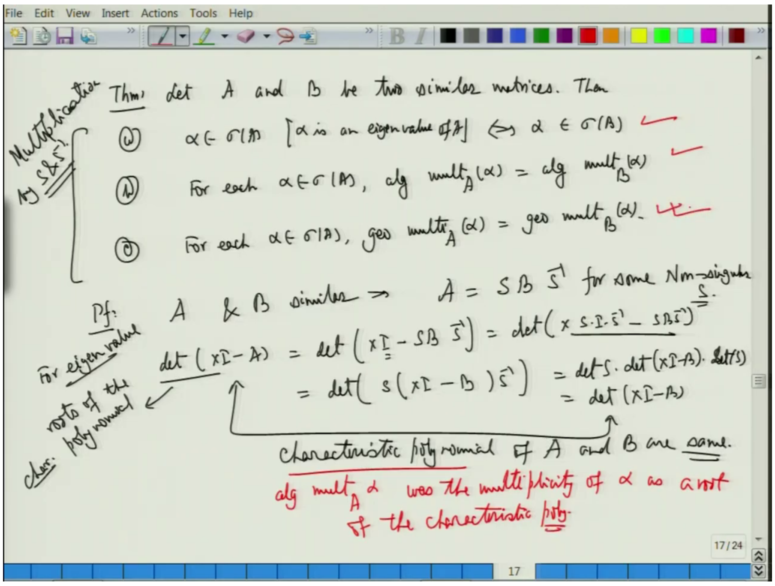
Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 55
Results on Eigenvalues and Eigenvectors

(Refer Slide Time: 00:17)



So, let me write what I just did. So, let me write it as a theorem, so theorem. Let A and B be two similar matrices, fine. Then a, alpha belongs to sigma A that alpha as an eigen value, alpha is an eigen value of A, if and only if alpha is an eigen value of B, fine. That is one thing two or b part, for each alpha belonging to sigma of A fine; algebraic multiplicity of alpha as an eigen value of A is same as algebraic multiplicity of alpha as an eigen value of B that is also same.

And c, for each alpha belonging to sigma A, geometric multiplicity is also the same; multiplicity of alpha as an eigen value of A has a same geometric multiplicity as B, is that ok. So, all the three are true. So, whenever I am looking at symmetric transformation, everything is nice fine; because I am multiplying by invertible matrix that is more important. So, multiplication of, multiplication by s and s inverse, that is what is, alright. So, once let us again understand it nicely.

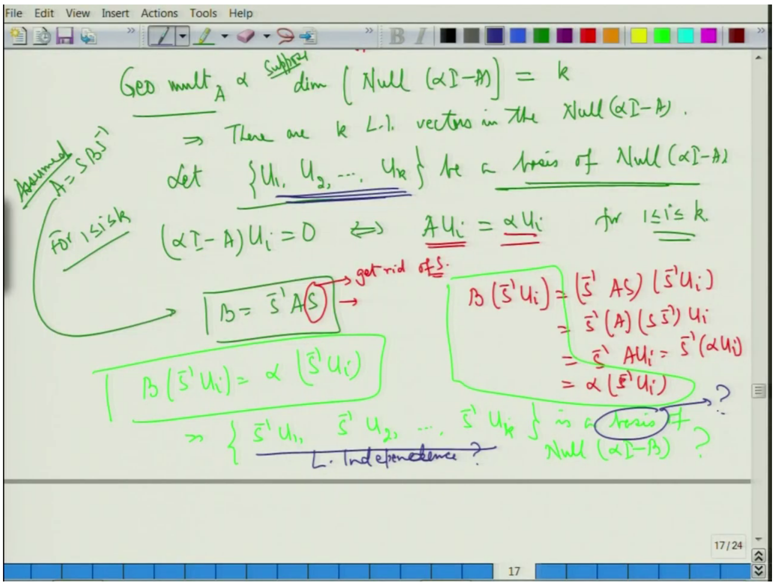
So, proof, what we did was that, we looked at determinant of. So, A and B similar, A and B similar implies, determinant of implies A is equal to S B S inverse for some non singular matrix, non singular S or invertible matrix S whatever you want to say. So, for we need to compute, so for eigen value, for eigen value; we need to compute determinant of x I minus A. So, eigen values, for eigen values roots of the polynomial alright, characteristic polynomial.

