Linear Algebra
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Lecture – 58
Schur's Unitary Triangularization (SUT)

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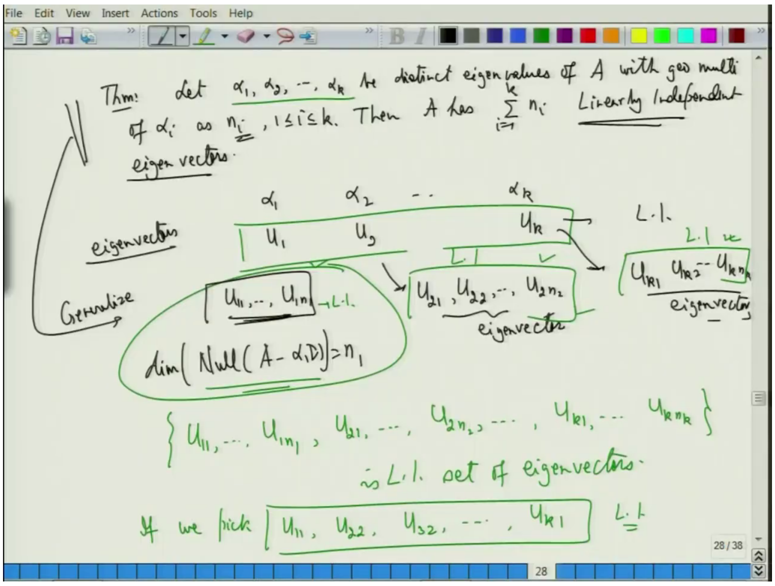

Alright. So, let us look at this result that we had done earlier. What we had said was that, if alpha 1 to alpha k at distinct eigenvalues and their geometric multiplicity is n i's; then you have summation n i number of such eigenvectors. So, we had so many eigenvectors with us; this is what we have showed. So, therefore, form here as a corollary you can say that, I forgot to say was that as a corollary, corollary.

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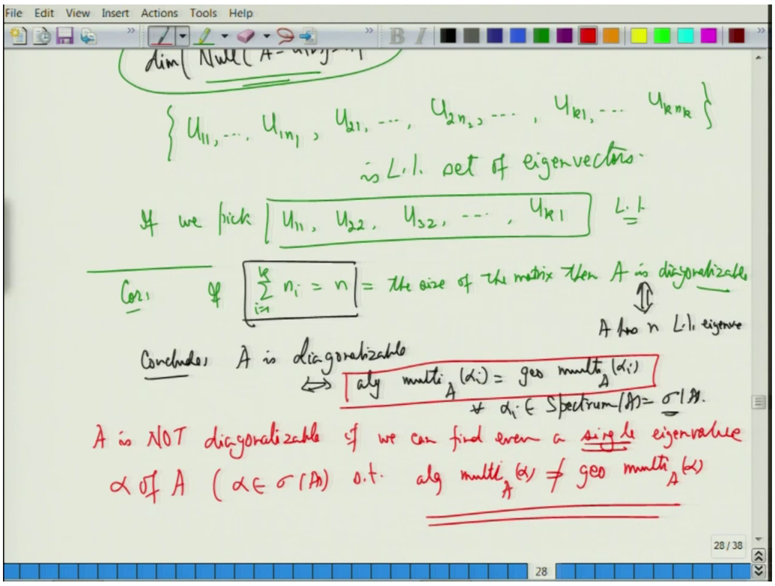

If summation n i, i is equal to 1 to k is n, which is equal to the size of the matrix, the size of the matrix; then A is diagonalizable, alright. Because the idea was that, A is diagonalizable if and only if it has n linearly independent eigenvectors, alright. So, diagonalizable if and only if A has n linearly independent eigenvectors, fine.

So, the question is that, when can this happen? What you know is that, there is a notion of algebraic multiplicity; they are the algebraic multiplicity of alpha is they will add up to n, because they are the roots of the characteristic polynomial and the degree of the characteristic polynomial is n, alright.

So, from this I would like you to conclude, conclude that, that A is diagonalizable, diagonalizable; if and only if algebraic multiplicity of alpha i is same as geometric multiplicity of alpha i, for all alpha i belonging to the spectrum of A which you wrote it as sigma of A, alright.

