Linear Algebra
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Lecture – 57
Diagonalizability Continued…

So, we are defined diagonalizity in the last class, we will gave examples to show that some metrics are not diagonalizable. We also had this theorem, they are in some sense we did not write it that A is diagonalizable if and only if A has n linearly independent eigenvectors.

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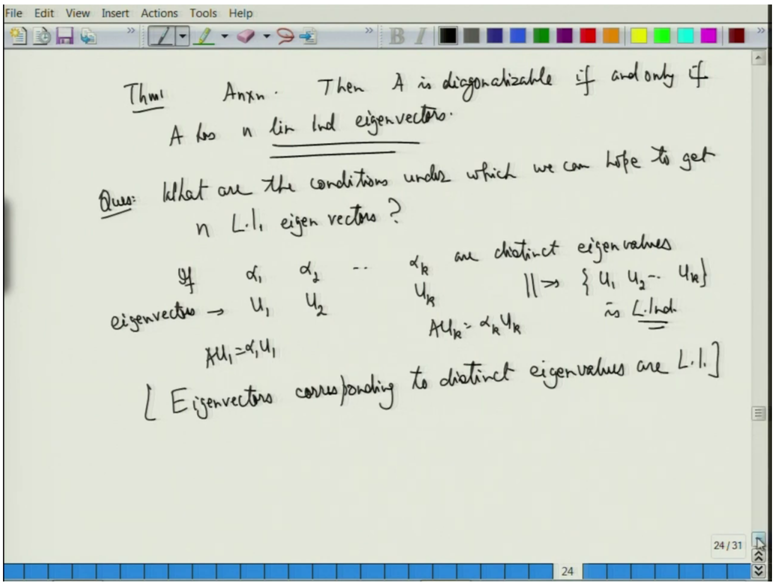

I did not state it so; let me write this as a statement theorem. A n cross n, then A is diagonalizable lizable if and only if A has n linearly independent eigenvectors. So, this we had proven using writing the matrix p as eigenvectors u 1 u 2 u n and so on. alright.

So, I will just stop that part that part I hope you have understood think about it again now. So, the next thing that we would like to do is that, when can I find linearly independent eigenvectors. So, the question what are the conditions, the conditions under which we can hope to get n linearly independent eigenvectors this is what we need alright.

So, that come we will get to that after a bit after some time. What we want to say is that we have seen till now looked at examples, where the diagonalizability has failed, because of the eigenvalue being the same in some sense fine. So, would what would like to say is that if I pick up eigenvalues which are distinct look at the corresponding eigenvectors can I say that they are linearly independent. So, the next thing that would like to say that if so, if alpha 1, alpha 2 alpha k are distinct eigenvalues fine.

And you have corresponding eigenvectors u 1, u 2, u n, u k can I say so, these are eigenvectors fine. So, what we know is that alpha 1 u 1 is equal to a u 1; similarly A u k is equal to alpha k u k. So, these are eigenvectors alpha is are distinct can I say that this will imply that this set u 1, u 2 u k is linearly independent alright.

So, what we are saying is that the eigenvectors corresponding to distinct eigenvalues are linearly independent. So, we are saying eigenvectors corresponding to distinct eigenvalues are linearly independent that is what we are claiming here like. So, let us try to prove it.

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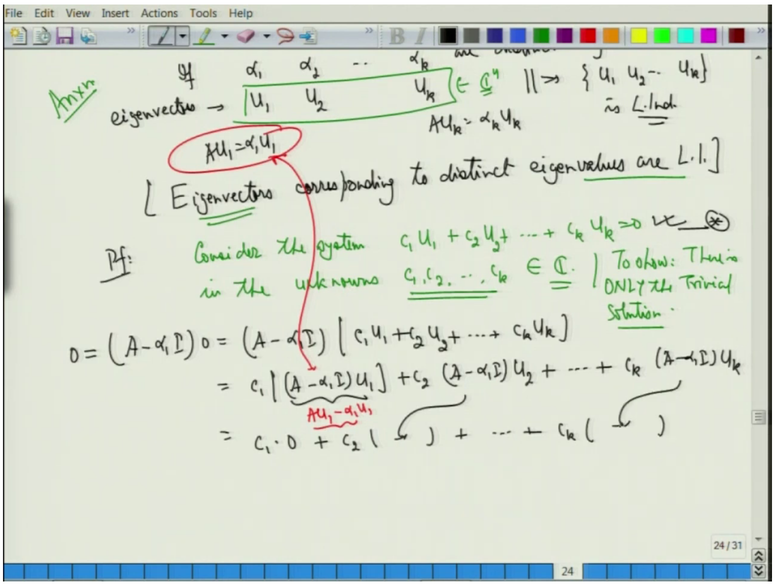

