Linear Algebra

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Lecture – 06

Alright, so having learnt matrix multiplication, I would like you to understand and spend some more time just to understand the whole thing. It is not just saying that I understand matrix multiplication or I have seen the examples. Try them out yourself, so that you understand them better because throughout we will just concentrating on matrix multiplication again and again different ideas will come.

So, let us go with inverse now, what is call the inverse of the matrix, alright and we have done some inverse in our school also 2 by 2, 3 by 3, but I do not want to go into the determinant of a matrix and then understand the inverse. I want to look at the inverse from the point of view of matrix multiplication. So, therefore I am concentrating on matrix multiplication itself.

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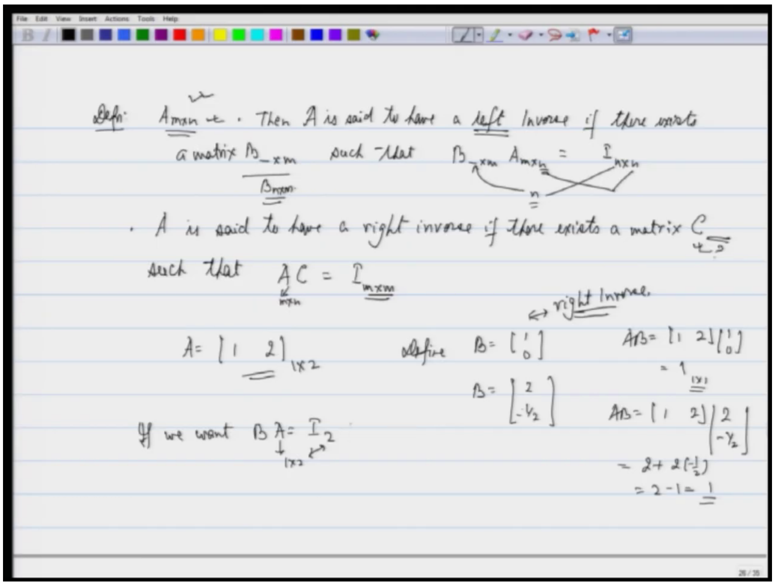

So, again let us go back to, so definition alright. So, I have a matrix A which is m cross n. I am taking real entries and so on, fine. So, A is a matrix of size this, then A is said to have a left inverse. So, left inverse A is said to have a left inverse if there exist a matrix B, alright. So, look at the size here m. So, I want here something times m such that B of something times A which is m cross n, I want it to be identity alright. So, since I want identity, identity is defined only for a square matrices.

So, identity has to be of the size m cross n because this is the last part that I have. So, this n has to match with this. So, therefore this forces me that this should be equal to this part as such, so this should also be n as such, fine. So, I need B of the size n cross m alright and this n is dictated from the idea that B is on the left and the product should be identity. So, B is on the left.

So, left inverse in many books they may write it differently, they may put it as right inverse, but for us left means multiplying on the left, alright. Similarly A is said to have a right inverse if there exist or if there exist a matrix C again. Look at the size yourself matrix C, such that AC is identity what is the size of this? Since A is size m cross n. So, m has to come here. So, therefore it has to be size n cross m.

So, from there you can define what should be the size of C as such, fine. So, let us look at examples to see that I may have such conditions coming into play because A is a rectangular matrix. So, for example if I take A as 1, 2, I can define B as 1, 0. So, if I see that AB is equal to 1 2 times 1 0 which is equal to 1 and which makes sense because A here is 1 cross 2. So, this has to be 1 cross 1, fine. I could have taken B to be also equal to say 2 and no, 2 and 1 will not work. So, two and half with a minus sign. So, if I look at this A B will be equal to 1 2 times 2 and minus half which will give me 2 plus 2 times minus half which is 2 minus 1 which is 1.

So, there can be lot of choices for me fine, but in this case we will see that I cannot have something on the left. So, this was looking at B was on the right inverse. So, this is getting right inverse, fine. If I want to get something, so I want. So, if we want BA to be identity I 2, alright. So, look at this A is the size 1 cross 2, therefore this is 2.

