Linear Algebra
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Lecture – 56
Diagonalizability

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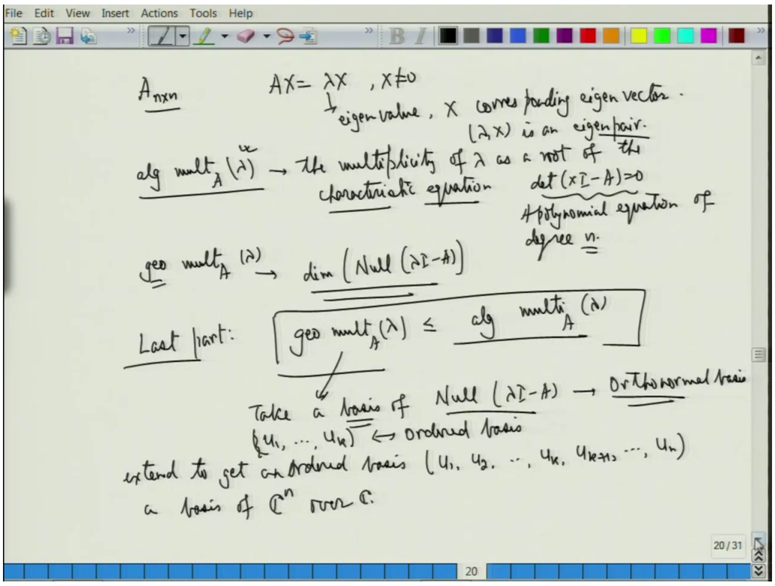

So, let us recall what we have done in the previous class. So, the ideas were that you have a A which is an n cross n matrix alright. Then you had a X which has nonzero such that A X is equal to lambda X, X not zero, so that gave me eigen values eigenvectors.

So, lambda is Eigen value and X corresponding eigenvector fine. And we have said that lambda comma X is an eigen pair. Then there is a notion of what is called algebraic multiplicity of eigen value of lambda as an eigen value of A. So, this was the multiplicity of root city of lambda as a root of the characteristic equation.

So, the equation if we remember you it was nothing but determinant of you can write X I minus A is equal to 0, a polynomial equation, polynomial equation of degree n alright. So, it has n roots over complex numbers, and we are looking at only everything about complex numbers fine.

So, when assuming that lambda is an eigen value and then I am looking at an algebraic multiplicity. If lambda is not an eigen value, its algebraic multiplicity will be 0, it will not be a root of the characteristic polynomial fine. So, now we had a notion of what is called a geometric multiplicity of lambda as an eigen value of A. So, geometric means some space has to come.

So, therefore, this one nothing but a dimension of null space of lambda I minus A, there was a dimension alright. And what we showed at the last thing that we showed you yesterday was last part was that geometric multiplicity of lambda as an eigen value is always less than equal to the algebraic multiplicity of lambda fine. This is what we showed that was, the last thing we showed.

And the idea was very simple which you have been doing again and again in some sense that we take a basis of this, take a basis of null space of lambda I minus a fine once you have taken a basis. So, let the dimension be K. So, basis and then we make it orthonormal basis, there is no need for this orthonormal basis. We do not need it at this stage, but in general sometimes we will need it fine.

Once you have taken a basis of this, extend it to form a basis. So, take a basis u 1 to u k of this. You can make it ordered basis. So, there is a bracket that comes into play here that we have ordered basis fine. Extend it extend to get an ordered basis u 1, u 2, u k u k plus 1 to u n a basis of C n over C fine.