So, proof fine. So, if I want to prove that this is linearly independent, I am supposed to form a system of equation let us form a system of equation. So, consider the system c 1, u 1 plus c 2, u 2 plus c k, u k alright is equal to 0 in the unknowns c 1, c 2 c k belonging to c complex numbers. Because u is are complex vectors alright and I am assuming that A is n cross n and these are so, u is are in c n fine so, that it makes sense.

So, I am solving the systems c 1, u 1 plus c 2 u 2 where the c i's are unknowns. I want to find them or since I want to prove that these are linearly independent the eigenvectors are linearly independent alright, I have to show that the only solution here is the trivial solution. To show there is only the trivial solution alright. So, what we can do is that? We can multiply this matrix this vector equation by certain matrices alright. So, let us try to multiply this by something alright.

So, I want to multiply this by say A minus alpha 1 I, I want to multiply this a star by this. If I do that what do I get on the left hand side, I will get this time 0 is 0 fine. The other side I will get it as A minus alpha 1 I times c 1, u 1 plus c 2, u 2 plus c k, u k. Now, what it gives me is just put inside c 1 times A minus alpha 1 I u 1 plus c 2 times A minus alpha 1 I u 2 plus c k A minus alpha 1 I u k fine this is what I get.

Now, this is equal to now what is u 1, u 1 is an eigenvector corresponding to alpha 1 or we already wrote here that A u 1 is equal to alpha 1 u 1. So, if I use this idea here what do I get? I get A u 1 if you rewrite it is A u 1 minus alpha 1 u 1. So, therefore, this part tells me that this is a 0 vectors. So, what I get is here? c 1 times 0 plus c 2 times something that I am not bothered about for the time being that will write afterwards plus c k times something here is that ok.

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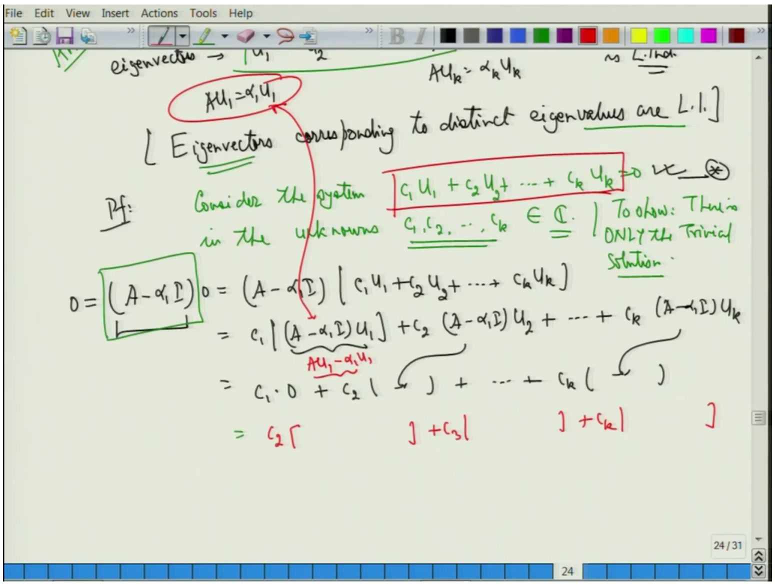

So, by multiplying by this here alright so, by multiplying by this A minus alpha 1 I, I have been able to get rid of c 1 is that ok. So, c 1 has lost. So, what I got now is that? Earlier I had equation which was in c 1, c 2, c k. Now, I have equation only in c 2 times something plus c 3 times something plus c k times something. So, the number of variables or unknowns has reduced fine.