So, for every eigen value I need that, for every eigen value I need that, algebraic multiplicity should be equal to geometric multiplicity. If there is even one eigen value for which algebraic multiplicity is not equal to the geometric multiplicity, I have a problem, so alright. So, let me write that also.

So, A is not diagonalizable, diagonalizable if we can find even a single eigenvalue value alpha of A or alpha belonging to sigma of A, such that algebraic multiplicity of alpha is not equal to the geometric multiplicity, is that. So, this is very important that, you can go the other way around and say that; since we do not have enough number of eigen vectors corresponding to the algebraic multiplicity, I do not have diagonalize with me, is that ok. So, that is very important.

So, keep track up that and understand it, fine. Now, the next thing that would like to look at what are called Schur Upper Triangularization, Schur Upper Triangularization, alright.

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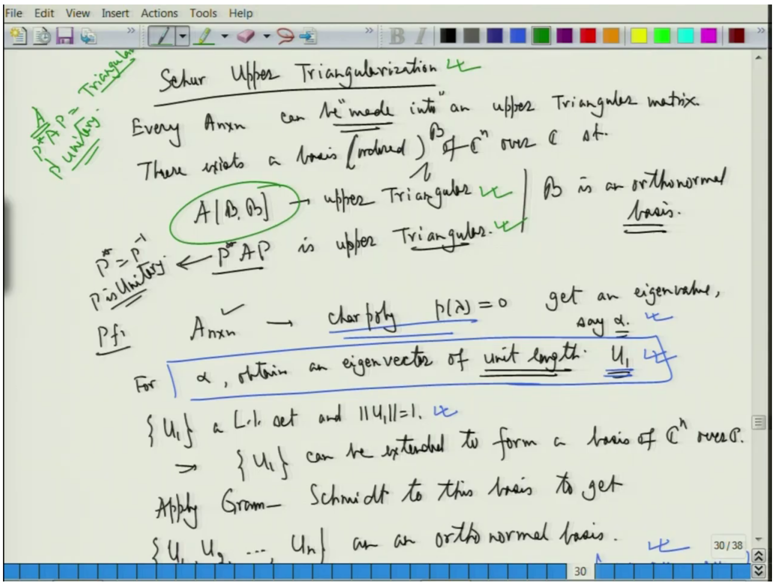

So, recall we have done A equal to l u, A equal to b c, the rank decomposition A equal to q r; we have also saw that, I can write A as s d s inverse for some diagonal matrix d, but then there was an problem in the sense that not all matrices can be made into a diagonal matrix it is, alright.

I cannot have a basis for every matrix where it is a diagonal matrix, fine. There lot of a estragon condition in the sense that, I need to find n linearly independent eigen vectors. So, it is a bit tough and we cannot do it in a general setup always. So, what you go to what are called Schur Upper Triangularization; it says that every matrix, every A square matrix can be made into an upper triangular matrix. So, made into means? So, there is this word made into.

So, what it is says is that, their exist a basis, ordered basis of C n over C, such that ordered basis C, ordered basis such that A with respect to B, B is upper triangular or in the language of matrix, it is P star A P is upper triangular.

So, in general it is not supposed to P star, it is supposed to be P star; in place of P star, so it should be P inverse. So, it should be here s inverse A s in general; but for us since I am looking at come alright, forget about this part I think. So, here P star is equal to P inverse, so that P is unitary, alright.

So, what we are saying is that, every matrix have this form upper triangular form, and here B is an orthonormal basis. So, B is an orthonormal basis, fine. So, let us try to prove it, proof requires similar idea as we did earlier.