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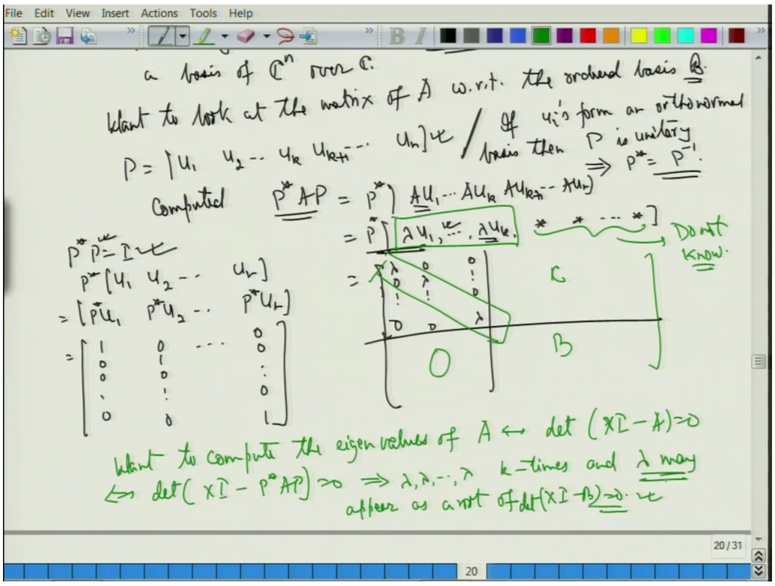

Once you have done that, then we looked at. So, we take this as it is. This is our ordered basis B. We want to look at the matrix, want to look at, we want to look at the matrix of A with respect to the ordered basis B. So, there are two ways of writing it understanding it. One was that we did yesterday was you looked at we defined P as u 1, u 2, u k, u k plus 1 till u n. And we computed P star A P we computed. And we took orthonormal basically because you wanted to compute P star A P.

Because if B is an orthonormal basis, then the matrix P, so if u i's are u i's form an orthonormal basis normal basis, then the matrix P that I have defined here P is unitary. And P is unitary implies P star is same as P inverse alright, fine. Is that ok? So, we have this idea that P star same as P transpose not inverse, it is not only that unitary. So, P star is P inverse.

So, let it be alright fine. And then from there we wrote it as P star, we put A inside. So, we got A of u 1 till A of u k A of u k plus 1 till a of u n.

This was nothing but P star of A of u 1 was lambda 1 u 1 or you are fixing lambda here. So, let it be lambda itself lambda u 1 till lambda u k, and I do not know what we have here, they are something that you do not know.

So, when I put in P star here again, so what you are saying is P star is P inverse; you just have to multiply by it fine. So, just look at P star. So, P star into P is identity. So, P star times P, P is your u 1 u 2 u n will give me P star of u 1 P star of u 2 P star of u n.

And since this is identity fine, I sub n what I get is this is the first, so 1 0 0, this will be 0 1 0 0 0, and this will give me 0 0 0 1 alright fine. So, this is what I will get. So, what you see here that you got A P star u 1 here, so I will get here lambda 0 0 0. Again it will be here which will be P star u 2; therefore, again I will get 0 lambda till 0 fine. This will give me 0 0 0 k, sorry 0 0 0 lambda fine.

So, this part is taken care of. So, this part is fine. There is no problem, because everything here is 0 for us. Now, this part, I do not know what it is do not know, I do not have any information about this part. So, I cannot do anything, I can just leave it as it is some say C here and B here. This is what we do fine.

And therefore, when I want to compute, so want to compute, you want to compute the eigen values of A, we near to look at determinant of X I minus A equal to 0 which was same thing as saying that we saw that the determinant of P star or the same as X I minus P star A P equal to 0, which will again imply that these lambdas will give you lambda, lambda k times and lambda may appear I am not sure what will happen about. But lambda may appear as a root of X I minus B determinant of this is equal to 0 alright fine. So, I do not know about this part. So, lambda can be more. So, algebraic multiplicity can be more alright.