This is more important you are trying to again go back to your Gauss elimination method, where at each a step we have tried to reduce the number of unknowns. So, we removed c 1. Now I want to remove c 2 fine.

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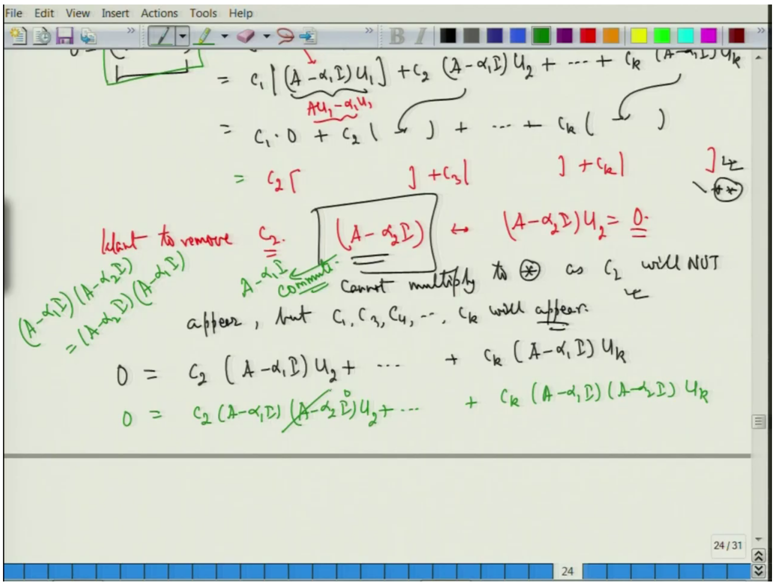

If I want to remove c 2. What we need to do? I need to so, if I want to remove want to remove c 2 I need to multiply A minus alpha 2 I two the first equation or to this equation alright, because what I know is that you have multiply this.

Then, A minus alpha 2 I of u 2 will give me 0 fine. So, I need to multiply by this I need to multiply by this, but I cannot multiply to the first one the star cannot multiply cannot multiply to a star as c 2 will not appear not appear, but c 1, c 3, c 4 till c k will appear alright so, will appear. So, what happens is that? I can kill c 2, but I cannot kill c 1 alright. So, that is the problem.

So, what I would like to understand is can I do this itself multiply this to the equation that I have got here which is say double star for me alright. So, let us write the double star again. So, double star is nothing, but 0 is equal to c 2 times A minus alpha 1 I times u 2 plus so on plus c k times A minus alpha 1 I u k this is what I had fine. Let us multiply A minus alpha 2 I 2 this. If I do this alright, if I do this and also you observe we are there this is very very important observation that this matrix times this matrix is same as A minus alpha 2 I times A minus alpha 1 I.

That is these two matrices commute. So, this and A minus alpha 1 I they commute. Why do they commute? Because, I am just multiplying by a itself and by identity, identity commutes with everything the scalar matrix alpha 1 I or alpha 2 I they are a scalar matrices, they commute alright. So, I would like you to check that they commute here. And therefore, I can just write this as 0 times c 2 of A minus alpha 1 I times A minus alpha 2 I u 2 and so on, I can go on plus c k times A minus alpha 1 I into A minus alpha 2 I u k.

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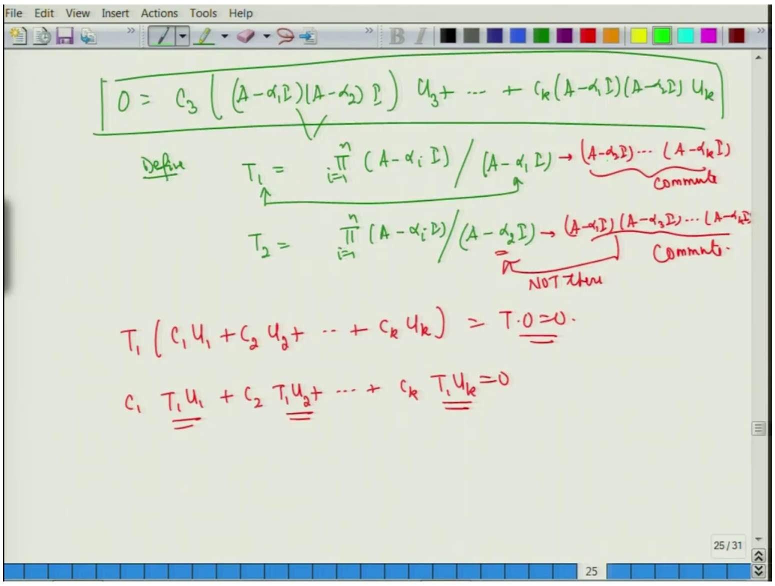

