here, alright. I can take over this root 2 1 plus i, 1 minus i and root 2 here. Another matrix A as this. If I look at this matrix, again look at the determinant of A, determinant of A is 2 minus 1 plus i into 1 minus i which is 2 minus 1 minus i square which is 0, alright, fine. So, 0 is again an eigen value. 0 is again an eigen value, eigen value.

If X is an eigen vector of A corresponding to 0 implies A X is equal to 0 X and this implies X star A X is equal to X star 0 X is 0, alright. So, I do get 0 here, is that ok. So, this is what you have to be careful you just have to look at eigen values from there you can get eigen vector, and then you can get all those things, alright, fine.

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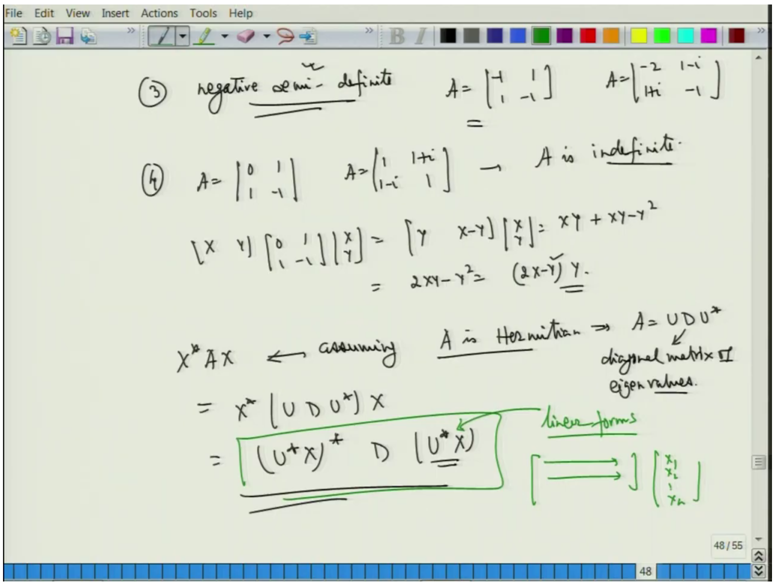

Third example. So, that I gave you positive semi-definite, now let me give you a negative semi-definite, negative semi-definite. I hope I have written it correctly. So, my A is 1 here minus 1, 1 and 1 minus 1, this that what I have wrote here or you can take a to be equal to minus 2, 1 minus i, 1 plus i, minus 1. Check that this is negative semi-definite. So, the one way to check is look at this part at least I can try this out myself. The determinant of this is negative as well as the trace is negative said to be negative yes. So, try that out yourself, fine.

Fourth one, look at the trace is minus 2 and determinant is 1 minus 1 which is 0, is that ok. So, 0 gives you semi-definite here, otherwise it would have a negative definite, fine. So, you can just take a larger number, make it negative and make it a negative definite yourself, fine. A which is 0, 1, 1, minus 1 or if I take A as 1, 1 plus i, 1 minus i, 1, alright then this is a is indefinite, alright.

So, let us check this part. So, x, y; 0, 1, 1, minus 1 times x y, I want to compute this is nothing, but this into this is y this into this is x minus y. So, x minus y times x y which is x y plus x y minus y square which is same as 2 x y minus y square which is 2 times 2 x minus y into y, fine.

So, there are choice of y, x and y where you can see that this is positive and if I take y to be negative or just take y to be very large as compare 2 x even if it is positive this becomes negative, alright. So, this is an example of a negative definite matrix, is that ok.

So, I would like you to construct these examples and understand the importance of eigen values and eigen vectors in this study, alright. So, I would like you to see that all the calculations are going to be based on eigen values and eigen vectors here, fine. Because I am looking at X star A X, since I am looking at X star A X we are also assuming A is Hermitian, fine. So, A is Hermitian implies that I can write A as UD U star, D is diagonal matrix of eigenvalues, fine.

So, I can replace this by that I did earlier also X star UD U star of X which is nothing but U star X whole star D times U star X, alright, fine. So, you can see that has come into play here, fine. The way we did for the previous example I think I did it somewhere I did it here, alright. So, you can see that this is the way I have played around here to get my things, alright. So, this is a play that I have done, fine.

Same play is to be done here also in the general setup that I have to find a diagonal and then everything becomes nice in some sense. And these are called what are called the linear forms and so on, alright. Linear forms in the sense that you have this matrix U and then you are multiplying by x 1, x 2, x n. So, this times this one will give you the first linear form, this row into this will give you another linear form. So there are n forms that you can get here and things like that depending on what your U is, fine.

So, this is about the definitions and so on. Let us now concentrate slightly in what does it mean to say that A is a positive definite matrix or A is a positive semi-definite matrix and so on, is that ok. So, let us look at some theorems on this part theorem.