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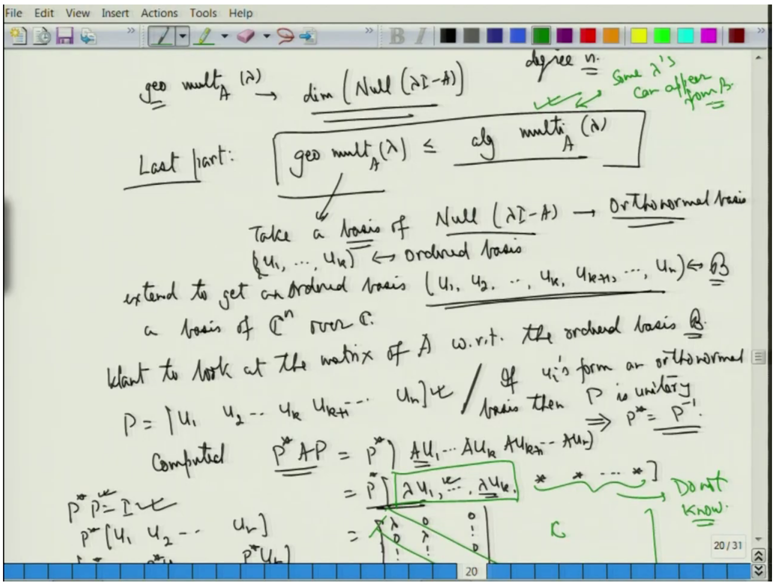

So, this is what we wrote here that algebraic multiplicity can be more, because this can come from some lambdas can appear from B alright. So, this is what we had. And this is what happens in most of the examples when we look at what are called nilpotent matrices.

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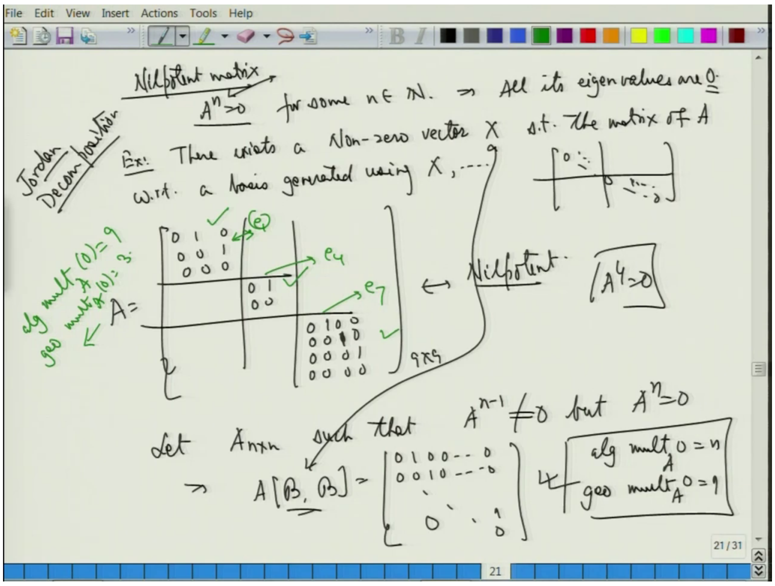

If I look at the nilpotent matrices, nilpotent matrix alright, what we have in nilpotent matrix means A to the power n is 0 for some n belong to the natural number alright. And therefore, so this implies all its eigen values, eigen values are 0 fine.

So, I would like you to show as an exercise show that there exists a nonzero vector X such that such that the matrix of a with respect to a basis generated using X and things like that there is a question mark that I need to put because I have not made my statement completely correct alright. It looks like 0s here on the diagonal and off diagonal, there will be some ones here somewhere ones here and so on alright. So, we are taken an example for you if you remember we had a matrix here say its 0 1 0 0 0 1 0 0 0 for example, I can take this.

So, I can for example, I can take this matrix which is of the size 4 plus 3 plus 2 is 9 cross 9 matrix, and you can say it is nilpotent. And not only that you can see that if I write this as A, then A to the power 4 is 0 alright. So, there is a notion of what is called the degree of nil potency or index of nil potency, and accordingly you can build up fine. So, I will not get into that part fully because that is going out of the syllabus that is what is called as Jordan decomposition fine.

