Linear Algebra
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Lecture – 59
Applications of Schur's Unitary Triangularization

Alright. So, let us proceed with the ideas that we had. In the previous class about unitary triangularization alright some applications of that.

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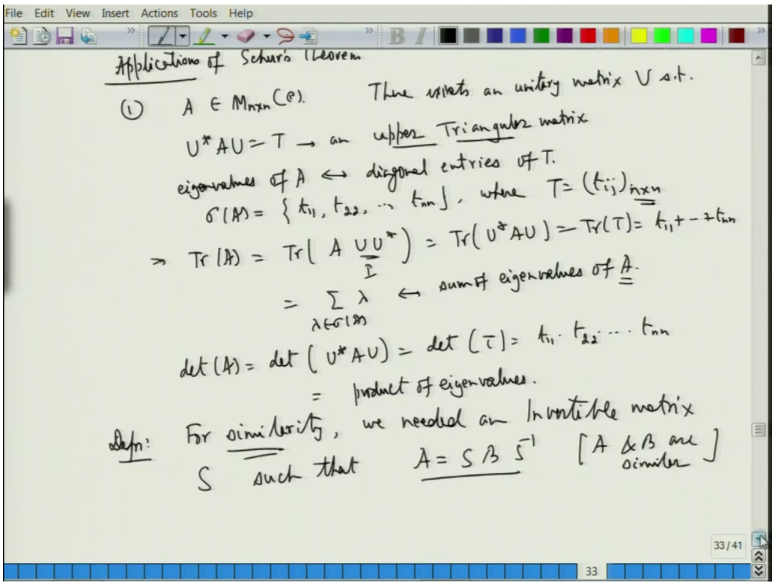

So, let me do one after the other so, some applications. Applications of Schur's Theorem one. So, what we have seen was that if. So, all our matrices are n cross n complex matrices fine. So, the theorem says that look at the this sigma A. So, A is U star or we wrote I think we wrote that there exist an unitary matrix U such that U star AU is T an upper triangular matrix fine; an upper triangular matrix also eigenvalues of A was same as diagonal entries of T fine.

So, sigma A is same as t 11, t 22 t n n. So, sigma A is just this where T is t ij alright fine n cross n. So, from here we can easily conclude that this will imply that if I look at trace of A trace of A is same as trace of U star AU or trace of A times UU star because UU star is identity which is same as trace of U star AU which is same as trace of T.

Which is same as t 11 till t n n alright, which is same as summation over lambda belonging to sigma A or sum of eigenvalues of A. So, when we trying to prove it to the first time there was a quite complication involved, but using the Schur's lemma everything is nice what about determinant of A? Determinant of A is same as determinant of U star AU by the same argument that we did here the determinant of A is same as determinant A into I and then we can multiply them out.

So, which is same as determinant of T, which is same as t 11 into t 22 into t n n which is nothing, but product of eigenvalues. So, you could do them directly. So, simply alright there is nothing much that needs to be done here fine. So, this is important that I would like you to understand. The second thing I would like to say here is definition. So, there is a difference between what is called unitarily similarity and similarity. For similarity for similarity what we wanted was similarity.

We needed an invertible matrix P or invertible matrix S such that A is equal to SBS inverse alright. So, they say that A and B are similar this is what we said alright. At no place we said that we need unitary we just needed similarity means invertible matrix S S invertible that is all we needed fine. Now, we are saying that unitarily similar. So, when we say unitarily similar we want we need S to be a unitary matrix.

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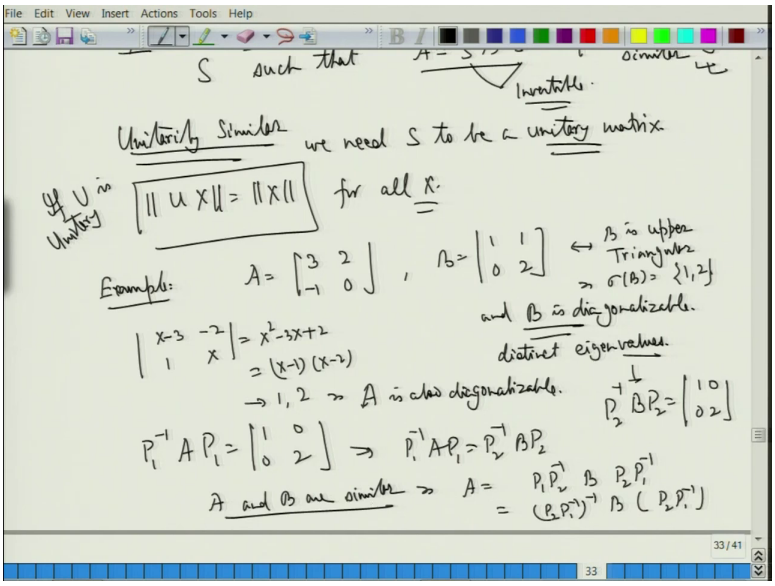