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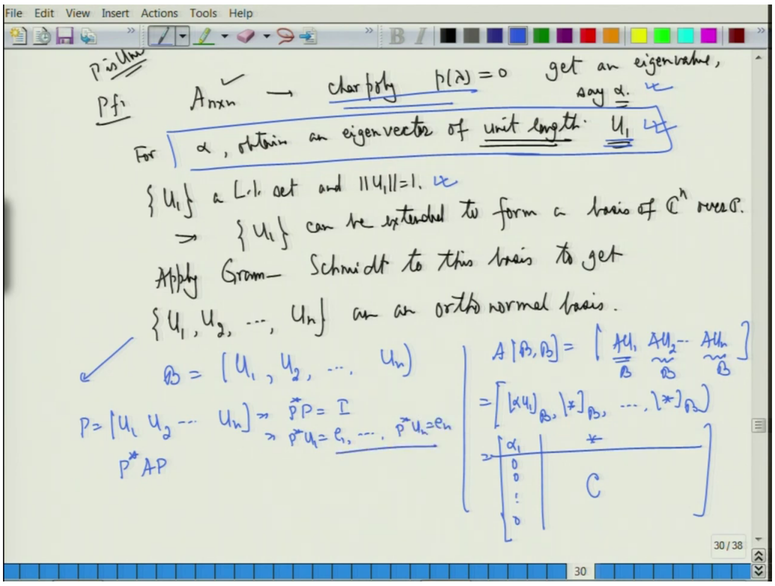

But let us try to go it; we did the idea, similar idea in what is called looking at algebraic multiplicity is greater than or equal to geometric multiplicity. So, let us proceed coolly and carefully.

I have matrix A, which is n cross n, fine. I do go to it is characteristic polynomial p lambda; substitute it to 0 will get an eigenvalue, get an eigenvalue say alpha, fine. Once you have got an eigenvalue for alpha, obtain an eigenvector, eigenvector of unit length. So, I can get a vector, eigenvector; now I can divide it by the length of it to get it a unit vector, alright. So, now, I have a vector u 1, which is of unit length, fine.

So, I have the set u 1 a linearly independent set and length of u 1 is 1, fine. So, this will imply that, this set. So, this can be extended to form a basis of C n over C, fine. Apply Gram Schmidt's to this basis to get u 1, u 2, u n as an orthonormal basis fine, is that ok? So, understand these things nicely. So, I had a characteristic.

So, I have the matrix A, from A I went to characteristic polynomial; from characteristic polynomial we got one eigen value alpha, for that alpha we got one eigen vector. Once I have that eigen vector, I can make it of length one and then I can extend it to form a basis of C n. And then I can apply Gram Schmidt's to get an orthonormal basis, alright.

So, orthonormal basis is not important now, but we will see it is use afterwards, fine. So, from here what we get is, I get a basis B which is u 1, u 2, u n; I get a basis like this. Now, what is the matrix of A with respect to this basis? So, again we will write it twice, so that there is a clarity again for you.

So, A of B B I want to compute, which will be equal to A of u 1, A of u 2, A of u n and this has to be computed with respect to B, this with respect to B, this with respect to B. So, this is same as A u 1 is here look at this alpha is an eigen value for u 1. So, I get that is alpha u 1 with respect to B; I do not know what is A u 2 is, so this with respect to B and so on, this with respect to B.

So, alpha u 1 is nothing, but alpha 1, 0, 0, 0, fine. I do not know the rest what the rest are. So, there will be something here and there will be a matrix C here that I get, fine. This is one way of understanding it; the other way will be to look at as I said, I can from here write P as u 1 u 2, u n, fine.

Let us look at P star A P. So, this implies that P star P is identity, fine. So, this will imply that p star of u 1 is e 1 and p star of u n is e n; I will not be using so many of them, but at least you should know what they are.

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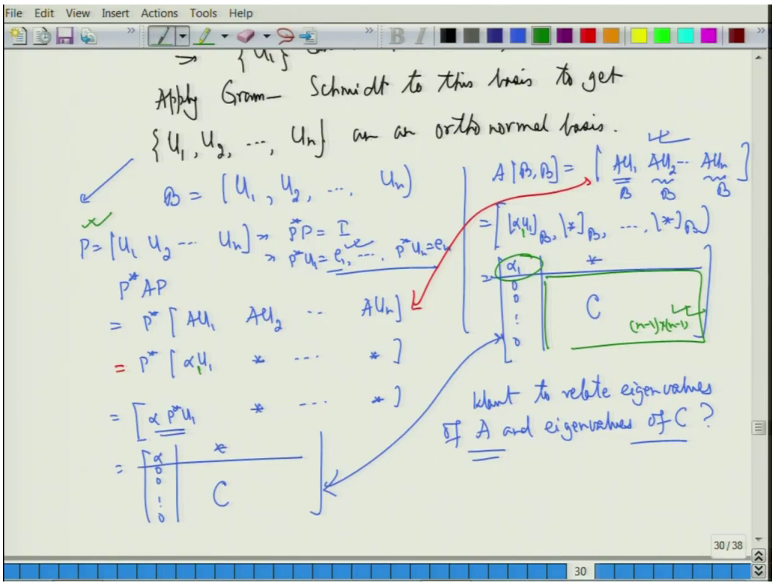