But what you can show if you say that so let A be n cross n such that A to the power n minus 1 is not 0, but A to the power n is 0 will imply A with respect to a basis B, there exists a basis B which comes from the previous idea itself, this X will look like 0 1 0 0 0 0 1 0 and so on.

So, 0 will be on the diagonal and 1 here alright. So, this is a way to look like. You can do it yourself. You can try to prove it yourself fine. So, I will not get into that part, but you can do it. And here in this case, you can see that the algebraic multiplicity of this is n algebraic multiplicity of 0 as an eigen value of A is n. But if you look at the geometric multiplicity of 0 here, this will be just 1 alright.

In this example if I am looking at, this example this A then the algebraic multiplicity of 0 as an eigen value of A is 9, because this is the 9 cross 9 matrix with 0s on the diagonal. It is an upper triangular matrix. But the geometric multiplicity here is 3. 1 coming from here, 1 from here, 1 from here alright fine.

So, you can check here that e 1 is there with you which will give you an eigenvector. Here this will be e 4, and here it will give you so 4 and 6, it will be e 7, 8, 9, 10 yeah. So, these will be the eigenvectors e 1, e 4 and e 7. These will be the eigenvectors and those are the only eigenvectors alright. So, I have given you idea of this. So, let us go back. So, the idea was to look at this part. This is one way of doing it which is P star A P fine.

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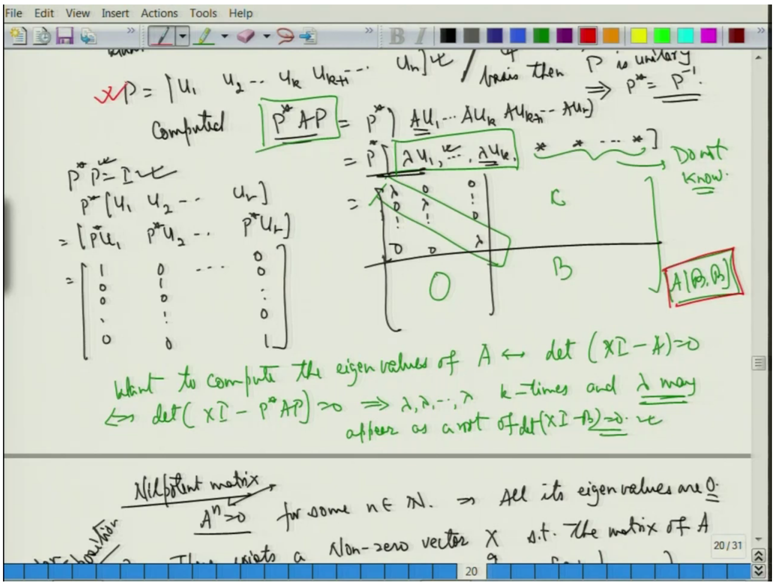

The other way will be to write A as B, B alright. So, I think I should, so the other way will be to look at A of B, B, and then do things. So, let me write that also with respect to, so my P is remains the same. So, A B, B, and my that is my B. So, let me write that part.

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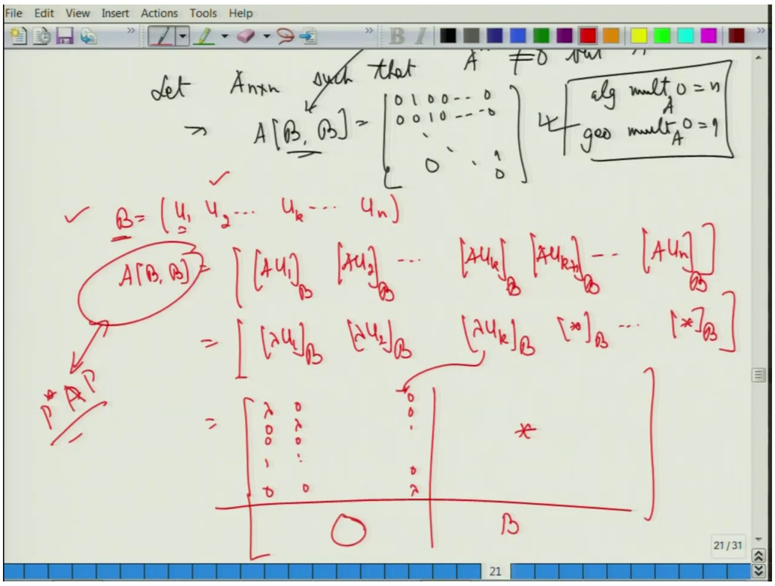