So, we are putting a an stringent condition that, in place of S being just non singular we want S to be unitary itself alright and why do you want it? Because unitary always helps us in trying to understand something more that if I have a unitary transformation S.

So, if U is unitary then the length of any vector x remains the same this for all x alright fine that is not true when I look at the similarity here with respect to invertibility. So, in some sense unitary similar is more stringent and we follow mostly in our calculations because the norm of the vector does not change fine.

It may happen that some of the entries after calculation may become very very small, but it may get taken care of by something which are larger and so, on, but everything cannot become smaller and some things are basically there are some things which are positive about it that the size does not change in some sense alright fine.

So, as a example let us look at this this example to differentiate between the two example, let us look at this matrix A which is 3 2 minus 1 0 and B as 1 1 0 2 fine. So, I would like you to see that the eigen values of these two. So, eigen values are this is already B is upper triangular. And therefore, sigma of B consists of just 1 2 and B is diagonalizable why it is diagonalizable? B is diagonalizable because it has distinct eigen values fine.

So, I can look at it what about A? What are the eigen values? Let us compute the eigen values of A. So, x minus 3 minus 2 1 x determinant of this is x square minus 3 x plus 2 which is same as x minus 1 into x minus 2. So, this is also 1 2. So, A is also diagonalizable A is also diagonalizable. So, what we have is for A I will get a matrix a P 1 such that P 1 star P 1 inverse A P 1 will be equal to 1 0 0 2, for B I will get P 2 inverse BP 2 will also is equal to 1 0 0 2.

And therefore, I can say that this implies that P 1 inverse AP 1 is equal to P 2 inverse BP 2 and that will imply that A is equal to P 1 P 2 inverse B times P 2 P 1 inverse which is same as P 2 P 1 inverse whole inverse B times P 2 P 1 inverse alright. So, A and B are similar A and B are similar alright can we say that they are unitarily similar or not?

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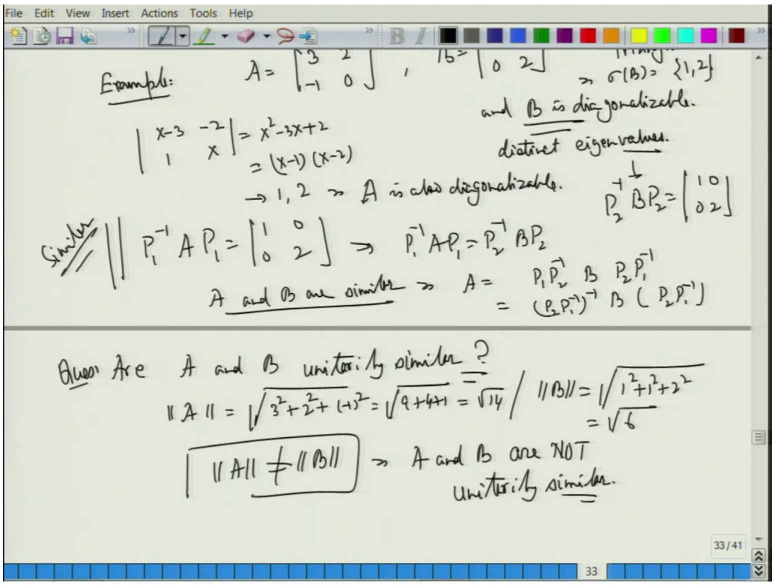

So, are question are A and B unitarily similar? Alright. So, I would like you to see that they are not unitarily similar because their sizes are different look at the size of A. The size of A if I look at in some sense I am not giving you what are the idea of size is, but I want to look at the size of A in some sense which is nothing, but if you remember our in matrices n cross n matrices we looked at the trace alright.