If I do this what will happen is? This has become 0 now I will be left out with 0 is equal to c 3 times A minus alpha 1 I A minus alpha 2 I fine times u 3 plus so on plus c k times A minus alpha 1 I into A minus alpha 2 I times u k fine. So, this is what I will be left out with this is what you have to understand fine. So, in general what I am doing let us rewrite this in a nice form so, that you understand it fine. So, I would like to write these matrices.

So, I want to write some matrix. So, I define T 1 is equal to product A minus alpha i of identity I is equal to 1 to n divided by A minus alpha 1 of I. So, this one comes with this one for me is that ok. So, I am defining my T 1 like this. Similarly I can define T 2 as product I is equal to 1 to n a minus alpha i of I and then divide by A minus alpha 2 of I fine. So, if I look at this what I am trying to say is that this corresponds to looking at A minus alpha 2 I so on till A minus alpha k of I.

This corresponds to looking at A minus alpha 1 of I, then A minus alpha 3 of I alpha 2 of I is lost here not there. And then I am going all the way to A minus alpha k of I fine. And what is know is that these matrices commute, they commute here fine. Because, I am just multiplying a with identity so, they commute. So, if I multiply so, if I multiply T 1 to c 1 u 1 plus c 2 u 2 plus c k u k. What I am doing is? I am multiplying this so, this was T times 0 which is 0.

So, what I am doing here is that I am looking at c 1 times T of u 1 T 1 of u 1. So, let me write c 1 into T 1 u 1 plus c 2 times T 1 u 2 plus c k times T 1 u k is 0. Now, what is T 1 u 1, what is T 1 u 2 and T 1 u k? Fine. So, understand this very nicely this very very important idea.

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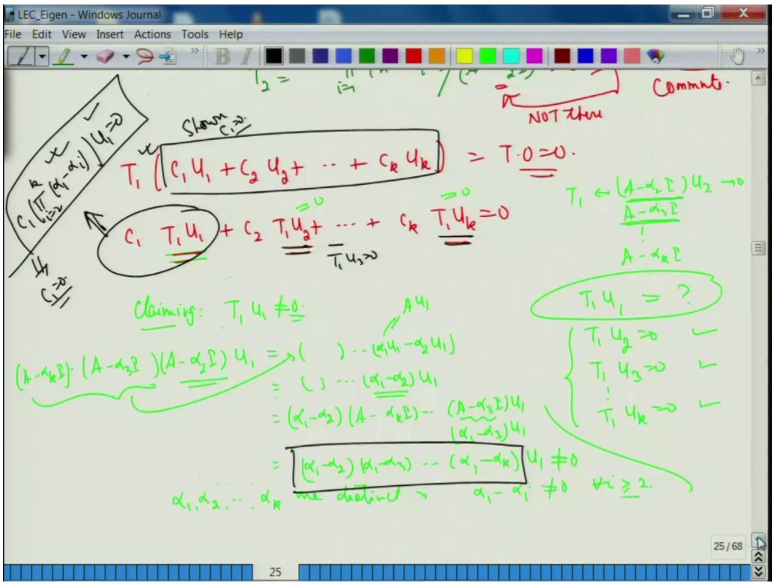

So, if I want to look at T 1 T 1 has A minus alpha 2 I, it has A minus alpha 3 I, it has a minus alpha k I fine. They commute so, at some a stage since they commute at some a stage, if I multiply this with u 2 this will give me 0 alright again when I multiply T 1 with u 3. So, T 1 with u 1 T 1 with u 1 I do not know what that will be, but what I know is that T 1 with u 2 will be 0, because there will be somewhere where A minus alpha 2 of I will come. Similarly T 1 of u 3 will be 0, basically because A minus alpha 3 is there. So, what will happen is that all of them T 1 of u k will be 0 alright.