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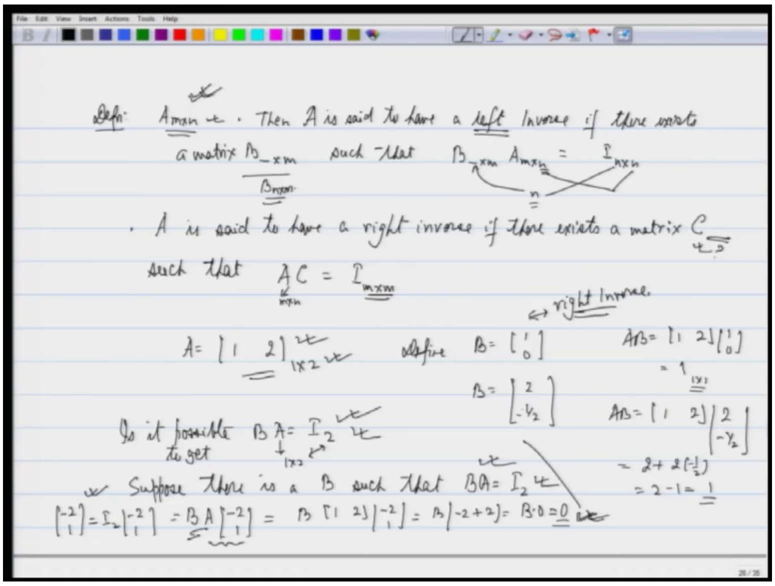

So, if you want this or if you want this, then can we get such AB.? So, is it possible to get such AB? Is it possible; is it possible to get B satisfying this property? Alright.

So, let us understand this. Suppose I get it. So, suppose there is AB such that B times A is I 2 fine, then what I see here is that BA if I multiply this a look at this minus 2 and 1, alright. So, that I multiplied then what do I get? I get B times A times, this is A is 1 2 times minus 2 1 which will be B times minus 2 plus 2 minus 2 plus 2 which is equal to B times 0 is 0 alright, but at the same time we have written that BA is I 2 and therefore, this will get I 2 times minus 2 1 which will turn out to be minus 2 and 1.

So, what we are saying here is that the vector minus 2, 1 is same as the zero vector which does not make sense for us and therefore, such AB that is given A is matrix size 1 cross 2. I cannot get a matrix B such that B times A is I 2 because if B times A is I 2, then I can write minus 2, 1 as I 2 times minus 2, 1 I 2 is already written as B times A. Just multiply it out, use associativity and what we see is that 1 two times minus 2 1 is 0 and therefore, minus 2, 1 that vector becomes the zero vector which does not make sense, alright.

So, I may have left inverse, I may not have left inverse, I may have right inverse, I may not have it. All these questions come into play. So, but this is true only when we are looking at A which is m cross n or rectangular matrix. Now, what happens if a is a square matrix?

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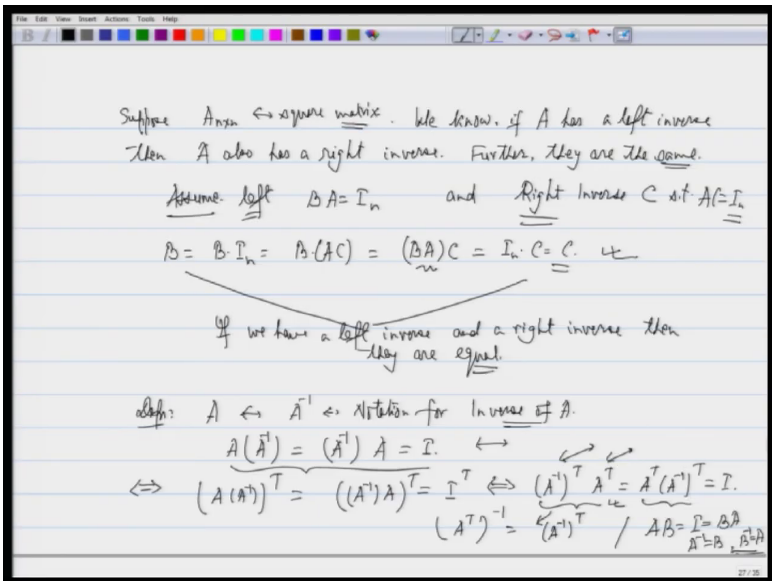

So, suppose A is n cross m, so A square matrix. Then we will see that or we know at present that if there is a left inverse, then there is also a right inverse. So, we know we know if A has a left inverse, then A also has a right inverse. Further they are equal, they are the same.