The inner product was in a bit product between A and B was B transpose A compute that and look at the trace of that which also correspondent to looking at dot product of each entry itself. So, here it will be nothing, but a square root of 3 square plus 2 square plus minus 1 square which is same as the square root of 9 plus 4 plus 1 which is the square root of 14, but what about norm of B? Norm of B is nothing, but a square root of 1 square plus 1 square plus 2 square which is the square root of 6 alright.

So, they are not equal norm of A is not equal to norm of B alright; this will imply that A and B are not unitarily similar fine. So, again understand it if they were unitarily similar that will imply that unitarily similar will imply that the length of A and the length of B have to be the same fine here length of A is not the same as the length of B therefore, there is a contradiction fine, but they are similar as such this is what we saw that they are similar fine.

So, similarity may not imply unitarily similar, but unitarily similarity will imply similarity alright. So, you need to keep track of that this is very very important idea. So, I would like you to look at some examples where you can prove that two matrices are unitarily similar by some unitary matrix and so on fine. Now let us look at the next idea where we look at now diagonalizity of a special matrices.

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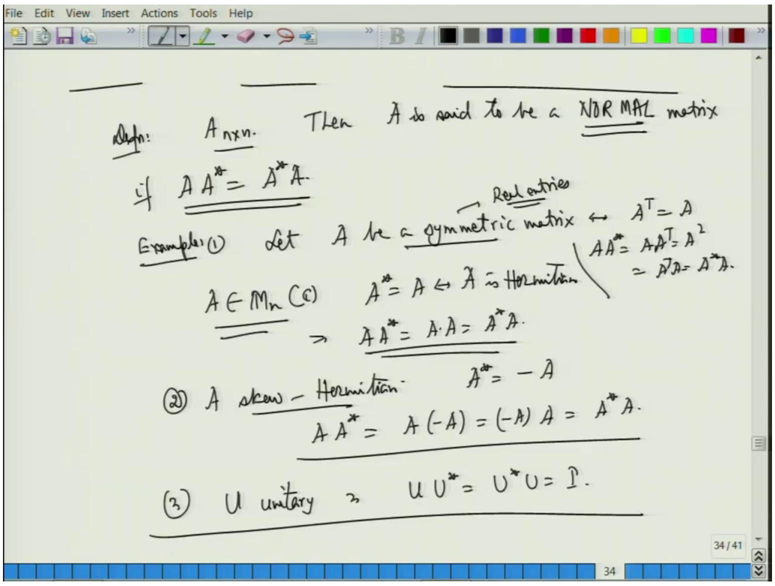

So, definition of what is called a similar matrix definition A is n cross n, then A is said to be a normal matrix if A times A star is equal to A star alright. So, a is a square matrix and we are putting this stringent condition that A times A star is equal to A star A fine example.

So, look at any. So, let A be a symmetric matrix what does symmetric matrix mean? This means that A transpose is equal to A and symmetric means I am looking at real entries A with real entries. And therefore, A times A star will be equal to A times A transpose which is A square itself which will be equal to A transpose which is A star A fine.

If I have A with complex entries fine then I need A star to be equal to A that is A is Hermitian fine. If A is Hermitian this will imply that A times A star will be equal to A times A which is same as A star A fine. So, every Hermitian matrix is normal what about a skew Hermitian; A skew Hermitian?

So, skew Hermitian means A star is equal to minus A. So, if I want to compute A times A star, it will be equal to A times minus A which will be same as minus A times A which will be same as A star A. So, again this is also true for me alright. So, whether I am looking at Hermitian matrix or a skew Hermitian matrix they are normal matrices itself you also have unitary matrices which are which also satisfy this because for unitary matrix 3, U unitary implies U times U star is equal to U star U is equal to identity alright.

So, unitary matrices are also normal matrices and so, on. So, there are lots of examples of normal matrices. So, we want to prove that every normal matrix can be diagonalized alright. So, let us try to prove that part.