So, my P, so my B is u 1 u 2 u k till u n, I want to get the matrix of A with respect to B alright. So, if I want to get this, what I am suppose to do is, recall we did everything with respect to linear transformation, but you can do with matrix also by defining T of X is equal to A X. So, we can just look at A of u 1 A of u 2 A of u k A of u k plus 1 till A of u n fine.

But to get the matrix, we need to evaluate with respect to B, we need to find out the coordinates with respect to B. So, we need to find out this B, this what you need to do. This is same as A u 1 is lambda u 1 so with respect to B, lambda u 2 with respect to B, lambda u k with respect to B. And then I do not know what they are. As I said earlier also in the previous part that I do not know what they are. So, I can just leave it as it is fine.

And this in the ordered basis B itself u 1 is a first vector. So, it is nothing but lambda 0 0 0; lambda u 2 u 2 is a second vector, so it is 0 lambda 0 0 0 fine, so whole 0 here. Similarly, this u k will give me 0 0 0 and lambda here fine. So, this whole thing is 0 for me. I do not know what I have, I do not know what I have, but it is a matrix B.

And therefore, you can see that either you write like this or you write it as P star A P, the two ideas are same. We are looking at writing a matrix with respect to the ordered basis and the other is trying to look at similarity they are same and nothing else. There is no change. The ideas are same, but you write it in a different way that is all fine. So, now, we go to the next idea what is called diagonalizability.

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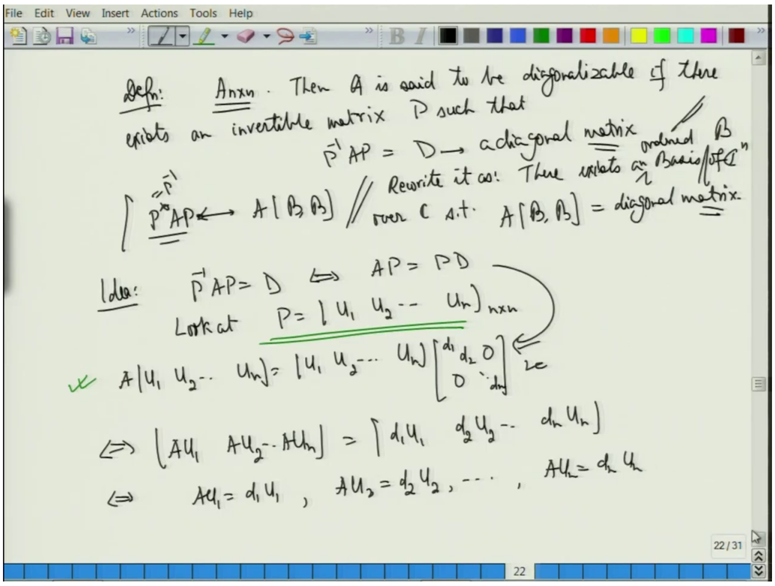

So, definition a n cross n remember that we have done LU decomposition, rank decomposition, QR decomposition, then the next idea was diagonalizability that when can I diagonalize a matrix. So, let A be an n cross n matrix, then A is said to be diagonalizable if there exists an invertible matrix P such that P inverse A P is D a diagonal matrix alright that is the definition.

So, if you look at the previous one, in the previous example, what we had was we had P star AP we looked at this part, and this was related with what is called looking at A of B comma B fine. Similarly, here also the same idea comes because P star was nothing but P inverse. So, here also you have same thing that I want so I can rewrite this.