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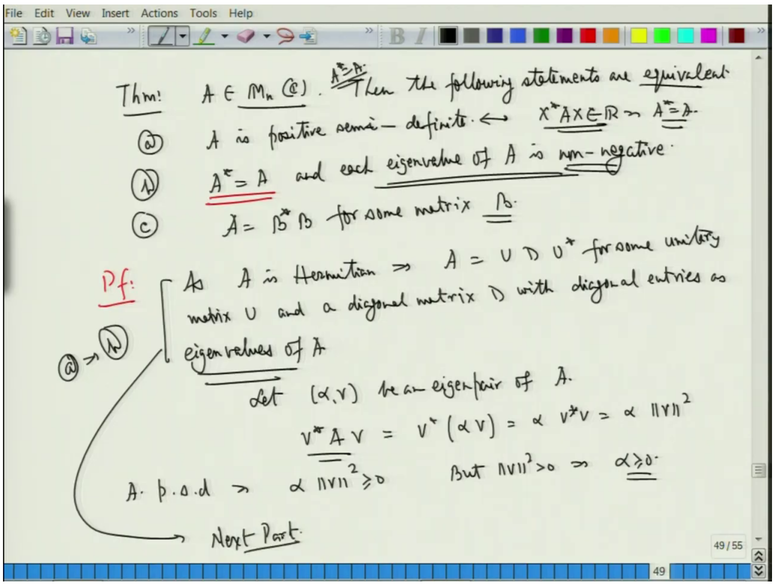

So, A belongs to M n of C, so as I stated I am assuming that A is Hermitian. Otherwise, you will have to prove it yourself, then the following statements are equivalent, are equivalent. a, A is positive semi-definite. b, A star is equal to A, so I am not stressing on this because I am already assuming, alright and each eigen value of A is non-negative. c, A is equal to B star B for some matrix B, alright.

So, this is what our assumption is, but; so, this part is our assumption I do not want to waste my time proving it you can try it out yourself, fine. So, proof. So, we are assuming that A is positive semi-definite, and we are taking that A star is A, we are already taking A star is A. Otherwise I will have to prove that positive semi-definite or I will have to prove that X star A X is belongs to R implies A star is A. I will not prove it. So, try that out yourself, alright. I do not want to waste my time, fine.

So, as A is Hermitian implies A is equal to UD U star for some unitary matrix U and a diagonal matrix D with diagonal entries as eigen values of A, is that ok. So, that is more important that they are eigen values of A, alright, fine.

So, let us look at something very carefully. So, let alpha comma v be an eigen pair of A, fine. Then, I want to look at v star A v. So, v star A v is equal to just look at this part v star A v is alpha v, fine. Since, it is Hermitian every eigenvalue is a real number. This is what we had seen already. So, this is same as alpha times v star v which is (Refer Time: 23:12) think about norm of this, fine. This is a positive quantity. We are saying that a is positive semi-definite.

So, A positive semi-definite implies alpha times norm of v square is greater than or equal to 0. But norm of v square or norm of v itself is positive because v is an eigen vector, this implies alpha is greater than equal to 0, alright. So, you have proved that each eigen value of A is non-negative, fine. From there I want to go to see that A is equal to B star B is for some matrix B, fine. So, this I have shown that a implies b is done.

Again have a look at it nicely, A is Hermitian implies A equal to UD U star for some unitary matrix U and then D is the diagonal of eigen values, fine. And, but anyway I have not used that idea at all, I should have used it. So, this is to be used for the next part. I had to show that eigen value part, so let alpha comma v be an eigen pair of A, then v star A v is this expression whatever I have written here, v star A v is this expression that you can compute yourself.

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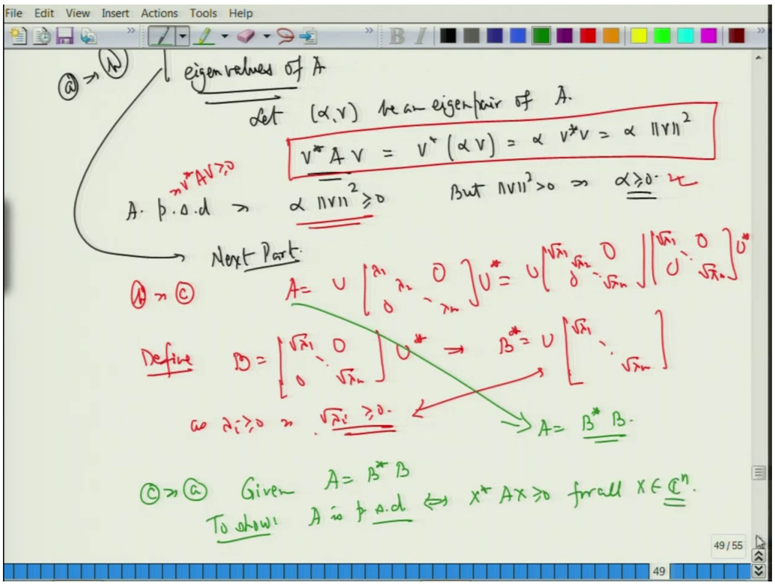

We have been given that A is positive semi-definite it means that v star A v is greater than or equal to 0 positive definite implies v star A v is greater than or equal to 0. So, therefore, this is greater than or equal to 0 and therefore, alpha is greater than or equal to 0, alright, fine.