So, here I will just prove that if there if the inverse exist then they are same. So, assume the, so assume I have a left inverse. So, that BA is identity and I also have a left inverse. So, sorry right inverse see such that AC is identity. So, this is the left and that is the right inverse, then I would like to show that B and C, they are same.

So, what you can do is that you can write B as B times identity which is B times identity same as A times C associativity helps me to write this as this. Now, I can use the idea of left inverse to write this as identity which is C. So, here what we see here is that if I have a left inverse and a right inverse, then they are equal.

So, what we are saying is that if we have a left inverse and a right inverse, then they are equal. We are not saying anywhere that given that we have a left invert that implies that it will have a right inverse and knowing that we have a right inverse, we will get a left inverse. We are not saying that that will come when you solve system of equations and derive results. We will say that whenever there is a left inverse, there is a right inverse and whenever there is a right inverse, there is a left inverse alright.

So, we will need to understand those system of equations and so on. We will have to wait for that, but if they exist then they have to be the same, fine. Now, since they are same, so what we do is that we define the inverse of A and denote it as A inverse given A, I have A inverse. So, this is the notation for notation for inverse of A, alright A to the power minus 1. Is that ok? Now, how does it behave with respect to. So, what we are saying here is that A times A inverse is A inverse times A is identity fine.

So, from here I can also conclude that. So, let us take, so from here I can take transpose on both the sides. So, I will get A times A inverse whole transpose is equal to A inverse a whole transpose which is same as identity transpose which is same as by the product transpose rule. It will be A inverse transpose into A transpose is same as A transpose into A inverse transpose identity.

So, what we see is that I have this condition being satisfied which was similar to the earlier one. So, it means that this matrix is the inverse of this matrix and vice versa. So, what we are saying is that, that a transpose has A inverse and the inverse of this matrix is nothing, but this matrix because their product is identity alright.

So, whenever we say that A times B is identity is same as BA, we are saying that A inverse is B or B inverse is A. We are writing either things. Is that? So, that is the way of writing it. So, let us look at some examples to understand when can we have inverse and when we cannot have. So, look at 2 by 2 example 3 by 3, we will not learn how to obtain the inverse, but at least understanding what does inverse mean.

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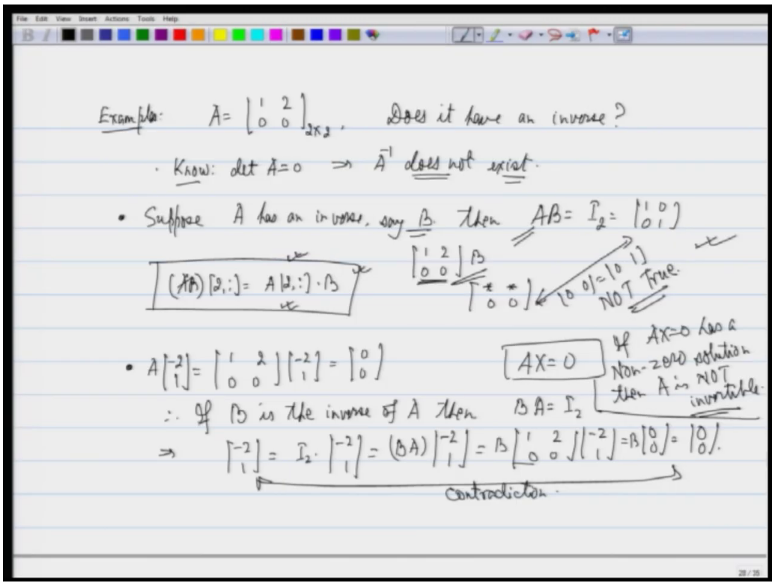

So, let us take examples. Suppose I have a matrix A which is 1 2 0 0 alright 2 cross 2 matrix. Does it have an inverse? So, from our school days we know that determinant of this matrix is 0 and therefore, it does not have an inverse.

So, no determinant of A is 0 implies A inverse does not exist fine, but I would like to understand it in a different way alright. So, two things that I would like you to understand here is that suppose A has an inverse say B alright, then A times B should be identity matrix I 2 which is 1 0 0 1 fine, but if I look at AB look at the matrix product.