So, rewrite it as so I am rewriting the definition, rewriting definition as there exists a basis an ordered basis of C n over C such that there exists an order basis B C such that A of B B is equal to diagonal matrix alright.

So, we will look at some examples later on, but first let me look at the idea behind it alright. So, let us look at the idea. So, idea, what are we saying exactly. We are saying that P inverse A P is diagonal alright. This is same thing as saying that A P is equal to P D fine. So, look at look at P and write it as say some u 1 u 2 u n. It is an n cross n matrix, I can write like this. Everything over C n over C, so it is just a column vectors of complex numbers. So, I can write P as this.

So, let us compute this both the sides. So, what is A P? So, therefore, A P, this implies it is nothing but a times u 1 u 2 u n is equal to u 1 u 2 u n times the diagonal matrix. Let me write the diagonal matrix as d 1, d 2 d n, this is 0, this is 0.So, now, I can look at this which is same thing as saying that A of u 1, A of u 2 A of u n is equal to now this diagonal matrix is being multiplied on the right. So, it is nothing but column transformations. So, it will be nothing but d 1 u 1 d 2 u 2 d n u n alright.

So, important please understand I am writing P as this. I have only done P writing like this. After that is all equivalent conditions. A P equal to P d is equivalent to this; this is equivalent to this alright. Again this will be equivalent to saying that A of u 1 is equal to d 1 u 1, A of u 2 is equal to d 2 u 2 and so on, A of u n is equal to d n u n.

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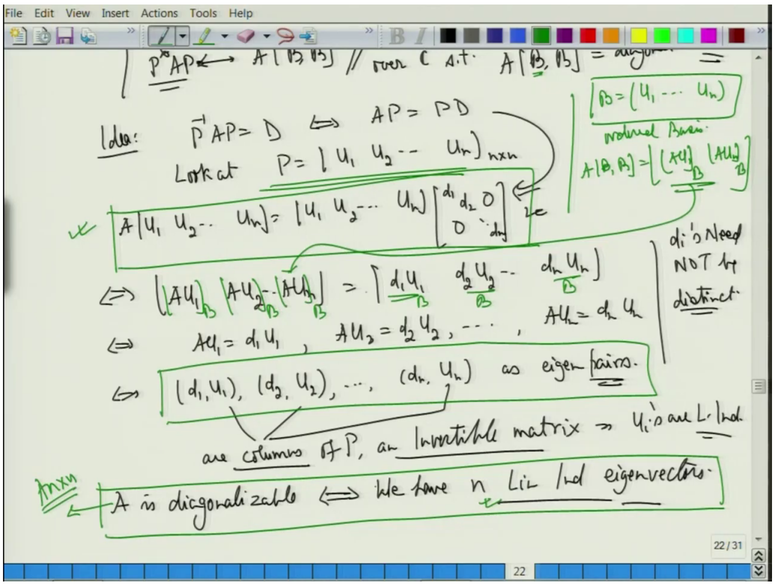

So, what we are saying here is that which is same thing as saying that I have got d 1, u 1, d 2, u 2, d n, u n as eigen pairs distinct; d i's need not be distinct, we are not saying that fine. But we are saying that I have got n linearly independent. So, if I look at these u 1, u 2, u n, they are columns. So, these are columns of P an invertible matrix..

So, P is an invertible matrix is very important. So, their columns of P they are invertible, and hence they are linearly independent implies u i's are linearly independent. Or what we are saying is that A is diagonalizable, A is diagonalizable if and only if we have n linearly independent eigenvectors.

Look at the definitions. So, we had here these are the eigen pairs. So, you have got u 1, u 2, u n, you have got n linearly independent eigenvectors alright. So, A is diagonalizable if and only if you have got n linearly independent eigenvectors.

And what is this n? n is the size of this matrix; A was n cross n alright. It was the size of the matrix fine. This is one way of understanding it using matrices. This is what we wrote fine. Or if I want to write in terms of our basis idea, so I can take my B here that I want to look at is nothing but u 1 to u n itself.