So, all of them there will be 0 I do not know whether this will be 0 or not. So, we are claiming now you are claiming that T 1 of u 1 is not 0 that is more important that we are saying that its not 0, we have already shown that T 1 u 2 is 0 u 3 is 0 u k is 0. So, what you are saying is that this is already 0 T 1 u 2 this is already 0. The only problem is that whether this is 0 or not so, let us look at that part. So, what our T 1 is T 1 had these terms.

So, the first term was A minus alpha 2 I, then there was A minus alpha 3 I and so on. A minus alpha k of I am supposed to multiply this with u 1, if I do this then what I get is that this part if I adequate I just leave it as it is for the time being. Now, this part gives me A u 1 A u 1 is nothing, but alpha 1 u 1 minus alpha 2

So, it will alpha 1 u 1 minus alpha 2 u 2, this is what I will get just multiply it A u 1 is alpha 1. So, this is nothing, but A u 1 is alpha 1 u 1 and then alpha 2 u 2. So, I can write it back as things like this and this is alpha 1 minus alpha 2 of u 1 again. This was a u 1 here alpha 2 of u 1 alright fine.

So, again this is a constant so, I can just put alpha 1 minus alpha 2 outside here. So, then I will get here A minus alpha k of I so on till A minus alpha 3 of I times u 1 again this will give me just multiply it out you will get here, alpha 1 minus alpha 3 times u 1. So, at each a stage what I am getting is that, we will just get its effect do the whole calculation what I will get is that I will get alpha 1 minus alpha 2 alpha 1 minus alpha 3. So, on till alpha k minus alpha 1 minus alpha k of u 1 alright fine.

And we are taken that the alpha 1 alpha 2 alpha k are distinct. And this implies that alpha 1 minus alpha I is not equal to 0 for all I greater than equal to 2. And therefore, this vector is not 0 is that fine. So, that is what is important that all these vectors T 1 u 2 is 0 T of T 1 u 3 is

0 and so, on till all of them are 0, only vector that is left out is this part, I know that T 1 of u 1 is not equal to 0.

So, what I get from here is that c 1 times some constant which is this number which is product of alpha 1 minus alpha i i is equal to 2 to k times u 1 is 0 fine u 1 is the eigenvectors so, that is not 0 this is not 0. Because, alpha is are distinct so, it implies that c 1 has to be 0 alright.

So, we have shown that now if I look at this part we have shown that in this part, that in this linear combination if I look at this linear combination c 1 is 0 you have shown. So, shown c 1 is 0 by applying T 1 alright. I can apply T 2 again to the whole thing to get c 2 is 0 and so on.

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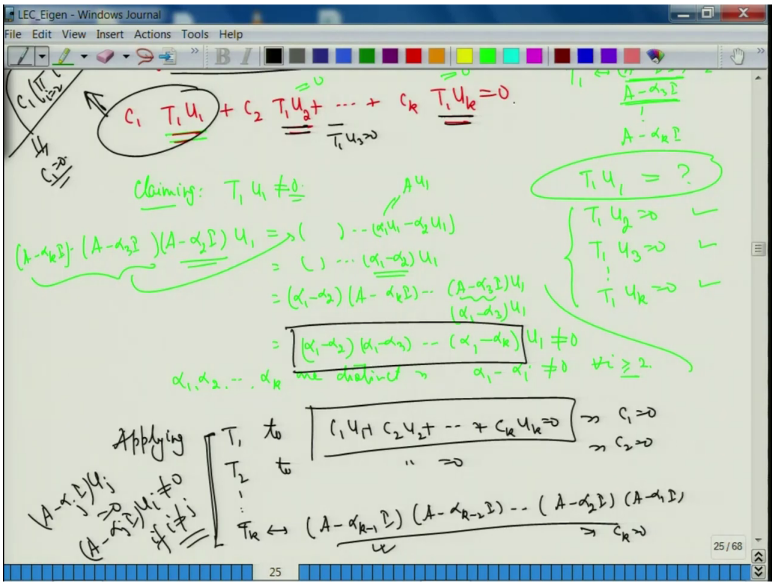