So, this which is same as determinant of we had done it, X I minus A is nothing, but S B S inverse, which is same as determinant of; I can write X as it is, replace I by S identity S inverse minus S B S inverse which is same as determinant of. I can take common here S this side, S inverse this side; I get here X I minus B here, which is same as determinant of S into determinant of X I minus B into determinant of S which is same as determinant of X I minus B.

So, what we are saying here is something more that characteristic, characteristic polynomial of A and B are same. Since, their characteristic polynomial is the same and therefore, we get A as well as B both together at a time because algebraic multiplicity was only related with the characteristic polynomial; the roots of the characteristic polynomial, alright.

So, this gave us algebraic multiplicity of alpha as an eigen value of A was the multiplicity of alpha as a root of the characteristic polynomial, alright fine. Now, let us look at the geometric multiplicity; how do I prove the geometric multiplicity?

(Refer Slide Time: 05:13)



So, for geometric multiplicity, geometric multiplicity of alpha as an eigen value of A; we need to look at dimension of null space of alpha I minus A, I need to compute this. So, suppose the dimension is k, suppose dimension of this is k; this implies I can choose a basis. So, this implies there are k linearly independent vectors in the null space of alpha I minus A. So, let u 1 u 2 u k this be a basis of null space of alpha I minus A, fine.

So, this is basis means; basis means that alpha I minus A multiplied to u i is 0, which is same thing as saying that. So, this is for 1 less than equal to i less than equal to k I get this, which is same thing as saying that A of u i is equal to; just look at this A of u i is equal to alpha u i for 1 less than equal to i less than equal to k, fine.

From here I want to go to B, I want to go to B and we have assumed that, A is equal to this. So, we have assumed; what we have assumed? A is equal to S B S inverse, alright. So, let us use this idea; I know about A, I want to go to B now, fine. So, I can rewrite this. So, I can rewrite this as, just have a look at it nicely; I can write B as fine S inverse A S, I can write like this, fine.

B is S inverse will go on the left and S will come on the right, fine. What I know is, I know about A of u i; then how do I multiply by u i? If I multiply by u i, there will be a problem here. So, what I can do is that, I can get rid of this S. So, get rid of S, if I want to get rid of S; I have to multiply this by S inverse, alright. So, let us look at B times S inverse of u i.

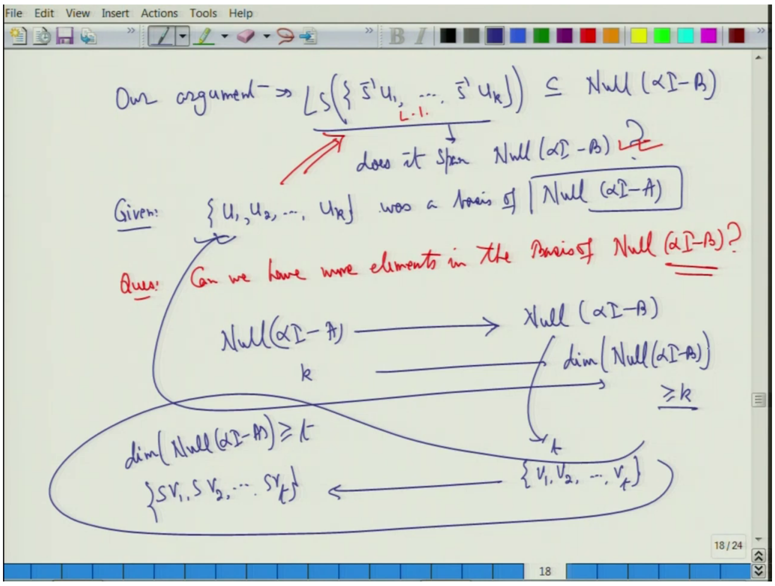
So, this will be equal to B is S inverse A S; you are multiplying by S inverse of u i and therefore, what you get is S inverse of, S and S inverse cancels out, A here S S inverse of u i which is S inverse of a u i, which is S inverse of A u i is alpha u i, which is same as alpha times S inverse of u i, fine. So, you can see here that, I have got here this that, B times S inverse of u i is equal to alpha times S inverse of u i, fine.