So, if I want to compute P star A P till the P star A of u 1, A of u 2, A of u n. So, you can see that I am doing the same thing as B, B here; I am not doing anything special, it is same thing doing here fine, which is same as I can put P star inside. So, blue ink. So, this is P star of A u 1 is alpha u 1; I do not know what it is, I do not know what it is, fine.

So, this is same as alpha times p star of u 1; again something herewe do not know what they are. But what I know here now is p star of u 1 is e 1, alright. Therefore, I get it as alpha 0, 0, 0 and I do not worry what they are, is that ok? So, whatever way you want to understand, understand it; but understand it, alright. Spend some time that they have the same thing in different languages; one is it with respect to the ordered basis, the other is matrix multiplication, alright fine.

So, once I have got this, now I want to understand the eigenvalues of A and eigenvalues of B; want to relate eigenvalues of A and eigenvalues of C, I want to relate the two, fine. So, what is the relationship, fine?

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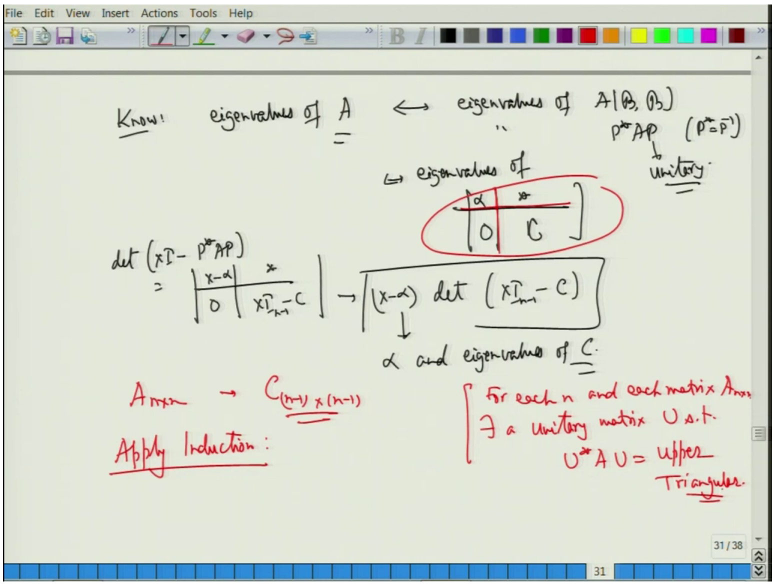

So, what we know? Know eigenvalues of A they are same as eigenvalues of this or same as eigenvalues of P star A P; because P star is same as P inverse alright, because P was unitary, alright.

So, therefore, I needed that part. So, unitary P was unitary, fine. So, the eigenvalues are same; so it means that eigenvalues of A are same as eigenvalues of this, eigenvalues of A B, B is same as eigenvalues of, this matrix is a alpha here, a star here, 0 here and C here, alright fine. So, I can see here that since there is 0 here; so if I want to compute determinant of X I minus P star A P, it will be nothing, but determinant of x minus alpha something here, 0 here, and X I n minus 1 minus C here, is that ok.

It will be determinant of this. And therefore, by expanding along the first column of this determinant, I get it is x minus alpha times determinant of this, alright. So, the eigenvalues of A are same as eigenvalues alpha. So, eigenvalues A are same as alpha and eigenvalues of C, is that ok? So, they are same and therefore, we can proceed further and then try to understand the next idea, fine.

So, we have got hold of eigenvalues of A and we have got this also, alright. And our matrix is this, this matrix we need to understand now. So, what we do; we do not worry about this part, we only bothered about C. We, so what we have done is from A which was n cross n; now I need to look at a matrix which is n minus 1 cross n minus 1, fine. So, in some sense I can apply induction.