So, let me write that now applying T 1 to c 1 u 1 plus c 2 u 2 plus c k u k is equal to 0 implied c 1 is 0. If we apply T 2 to the same thing I will get c 2 is 0 and so on. So, apply finally, T k and what was T k if you remember T k is nothing, but it is supposed to be A minus alpha k minus 1 I A minus alpha k minus 2 I so, on till A minus alpha 2 I into A minus alpha 1 I, if I apply this matrix to this equation I will get that c k is 0.

So, the idea is that each of them each of these matrices they play some role as per as we are concerned is that ok. So, A minus alpha k of I applied to u k or not u k alright u j let me write u j here, u j u j this is 0 fine. And if I am looking at A minus alpha j I of u I this is not 0. If I is not equal to 0 alright that is very important, because of that we are able to get this is that ok, generalization of this idea.

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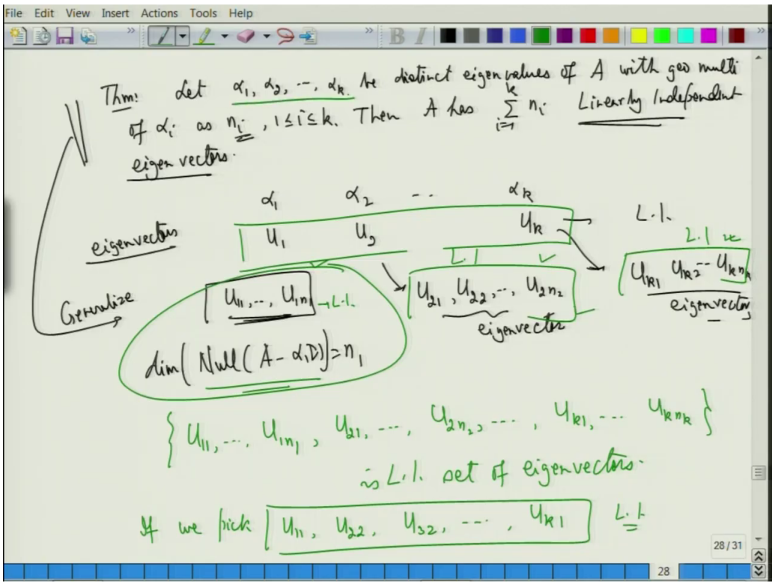

Let 0 let alpha 1, alpha 2 alpha k be distinct eigenvalues of A, with I do not know about algebraic multiplied with geometric multiplicity of alpha i as n sub i 1 less than equal to i less than equal to k, then A has summation i is equal to 1 to k n i linearly independent eigenvectors is that ok.

So, what exactly we have shown in the previous theorem was that if I have got alpha 1, alpha 2, alpha k I had u 1 here, u 2 here u k eigenvectors u 1 eigenvector. So, I took any eigenvector u 1, u 2, u k of alpha 1 to alpha k, then they are linearly independent. Now, what we are doing in this theorem is we are trying to generalize. So, what you are saying is that if I pick here eigenvectors as u 1 1 till u 1 n 1 because they are n 1 eigenvectors here.

So, n 1 dimension of A minus alpha 1 I null space of A minus alpha 1 I is equal to n 1 the sort you out assumption is so, I will get. So, many linearly independent eigenvectors here, I will get u 2 1 u 2 2 u 2 n 2 eigenvectors fine. And here I will get u k n 1, u k 1, u k 2 till u k n k eigenvectors. So, what we know is that know since dimension is this implies that these are linearly independent here the dimension is n 2. So, these are linearly independent similarly these are linearly independent alright fine.