So, this implies S inverse of u 1, S inverse of u 2, S inverse of u k; this is a basis of null space of alpha I minus B with a question mark. Why a question mark? Because we do not know; till now we have not shown that this is linearly independent, linear independence. So, there is a question of linear independence, alright.

So, recall that, if I have a linearly independence at u 1 and u 2, u k; if I multiply by any invertible matrix, the linear independence is maintained, alright. So, here S is an invertible matrix and I multiplying by S inverse which is an invertible matrix to a linearly independent set.

So, again emphasising, if you have a linearly independent set, multiply by an invertible matrix whether on the left or right whatever it is; if you are able to multiply, if your multiplication makes sense, then you again get a linearly independent set, alright. So, therefore, this will be a linearly independent set, I cannot say now. So, I have got that this is linearly independent set; how do I say it is a basis, alright? So, what?

(Refer Slide Time: 10:02)



So, our argument, our argument implies S inverse of u 1 till S inverse of u k is a subset linear L s of this is a subset of null space of alpha I minus B. Can it have more elements question? So, this; does it a span null space, that is the question, alright? So, what we know, we have been given.

So, let me just write it again. So, I started with. So, given what we had given that, this set u 1, u 2, u k was a basis of null space of alpha I minus B. From there we got that this will imply that, that this is linearly independent; these are linearly independent and the linear span is a subset of this, fine.

Question is, can I have more elements in the basis, fine? So, question, can we have more elements in the basis of null space of alpha minus B, alright? So, what we have done? I wrote it wrongly here, it is A here, alpha I minus A here, fine. So, from null space of alpha I minus

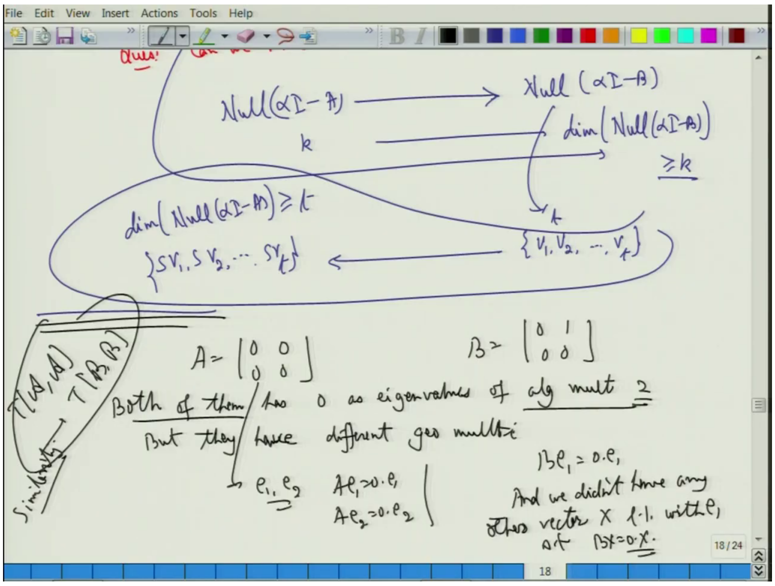
A, I went to null space of alpha I minus B, this is what I did. So, from k I got something. So, here what we are saying is that, what we have shown is that, dimension of null space of alpha I minus B is greater than equal to k, fine.

So, from k, I went to say that this is greater than equal to k; if I start with a t here, the same argument, alright. So, if I start with a basis here say v 1, v 2, v l; if I start with this here fine, I can go here and then look at s v 1, s v 2, s v l. I can go to this; again S is invertible.