So, I have gone from n to n minus 1; so you can apply induction, apply induction. So, how do I apply induction? So, let us understand what is the induction, mathematical induction; mathematical induction says that, I have a some statement that I want to make and it depends on natural number n, fine.

So, here my definition is or my statement is, my property is that, every n cross n matrix can be written in terms of or can be made into an upper triangular matrix alright that is my statement;

that the given n, every n cross n matrix alright has a basis or can be written in terms of a basis, such that it looks like a upper triangular matrix.

So, a statement let us write it, for each n and each matrix A, A which is n cross n; there exists an, there exist a unitary matrix, there exists a unitary matrix U, such that U star A U is upper triangular. So, that is my statement that I am making about positivity integer m.

So, I want you to look at the first step of induction, which is looking at n equal to 1. So, if I look at n equal to 1; then I know that everything is nice, because it is the 1 cross 1 matrix which is already an upper triangular matrix, alright. So, for 1 cross 1 we have already the result that, it is the 1 cross 1 matrix which is upper triangular; hence I do not do anything. The next step of the induction is that, assume the result to be true for n minus 1 and lowered size matrices, alright.

So, assume, next step is to assume that is true for the matrix of a smaller size and you want it to prove it for n. So, now, so let us do that. So, from A, I am going to n minus 1.

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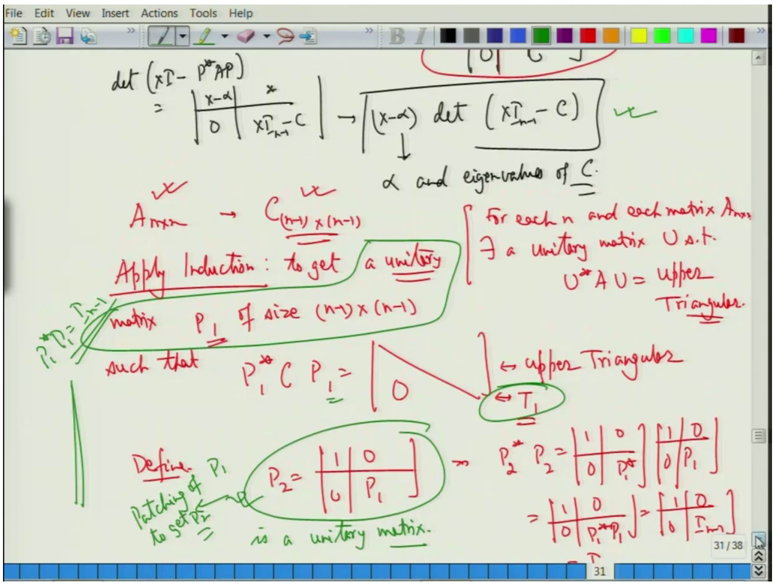

So, apply induction to get a unitary matrix, matrix u 1 of size n minus 1 cross n minus 1, such that u 1 star C u 1 is upper triangular alright, upper triangular. Let me write this matrix as T 1 fine, is that ok?

So, by induction I know the result is true for C, which is an n minus 1 cross n minus 1 matrix. So, for this matrix C, I can get a unitary matrix u 1; did what did I write, just a minute. So, there is u 1 here also, there is u 1 here also that will confuse. So, I need to write not u 1, but say some P 1. So, I have a P here, P 1, P 1, P 1 here, alright. So, I have a matrix P 1 which is a unitary matrix. So, that P 1 star C P 1 is 0, alright. So, now, from here I want to go to the matrix A, fine.

So, define P 2 is equal to 1, 0, 0, P 1, fine. So, this will imply that will let us look at P 2 star P 2; this will be equal to 1, 0, 0, P 1 star times 1, 0, 0, P 1, which will be equal to 1, 0, 0, P 1 star P 1, which is nothing, but 1, 0, 0, I n minus 1 which is I n, alright. Because we took that P 1 was a unitary matrix of size n minus 1 cross n minus 1; therefore, P 1 star P 1 is I n minus 1, alright fine.

So, therefore, what we see is that, P 2 is a unitary matrix. So, starting with P 1, I got P 2 j unitary matrix; we also took initially this matrix has unitary, this P as unitary, fine. So, now, let us multiply these two matrices; what do I get?