What we are claiming here is that all of them. So, what we are claiming here that if I look at the set u 1 1 till u 1 n 1 u 2 1 till u 2 n 2 so on till u k 1 till u k n k, then this is linearly independent set of eigenvectors, this what we are claiming. And the idea of the proof is what I know is that if I just pick any one from the first set, any vector from the second set, any vector from the k th set then that is linearly independent because of the previous theorem alright.

So, they are linearly independent because of this part that, if I have any collection of eigenvectors corresponding to distinct eigenvalues, they are linearly independent. So, if I pick u 1 1 u 2 2 u k 2. So, if i pick. So, if we if we pick u 1 1 u 2 2 u 3 2 so, on till u k 1 you have picked this collection then, this is linearly independent because of they correspond to distinct eigenvalues alright.

And hence all of them there are going to give me some eigenvector or the other this what is given to me, they are eigenvectors. And since they are coming from distinct eigenvalues, they are linearly independent. And among themselves they are linearly independent, because I chose them to be linearly independent, because they are n 1 of them here n 2 here and n k here is that ok.

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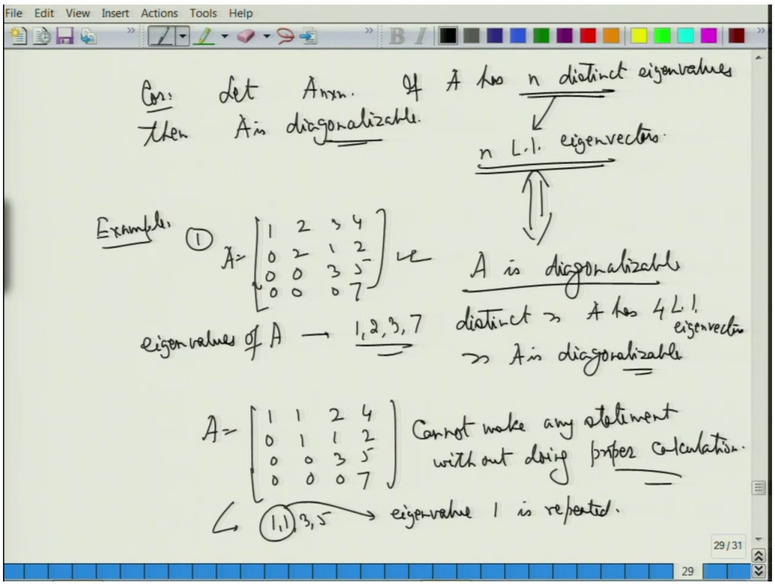

So, you need to understand that if I have distinct eigenvalues I can do things alright. So, as a corollary of this as a corollary let A be n cross n if A has n distinct eigenvalues, then A is diagonalizable, we are saying that it has n distinct eigenvalues. So, n distinct eigenvalues means there will be n linearly independent eigenvectors and what we saw was that whenever A is diagonalizable, if and only if it has n linearly independent eigenvectors alright.

So, this was same thing as or equivalent to saying that A was A is diagonalizable fine. So, what we are saying here is so, as a example or as an application of this idea, example look at this matrix A which is 1 2 3 4, 0 2 1 2, 0 0 3 5, 0 0 0 7 alright. So, if I look at the eigenvalues of A they are going to be 1 2 3 and 7. Because, this is an upper triangular matrix with diagonal entries 1 2 3 7 these are distinct implies A has 4 linearly independent eigenvectors, implies A is diagonalizable.

So, this is the (Refer Time: 24:34) that we have that as soon as we know that something is upper triangular, it has distinct entries on the diagonal, then that matrix is diagonalizable alright. So, here you I know it is diagonalizable, but if I want to look at this matrix 1 1 2 4, 0 1 1 2, 0 0 3 5, 0 0 0 7 cannot make any statement cannot make any statement, without doing proper calculation alright.

So, I will have to calculate thing here because, I do not have any theorem with me which will say that this is diagonalizable, this may be diagonalizable this may not be diagonalizable, but you have to compute basically because here the eigenvalues are 1 1 3 5. So, there is an eigenvalue which is repeated eigenvalue A 1 is repeated alright. So, whenever we have that something is not repeated everything is nice otherwise there is a problem alright. So, that is all for now.

Thank you.