So, in the matrix product A comes like this and B is here with u, fine. So, we are looking at the second row of A has all the entries 0. So, what will happen to the second row of AB? I do not know what will happen to the first row of AB, but the second row of AB has to be 0, alright. Why? Because we call that second row of AB is same as second row of A times B.

So, this is where the matrix product comes to our help to say that since the second row of AB is first row, second row of A times B and the second row of A is 0 0, we get that the second row of A B will be 0 and then, we are saying that 0 0 should be equal to 0 1 not true and hence, I cannot have such AB. This is one way of saying it, fine.

The another way of saying this will be that look at this matrix A. This matrix has the property that 1 2 0 0 is there, alright. So, A times if I multiply this matrix with minus 2, 1 which is same as minus 2, 1, this will turn out to be 0 0 alright. Therefore, if B is the inverse of A, then B times A should be identity I 2 and this will imply that minus 2, 1 is same as I 2 times minus 2, 1 which is same as BA times minus 2, 1 which is B times A which is 1 2 0 0 times minus 2, 1 same as B times 0 0 which is 0 0 fine.

So, this gives me a contradiction. So what we have done here is, we have looked at this as a matrix product and tried to get a contradiction. Here we are saying that look at the system A x is equal to 0. Let us try to solve it. If it has a solution, if A x is equal to 0 has a non-zero solution, then A is not invertible alright. This is a very important idea that we have.

We are able to relate, alright understand we are able to relate the inverse with solving a system of equations and what is the system of equation? It is a homogeneous system and we are relating it with a non-zero solution. Zero is always a solution of the homogeneous system because A times 0 is 0, fine. Let us look at another example. Here we looked at the row alright. Now, we look at the column alright.

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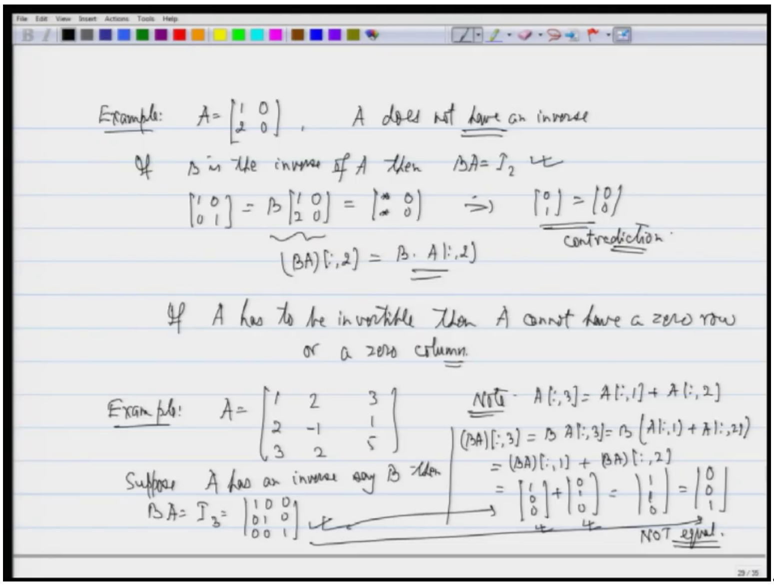

So, let us look at this example, another example. So, let me take A as 1 2 0 0, fine. Here again A does not have an inverse, does not have an inverse. Why it does not have an inverse? Because if I multiply with B, so I know that if B is the inverse of A, then B times A should be identity, but if I multiply B with 1 2 0 0, I will get it as something here and 0 here because recall that in if you look at BA, the second column of this, then it is nothing, but B times the second column of A alright fine, but we are saying that B times A is I 2. So, this should be equal to 1 0 0 1 and therefore, what we are saying is that this vector 0 1 should be equal to 0 0 and that is the contradiction, fine.

So, what we are seeing here is that A matrix A cannot have a zero row or a zero column. So, if A has to be invertible has to be invertible then A cannot have a zero row or a zero column, fine. You can again use the idea of system of equation, but you have to do it something else. So, another example which is 4 3 by 3 alright.

So, let us write A as 1 2 3 2 minus 1 2 and let us write this as the sum. So, it is 3 1 5. So, see here that the third row the third column. So, note that the third column of A is equal to first column of A plus second column of A.