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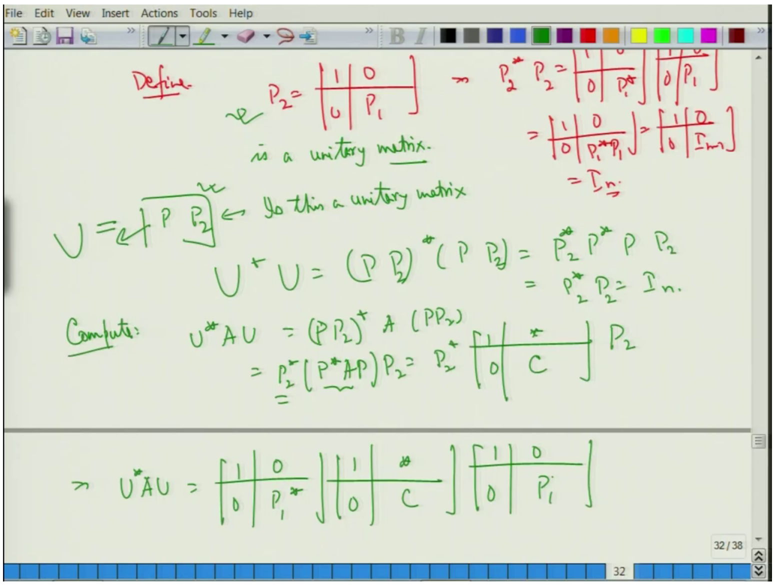

So, I want to look at what is P times P 2; is this a unitary matrix? So, let me write this as say some capital U. So, capital U star capital U is something, but P 2 of P star into P 2 of P, which is nothing but P time P star P 2 star P 2 P, which is same as P star P which is identity, alright fine.

So, therefore, if I take U as this, then I get everything nice. So, let us compute U star A U, fine. So, U star A U is nothing, but U is P 2 P. So, P will come outside so it is fine; look at it P 2 P for me. So, P 2 of P or should I written the other way around I think; I should have written the other way around, P here and P 2 here, because I wrote I think P 1 here. What did I write there? I wrote P on the right yeah.

So, I should have written P 2 P here. So, therefore, this should be changed, P 2 here, P 2 here. So, P 2 star P and P 2 will come here. So, P 2 star P 2 and this I will get, alright. So, I should have written here P times P 2 star A P, P 2, which is same as P 2 star P star A P P 2, which is same as P 2 star. Now, what is P star A P 1? P star A P 1 was 1, star here, 0 here, C here and this is your P 2.

So, this implies U star A U. What is P 2 star? Look at P 2 star here; this is your P 2, P 2 star is 1, 0, 0, P 1 star times 1, 0, star, C and then P 2 again. So, 1, 0, 0, P 1, alright. So, let us multiply it out.

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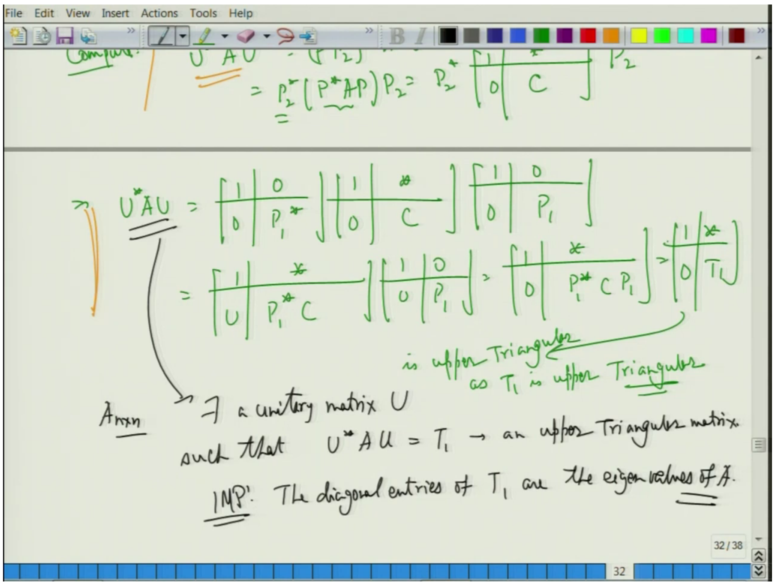

So, this will be equal to this into this is 1, just look at nicely this into this is 1; this into this is a star I do not worry about it, this into this is 0, this into this is P 1 star C 1, 0, 0, P 1.