So, I take my ordered basis as this ordered basis. So, when I want to compute A of BB, I am supposed to look at A of u 1 till A of u n, and then evaluate each of them with respect to B this what I am supposed to do.

So, if I look at this, I am supposed to look at this itself fine. And once I have done that, I am only supposed to look at with respect to B here, with respect to B, with respect to B. So, this again with respect to B this with respect to B this with respect to B. And this will nothing but d 1 1 1 1.

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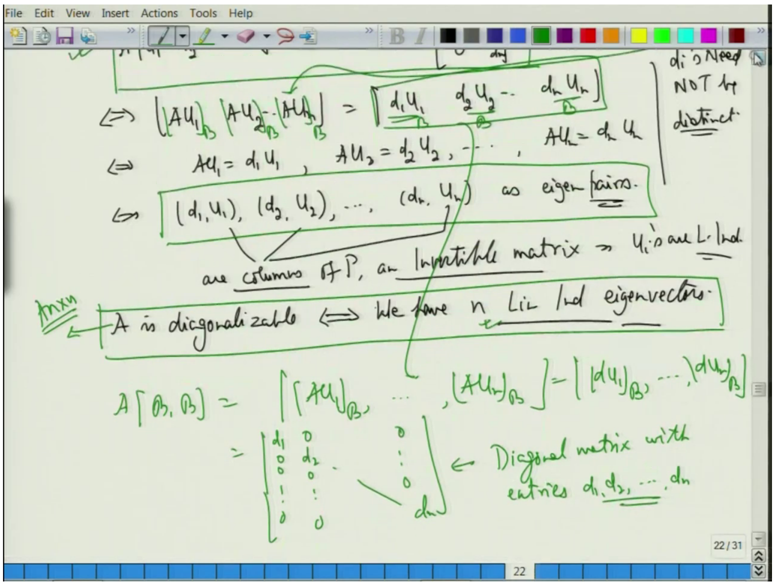

So, if I want to go further, so this part or I think there may be some problems, so clarity. So, let me write here. So, A of shall I have to compute A of B B will be equal to A of u 1 till A of u n with respect to B which will be equal to d of u 1 with respect to B till d of u n with respect to B which will be equal to d 1 0 0 0 0 d 2 0 0 0, and d n here 0 0 0.

So, it is a diagonal matrix, diagonal matrix with entries d 1, d 2, d n fine. This is what we wanted here. See here this is what we wanted that in a diagonal matrix with diagonal d 1, d 2, d n fine. So, what we are learning here is that whenever we have something is diagonalizable, it means that I have an ordered basis with respect to which the matrix is a diagonal matrix or we have n linearly independent eigenvectors fine, so that is important you have to keep track of things. So, now, let me look at examples.

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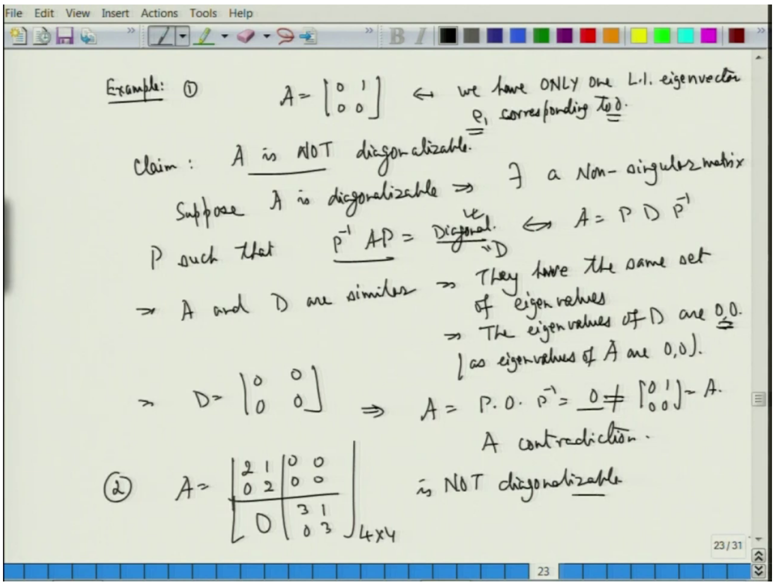

So, I just need to find linearly independent enough number of linearly independent eigenvectors alright. So, the first example you already have that if I look at this matrix 0 1 0 0 fine, we have only one linearly independent eigenvector which was e 1 alright. This was the only one fine which corresponding to 0 corresponding to 0 alright.