So, let us see 2 plus 1 is 3 1 3 plus 2 is 5, alright fine. Suppose I have an inverse. So, suppose A has an inverse say B, then B times A will be identity which will be 1 0 0 0 1 0 0 0 1 fine. Now, let us look at the third column of AB.

So, what is the my matrix multiplication? The third column of BA will be equal to B times the third column of A which is B times the first column of A plus the second column of A which by matrix multiplication is again BA times the first column of BA plus the second column of BA which is nothing, but the first column of BA from this part is 1 0 0 and from the same part it is 0 1 0. This is the first column of BA; this is the second column of BA. So, this gives me 1 1 0, but the third column of BA from here is 0 0 1 they are not equal alright.

So even though we do not have a determinant with us, we are able to say that if there is a column of A alright. So, if there is a column of A which is sum of two columns, I have a problem alright.

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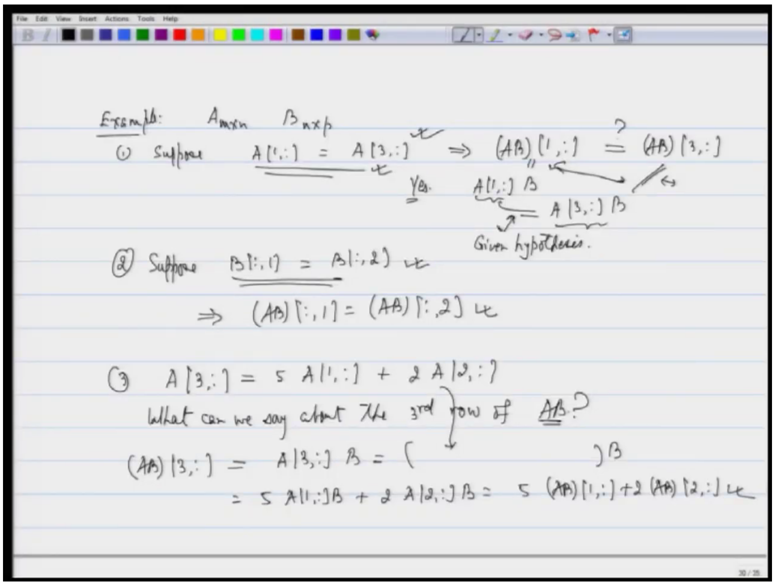

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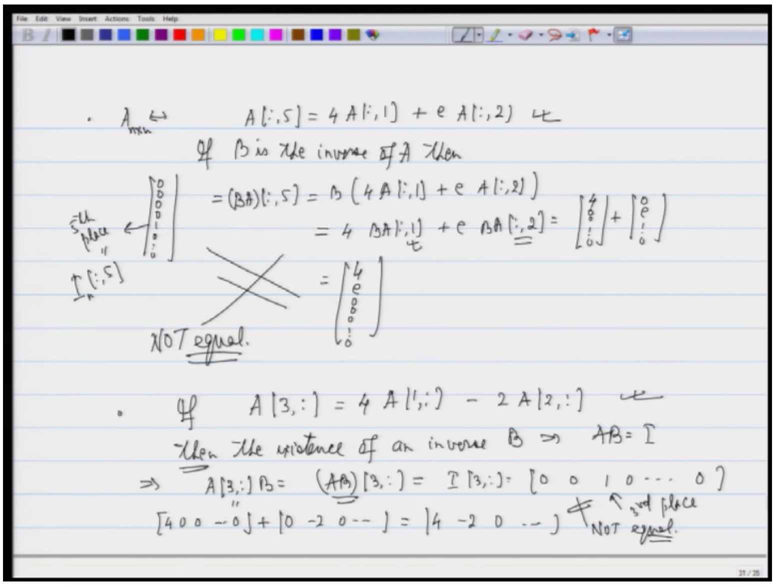

Not only that the same idea will tell you that if I have matrix A which is n cross n and suppose I am saying that it is 5th column is same as four times, the first column plus e times the second column, then what will happen to BA?

So, if B is an inverse, so if B is the inverse of A, then if I look at the 5th column of BA, 5th column of BA should be 0 0 0 0 1 0 0. I am not getting the size here. So, one at the 5th place because this is what the identity will be because this is coming from identity I n of 5, but there if I write in terms of this part, it is nothing but B times 4 of this plus e of this which will be equal to 4 times BA of this plus e times BA of this which will be equal to. So, it is 1, the first column. First column will be this and the second column will be 0 e and so on.