So, what we know is that all the eigen values are positive. So, D has all positive entries. So, I would like to write A as next part B implied C. I have A as U, D is lambda 1, lambda 2, lambda n, 0, 0, U star. So, I write it as U, a square root of lambda 1, a square root of lambda 2, a square root of lambda n this, times this itself root of lambda 1 till root of lambda n times U star, is that ok. So, I write like this.

Define. I wanted to show B or B star, B, alright. So, define B is equal to a square root of lambda 1, till a square root of lambda n, 0, 0 and U star define B as this. This implies B star is equal to U star star which is U and a star of this, this is a diagonal matrix it will remain same, I have taken complex conjugate, but lambda is ideal non-negative, alright.

So, each lambda is non-negative, so square root is a non-negative number and therefore, there is no complex number coming into play. So, it will be just lambda 1 till the square root of lambda n itself, is that ok. So, this is what we have because as lambda i is greater than and equal to 0 implies the square root of lambda i is also greater than or equal to 0, is that ok.

So, therefore, we are getting this part. So, you can see from here that A is equal to B star, fine. We have done that part. Now, c implies a, so given A is equal to B star B, to show A is positive semi-definite or which is same thing as showing that X star A X is greater than or equal to 0 for all X belonging to C. And this is what you have to show. So, let us compute that part.

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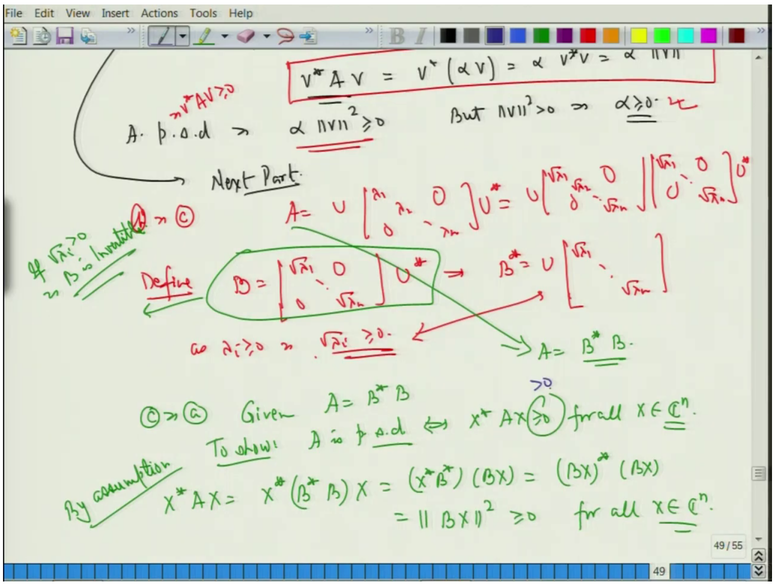

So, by definition or by what you mean by given assumption by assumption; X star A X is equal to X star B star B X which is same as X star B star into BX which is BX whole star BX which is nothing but norm of BX whole square and which you know that the length of anything is greater than or equal to 0.

So, this is greater than equal to 0, for all X belonging to C n, alright. So, we have shown that all the 3 are equivalent to us, is that ok. Nothing special we have done that part very simple. Now, I want to write it in terms of positive definite matrix.

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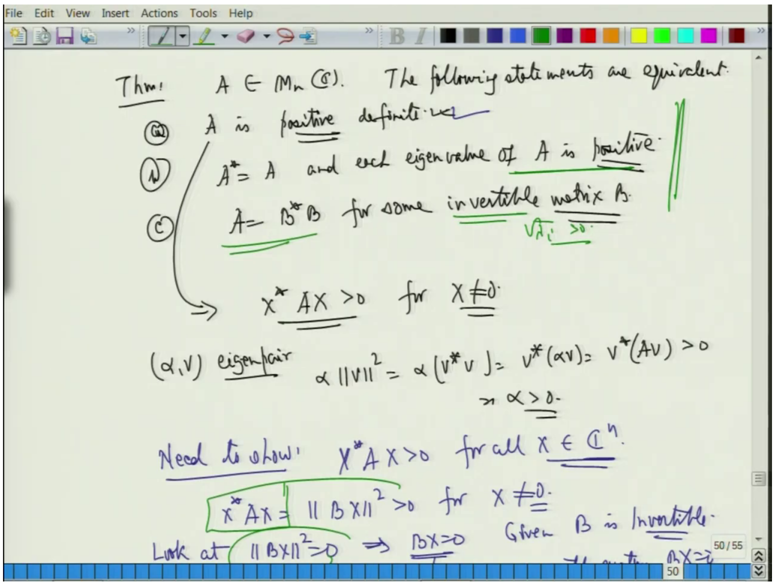