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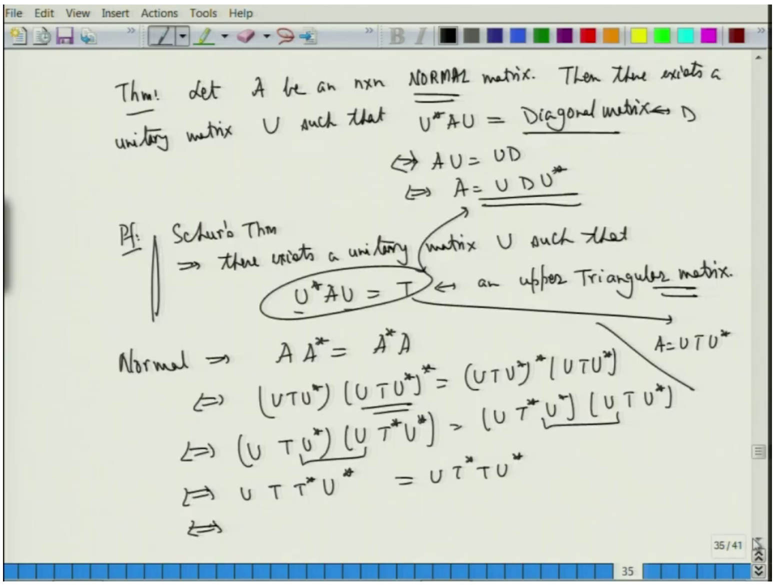

So, theorem let A be an n cross n normal matrix then there exists a unitary matrix U such that U star AU is diagonal matrix alright or same thing as saying that AU is equal to if I write this as D

It is UD or which is same thing as writing as A is equal to UD U star fine. This is important that you can write it in whatever way you like it depends on you and accordingly proceed with all the mathematics that you want to do fine. So, let us prove this, this is a very simple proof with what we have learned till now. So, what we know is by Schur's lemma by Schur's theorem implies there exists a unitary matrix U such that U star AU is T an upper triangular matrix fine.

So, we you have U star AU as an upper triangular matrix let us try to understand what do I mean by now using normality. At this stage when I am writing this I am not use the idea of normal now let us use the idea of normal. So, what does normality mean? So, normal implies A times A star should be equal to A star A fine; is that ok?

So, this will also imply and is implied by because U is a unitary matrix, I can just multiply by U on both the sides or I can write A in terms of whatever it is. So, if I write A here I already wrote if I look at this part here you can write like this. So, I can write A as UD U star I am writing. So, here it will become. So, let us write A here.

So, here A is equal to U T U star alright fine just multiplying by U and U star on both the sides. So, this is equivalent to saying that U T U star U T U star whole star is same as U T U star whole star into U T U star fine which is same thing I am looking at just multiply it out you get U T U star what about this? This is nothing but U T star U star by fine.

What about this part? This is U star star again U here T star here and U star will come here U T U star. So, this was just looking at the star taking a star both the sides now I can look at associativity of the matrix product these two matrices are identity.

So, I get U T, T star U star is equal to again here T U star U is identity. So, I get U T star T U star fine. So, now I can look at the U is an invertible matrix U is unitary. So, invertible and therefore, I can multiply by U star on the left and U on the right.

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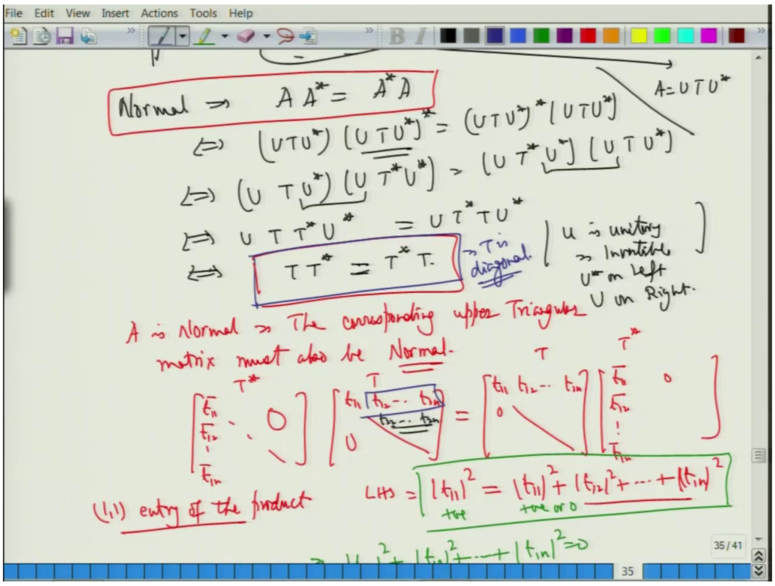

And therefore, this will imply that this is same as T T star is equal to T star T fine. U is unitary implies invertible or whatever it is you want to say just multiply by U star on left and U on right.