So, this will turn out to be 4 e 0 0 0 and so on. So, you can see that these two are not equal fine. We can make a similar statement when we are looking at say rows. So, if I know that, so if there is some row say 3rd row of A is equal to 4 times, the 2 row of A minus 2 times, the 2nd row of A, then I can multiply B on the right now, then the existence of B existence of an inverse B implies AB is identity and this will imply that A this of B which is nothing, but the 3rd row of AB will be equal to 3rd row of AB since I am writing identity 3rd row.

So, it will look like 0 0 1 0 0 0 this is the way it will look like one at the 3rd place and this condition will give me that this should be equal to 4 0 0 and so on plus 0 minus 2 0 and so on. So, this will give me as 4 minus 2 0 and so on and then you can see that these two are not equal, alright.

So, what we are arriving here is that whenever matrix A has the property that I am able to write a row alright in terms of other rows, then that matrix cannot be invertible. Similarly, if I am able to write a matrix A column of A matrix in terms of some other columns, I again have an issue alright.

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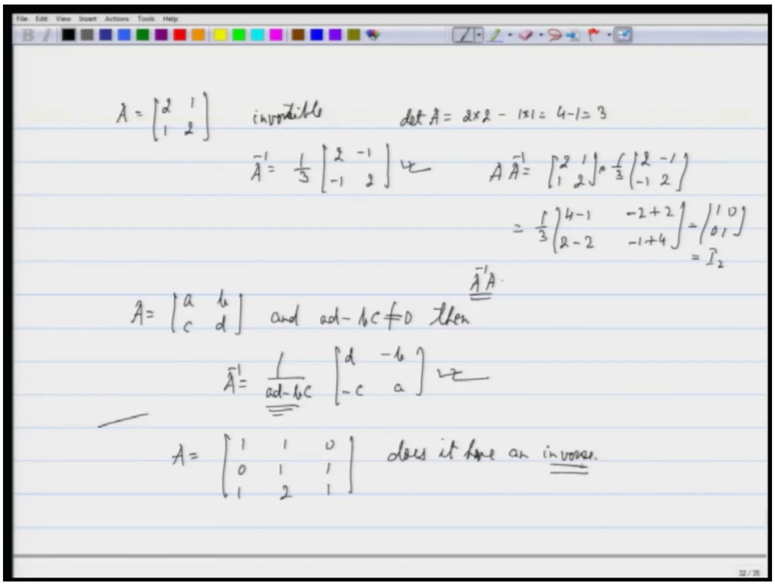

So, I will just give you I will finish it by giving you an example of a matrix which is invertible. So, let me write 2 1 1 2, this matrix. So, this matrix is inverse is invertible and the inverse is all let you know it that determinant of this matrix since the 2 cross 2 matrix is nothing, but 2 into 2 minus 1 into 1 which is 4 minus 1 is 3 and therefore, the inverse of this matrix turns out to be equal to 1 upon 3 times the adjoint of A. So, 2 2 minus 1 minus 1, alright.

So, you can check that A times A inverse is 2 1 1 2 times 1 upon 3 of 2 minus 1 minus 1 2 which is 1 upon 3 times multiply by 2. So, 2 into 2 is 4 minus 1 minus 2 plus 2, 2 minus 2 minus 1 plus 4 which is 1 0 0 1 which is I 2 fine. You can also verify what happens to A inverse A. Do it yourself.

In general if I have A which is a b c d and a d minus b c is not equal to 0, then A inverse is equal to 1 upon a d minus b c. So, this is the inverse that is the determinant times something.

So, I hope I am correct. I will have to check it out just to see that whether I am correct or not. So, a d minus b c is fine, minus a b plus b a is fine, c d minus d c and minus c a this yeah. So, this is the inverse of this.

So, we will not be bothered about computing the inverses as such, but what I would like you to understand is that just looking at matrices the inverse can be obtained. So, as an example I would like you to see what happens to this matrix. Does this matrix have an inverse? So, does it have an inverse? Alright. So, I just end it here.

In the next class what we will do is that we look at how this matrix product the inverse, this notion of inverse is going to help us to compute or to solve a system of linear equation. We will start with the basics, but I would like you to go back, spend some time with matrix multiplication, learn how to multiply things and come for the next class.

Thank you.