So, let us try to show that A is not diagonalizable. So, claim A is not diagonalizable. So, how will I do? Suppose A is diagonalizable, on the contrary suppose A is diagonalizable implies there exist a non singular matrix matrix P such that P inverse A P is diagonal alright. Now, this diagonal matrix and this P inverse A P, they are similar matrices. So, they are supposed to have the same eigen value alright fine. So, A is nothing but P D. So, writing D here with the P inverse alright.

So, this implies A and D are similar implies they have the same set of eigen values. Now, what are the eigen values of A? This implies the eigen values of D are 0 0 as eigen values of A are 0 comma 0 fine. So, see these are diagonal matrix and eigen values are 0. So, D must be the 0 matrix.

This implies D has to look like 0 0 0 0 this is what we saw in the previous slide if you see that d 1 to d n where the eigen values alright. So, whenever I have a diagonalizablity, then d 1 to d n are eigen values fine. And therefore, what I get here is that this will imply that A will look like P times 0 matrix time P inverse which is the 0 matrix, but A is not the 0 which is not equal to 0 1 0 0 which is A alright. So, there is a contradiction, contradiction.

Similarly, I would like you to check that if I write A as 2 1 0 0 2 0, or let me write one more here fine. So, 0 1 or say 3 1 0 3 if I write, this matrix say 0 here, 4 cross 4 a is not diagonalizable. So, think about it why it is not diagonalizable. Just so diagonalizable, so try that out yourself fine.

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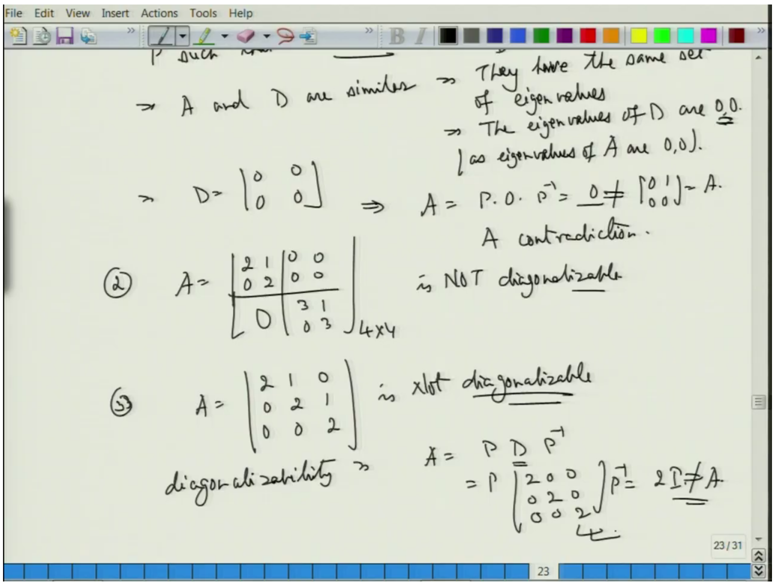

I would like you to also show that this is a bit difficult for you. But you can show that this matrix A which is 2 1 0, 0 2 1, 0 0 2. This is not diagonalizable because diagonalizability implies A is equal to P d P inverse which will be P times D eigen values here.

So, it will be 2 0 0, 0 0 2, 0 0 0 2 times P inverse which is 2 times identity which is not A alright. So, why should you get this is as the diagonal matrix? Think about it. I want you to understand that because very important for us. And we end this lecture with this idea itself, alright.

Thank you.