So, whatever way you want to understand, understand it. So, normality implies that normal tells me this part, but this part is equivalent to saying that the tried at the triangular matrix that I got should also be normal. So, A is normal implies the corresponding T corresponding upper triangular matrix upper triangular matrix must also be normal alright and now let us remember one result where we said that if I have an upper triangular matrix.

So, the T T star is T star T, then t has to be a diagonal matrix we have done it for the lower triangular also. So, let us try to understand this again. So, I have t here which is upper triangular. So, t 11 t 12 t 1 n now I am looking at. So, this is T I want to look at T star. T star we are saying this is equal to fine. So, T star is t 11 bar t 12 bar this is 0 here fine.

So, there will be 0 here and t 1 n bar this is what t star will be we are saying that this is equal to T here and T star here. So, what is T? t 11 t 12 t 1 n 0 here something here T star writes t 11 bar t 12 bar t 1 n bar something here and 0 here fine. So, just multiply it out look at the 1,1 entry 1, 1 entry of the product alright. If you look at the 1, 1 entry of the product this entry the left hand side LHS is equal to mod of t 11 square what about the right hand side? The right hand side is t 11 this square plus t 12 this square plus t 1 and this square alright.

So, absolute value of so many things. So, what we are saying is that, this should be equal to this, these are complex numbers and you are looking at the absolute value square of this. So, these are positive numbers or non negative numbers positive positive or say positive or 0 all of them.

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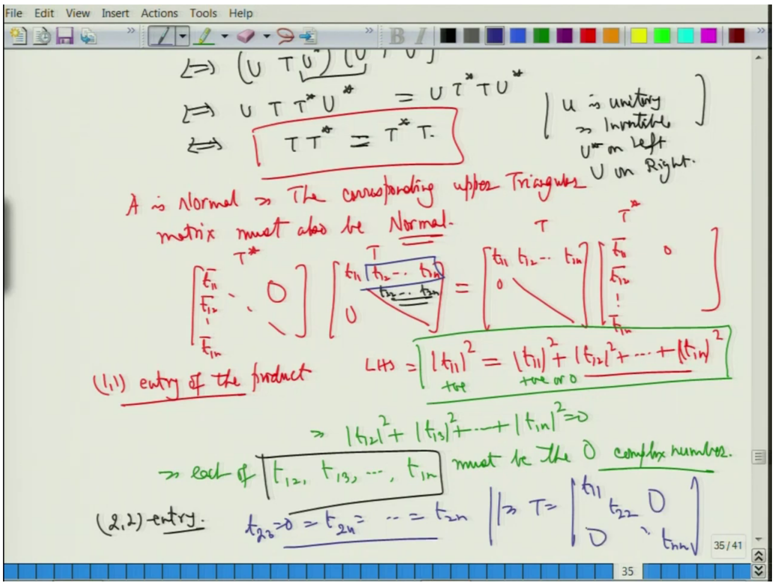

So, when I am saying that this is equal to this it means that. So, this implies mod of t 12 square plus mod of t 13 square plus mod of t 1 n the square is 0 implies each of these each of t 12 t 13 till t 1 n must be the 0 complex number fine.

Now, once you have done that then you can look at the 2, 2 entry after this you can look at the 2, 2 entry you can look at the 2, 2 entry and from there conclude that the 2, 2 entry must be 0. So, this 2, 2 till 2,2 n. So, they will also become 0. So, at the first stage we have shown that this is 0. So, that implies that implies that this part is 0 alright and the next stage t 2 3.

So, this will imply that t 2 3 will be equal to 0 equal to t 2 4 is equal to t 2 n alright fine. So, what we will see here is that. So, what we see this part implies that the matrix T basically looks as t 1 1 t 22 t n n alright the rest of the entries are 0 fine. So, we have shown here that because of normality a is normally implies T T star is T star T and T is upper triangular implies T is diagonal.

This was there in one of the exercise also in I think first or second slide or after matrix multiplication I think fine. So, you have to keep track of things. So, therefore, we have proved that if A is a normal matrix which implies that A A star is A star A which implies T star T is equal to TT star and hence U star AU is A diagonal matrix fine.

So, for any normal matrix diagonalization is true and you have to keep track of that that this is one of the most important results that we have here we look at its applications now.