So, theorem, again A belongs to M n of C, fine. The following statements are equivalent. a, A is positive definite. So, semi-definite has gone it is only definite that we are looking at b, A star is equal to A and each eigen value of A is positive. So, from non-negative we are going to positive. And c, A is equal to B star B for some invertible matrix B, fine.

So, again let us go back and understand that positive definite definition implies that X star A X is positive for X naught equal to 0, alright, fine. So, when I look at eigen vector, so if alpha comma v is an eigen pair, then what we know is that alpha times norm of v square as you computed earlier this is same as alpha times v star v which is same as v star alpha v which is same as v star Av, alright, fine.

This as it is positive, it means that this is positive. So, this implies alpha is positive, fine. So, just writing things the same thing again and again and nothing else; so, one implies to I have done A implies B. Now, let us look at B implies C here, fine. We want to show that every eigenvalue is positive implies that A equal to B star B, but we want that B to be invertible here, fine.

So, let us look at B here for us. B was this matrix, fine. So, when we are saying that the lambda is are positive, what does it mean? Lambda is positive means a square root of lambda 1, the square root of lambda 2 till a square root of lambda n, each of them is positive, and therefore, this matrix B is positive, alright.

So, if here itself if the square root of lambda is positive implies B is invertible, alright. So, we get that B is invertible and we have been able to write A equal to B star B, fine. Similarly, here also since you are going to assume that this is positive, so now, we are going to assume that this is positive, fine. So, we need to show, sorry we need to show, alright, A is positive definite, so need to show v star or X star A X is positive for all X belonging to C n.

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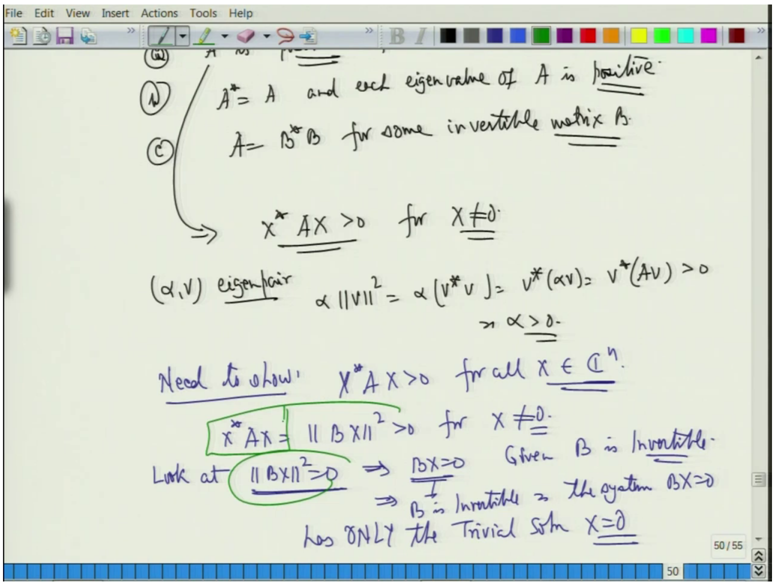

So, just compute X star A X which you computed earlier as B X whole square. I have to show this is this, fine. So to show, that this is positive for all X not equal to 0. This is what I have to show, fine. So, look at, so look at this equal to 0, alright. Now, norm of a vector is 0 only when the vector itself is 0. So, since that I am saying that the norm square is 0, so norm is 0, so implies that BX is 0.

But we have been given, given B is invertible. Well, what is invertibility mean? Invertibility means that this system B is invertible, implies the system B X is equal to 0 has only the trivial solution X is equal to 0, alright. So, what we have seen here is that, fine; that this can be 0 or this can be 0, X star A X can be 0 only when X is 0, alright.

So, you have shown that all these 3 conditions are equivalent. A is positive definite implies A star is equal to A. I have not proved that part you have to do it yourself. I have shown that the eigenvalues are positive. Eigenvalues are positive means this happens and invertibility comes because of each lambda i is positive, fine and from here we go to this because B is invertible implies the system B X is equal to 0 as only the trivial solution, alright.

So, I end this class here itself. We will look at the next ideas in the next class.

Thank you.