Which is same as 1, here some star, 0 here, P 1 star C P 1, which is nothing but 1, star, 0 I wrote it as some T, I think T 1; I wrote it as T 1 upper triangular T 1. So, it is T 1, alright fine. So, what we see here is that, this matrix is upper triangular as T 1 is upper triangular, fine. So, let us look at nicely what we had done, this is a very important theorem.

So, let me get into it nicely. First thing we did we wanted to put the Schur Upper Triangularization that every matrix. So, given a matrix A; they exist a unitary matrix, such that P star A P is triangular, you can make it lower triangular also, but we have looked at upper triangular here and P is unitary, P unitary, fine.

So, that I do not have to compute P inverse for me; otherwise everything is nice, no problem as such. Or what we are saying is that A of B, B is upper triangular or whatever you want to say, this what we wanted.

So, we looked at the eigenvalues of the matrix, we picked up the first eigen value that I could compute or I could find out; used it to get a matrix which was nice in the sense that I could differentiate alpha 1 here, but there was alpha 1 that I need right here, alpha 1 here and maybe somewhere else alpha 1, maybe alpha 1 here and things like that, fine. So, I have this here with me. So, I got the first eigen value alpha 1 and using that orthonormality of u 1 to u n or the ordered basis I could get a matrix of this size, alright.

So, from n I could go to an n minus 1 by n minus 1 matrix C; apply induction to it and get my results, alright. So, the important thing was that, I could get a metric C which is n minus 1 cross n minus 1, I could get it and the rest of the part if I look at everything was 0 on the first column other than alpha 1, fine. So, I was in the process of making it a upper triangular matrix, fine. Now, that was the first thing we observed. Second thing we observed was that, eigenvalues of A are same eigenvalues of C, fine.

So, therefore, when I get the final answer alright; the upper triangular matrix, the diagonal entries will be nothing, but the eigenvalues of A itself, alright. So, that is important; the final matrix that I have, it will have on the diagonal only eigenvalues of A, it will not be anything different, fine. So, once I have understood that the eigenvalues are going to be the same, I apply the induction.

So, I applied the induction here, this induction I had also applied; if you remember the similar idea was there, when we looked at what is called LU decomposition, alright. There also we have done the same thing that, we went into a matrix which of the type this block matrix; we defined block matrix 1, 0, 0, L, L for the lower triangular we had done and then we had done all the work, alright.

So, similarly, so the idea of this what is called a patching. So, you do for something for a smaller matrix and then you patch up with zeros and ones, so that you get some nice matrix. So, you have patched up. So, we have done what is called patching of P 1 to get P 2, which is the required property that we are looking at, fine. So, that was important for us that we wanted it.

Once you have done the proper patching, so that p 1 was unitary, so P 2 is also unitary; I applied to it, alright. So, from there we looked at this part; you defined U as P P 2, so that P is unitary, P 2 is unitary, so their product is unitary. We verified this also and then you just applied the matrix, whatever it is matrix multiplication; we just looked at matrix multiplication and get our results, alright.

You could have gone through the basis also and done some work; but that will require much more work, because I have to relate different basis, first I have the basis B, then another basis, another basis, alright.

So, in place of that, I followed the idea of matrix multiplication here. So, that the clarity is there just you multiply it; you can see that everything is coming nicely, alright. So, what we have shown is that, given any matrix A which is n cross n; I have a unitary matrix, there exists a unitary matrix U, such that U star A U is T 1 an upper triangular matrix, alright.

Important, very very important; the diagonal entries of T 1 are the eigenvalues of A, alright fine. So, we look at lot of application of this in the next class; what is called will also prove what Cayley Hamilton theorem using this idea, will use this.

We will also be able to say that, all symmetric matrices are diagonalizable; all Schur Hermitian matrices are diagonalizable, and all of them they come from a set what is called an set of normal matrices. So, normal matrices are diagonalizable, we will prove it; we will define what is called a normal matrix and prove all those things, alright. So, that is all for now.

Thank you.