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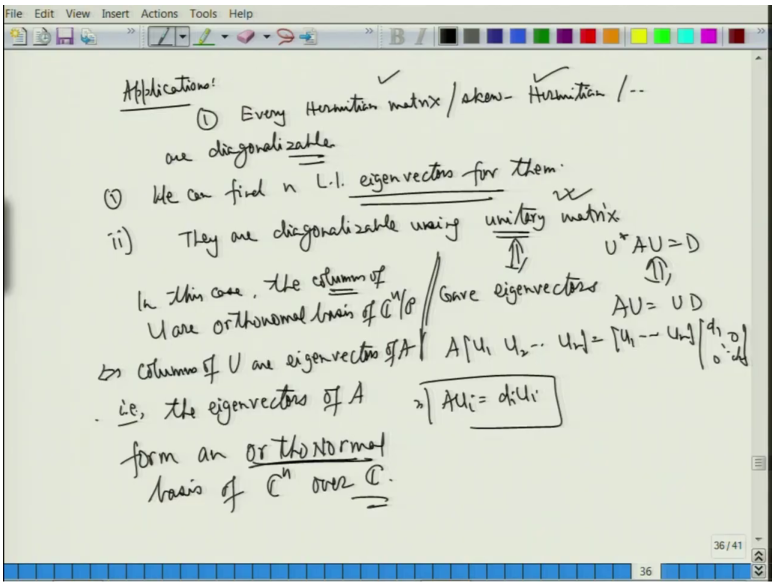

So, let us try to have an application of this. Applications 1: every Hermitian matrix oblique a skew Hermitian matrix or unitary matrix. So, on are diagonalizable alright this is very important that all of them they can be diagonalized fine.

And what do I mean by say diagonalized means they. So, we can find we can find n linearly independent eigen vectors for them is that fine and you saw that you can do it using unitary matrices one sorry first part second alright they are diagonalizable using unitary matrix what does it mean?

Recall this matrix will also give you eigen vectors alright they gave you eigen gave eigenvectors because when I am looking at U star AU is diagonal this is same thing as saying that AU is equal to UD. And therefore, A of u 1 u 2 till u n is equal to u 1 to u n times diagonal d 1 to d n implies A of u i is equal to d i u i alright fine.

So, when you are saying that they are using unitary matrix, it means that the rows of A of u or the columns of u both are orthonormal matrices. So, we are saying that in this case; in this case the columns of u are orthonormal basis of C n over C fine or which is same thing as saying that the columns of U are nothing but the eigen vectors columns of U are eigenvectors of A fine. So, we are saying that this.

So, what we are saying is. So, that is the eigenvectors of A form an orthonormal basis of C n over C is that ok? So, they form an orthonormal that is very very important that whatever we are doing we are able to get an orthonormal basis of C n over C and that is true for Hermitian matrix the skew Hermitian matrix or for that matter any normal matrix alright. In the next class we will try to understand them in a different language fine.

So, I want you to keep track of things here and what we mean by saying things. So, as the last part let me rewrite this part in a different language now. So, let us try to understand what we are saying.

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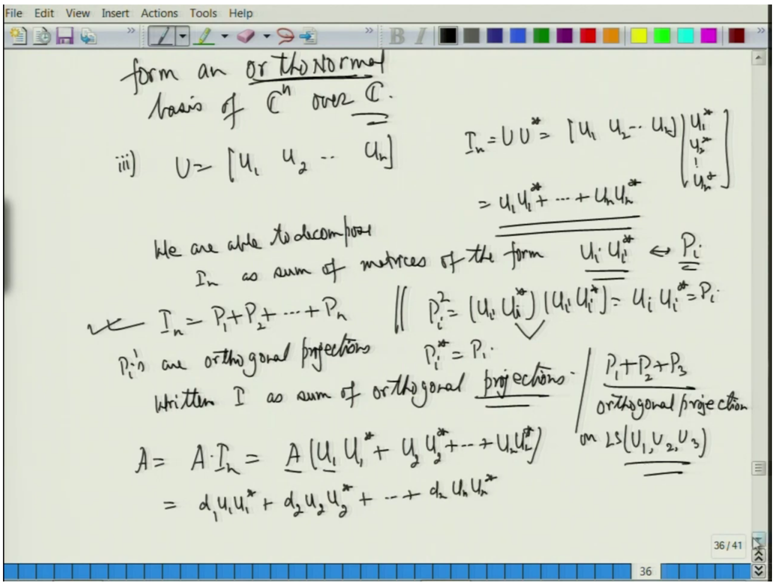

So, third part. So, what I am saying is that I have a matrix U which is u 1 u 2 u n let us multiply it and see what happens. So, I want to multiply u and u star I know that this is supposed to be identity because u is unitary matrix. So, this is same as u 1 u 2 u n times the transpose of it; so, u 1 star, u 2 star, u n star. So, I am writing identity I just multiply it you get u 1 u 1 star plus so, on till un un star fine I am writing like this.

So, we are able to decompose; we are able to decompose I n as sum of matrices of the form u i u i star fine. Let me write this as P i fine then what we see is that identity is P 1 plus P 2 plus P n; what is P i square? P i square is u i u i star into u i u i star, this part u i star u i is nothing, but the dot product or standard dot product of complex numbers. So, I get this as u i u i star itself which is P i and P i star is same as P i. So, we have written I as. So, this P i's are orthogonal projections fine.

So, we have written I as; written I as sum of orthogonal projections fine each of them will have rank 1. So, each P i will have rank 1. If I just want to look at say P 1 plus P 2 plus P 3 this is also an orthogonal projector; orthogonal projection on the subspace linear span of u 1, u 2, u 3 fine. Their rank will be 3, because they are orthogonal and they are linearly independent and so on fine.

So, I have got I n like this if I look at A, A is nothing, but A times I n which is same as A of u 1 u 1 start plus u 2 u 2 star plus u n u n star; I should written a star here because this is what we are looking at. So, this is equal to A times u 1 is lambda u 1 what is A u 1? It is an eigenvalue.

So, I wrote it as d 1 d 2 I think. So, A u 1 is d times u 1 u 1 star I think let just a minute I should not get confused. So, let me write this itself u 1, u 2, u 3 projection alright. So, d of this plus not d 1, it was yeah d 2 u 2 u 2 star plus d n u n u n star.

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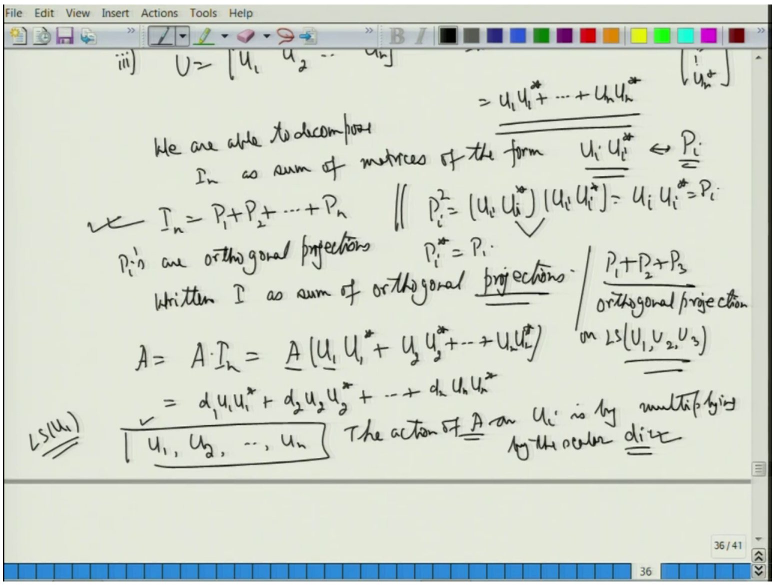

So, what we are saying here is that, I have these spaces u 1, u 2 u n; I have these vectors the action of A; the action of A if I look at what it does? It expands or it elongates or whatever you want to say it changes by multiplying by d i alright.

The action of A on u i is by multiplying by the a scalar d i; is that ok? So, this is what you have to be careful about that A is a matrix which may look quite different. But, if I look at the action of A on each of these subspaces, if I look at the action of A on the subspace u 1 or linear span of u 1 if you want to say of you want to be completely correct.

Similarly, linear span of u 2 that is just by multiplying by a scalar quantity and nothing else in place of multiplying by some vector we are multiplying by a scalar quantity; is that ok? So, you have to keep track of this we will look at this again when we come to Hermitian matrix; we will add something extra to this ideas alright. So, that is all for now.

Thank you.