So, this will be linearly independent set and this will imply that dimension of null space of alpha I minus A will be greater than equal to l, fine. So, from k I have got, this is greater than equal to k from this part, alright. So, this is t I wrote. So, t here, t here, t here, alright fine. So, from one I can go and say that dimension is greater than equal to k; I can go the other way around to say that its dimension is more.

So, the dimension of the two is the same, alright. So, you have to be careful, you have to understand this; this is very important fine. So, what we have shown is that, if two matrices are similar; then they have the same eigen values, there algebraic multiplicity is the same, their geometric multiplicity is the same, alright fine. What more?

(Refer Slide Time: 13:36)



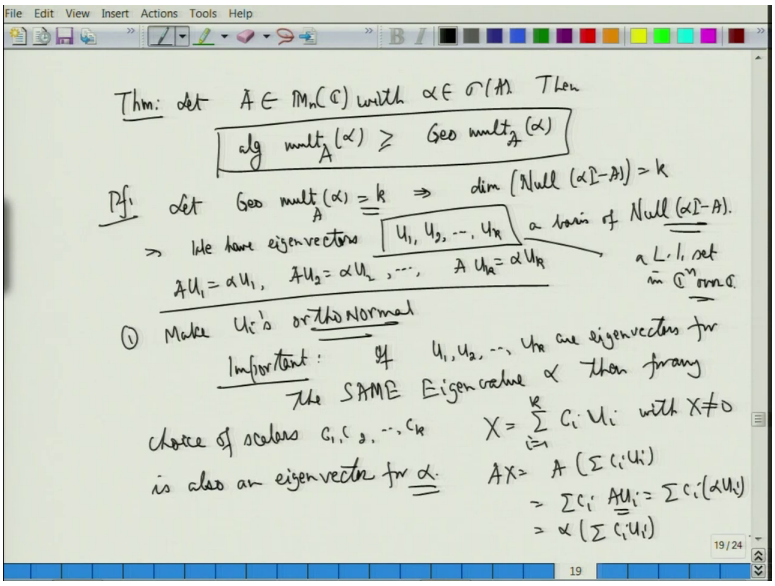
So, let us look at example which you have already done; I am just trying to recollect for you. Look at this matrix A which is 0, 0, 0, 0; the matrix B which is 0, 1, 0, 0, both of them have the same eigen values, both of them have 0 as eigen values of algebraic multiplicity 2 fine, both of them 0, 0, 0, 0, fine. But they have different geometric multiplicity, alright.

So, recall e 1, e 2 as my eigen vectors; because A of e 1 is 0 times e 1, A of e 2 is 0 times e 2. But in this case we will look at B times e 1 was 0 times e 1 and we did not have any other eigen, any other vector X linearly independent with e 1, such that B X is equal to 0 times x, alright.

So, understand them that, if you have similarity, everything is nice; if there is no similarity, there is a problem. And again recall, I am telling you again and again that, similarity means changing the basis; this is what we had done at the time of defining itself when we looked at linear transformations.

I think the last slide or last lecture in that part that, whenever we look at matrix of a linear transform from T of A, A to T of B, B alright; then the notion of similarity comes into play. So, we are just changing the basis and trying to get things. So, whenever I change the basis, properties do not change; everything is nice, fine. So, I would like you to keep track of this part, fine. As the last thing in this lecture, I would like you to understand this theorem.

(Refer Slide Time: 16:04)



And I want you to understand the argument here, because a similar argument will be used at different place also, fine. So, let A belong to M n of C and this with alpha belonging to a spectrum of A; then algebraic multiplicity of alpha as an eigen value of A is greater than equal to geometric multiplicity of alpha, is that ok.

Proof, very important idea, it gives you idea of how do we play around with matrices eigen vectors, eigen values, linear independence, dependence and so on; how are we playing things that is very important. So, understand it nicely, alright. So, let geometric multiplicity of alpha as an eigen value of A be k, suppose it is k.

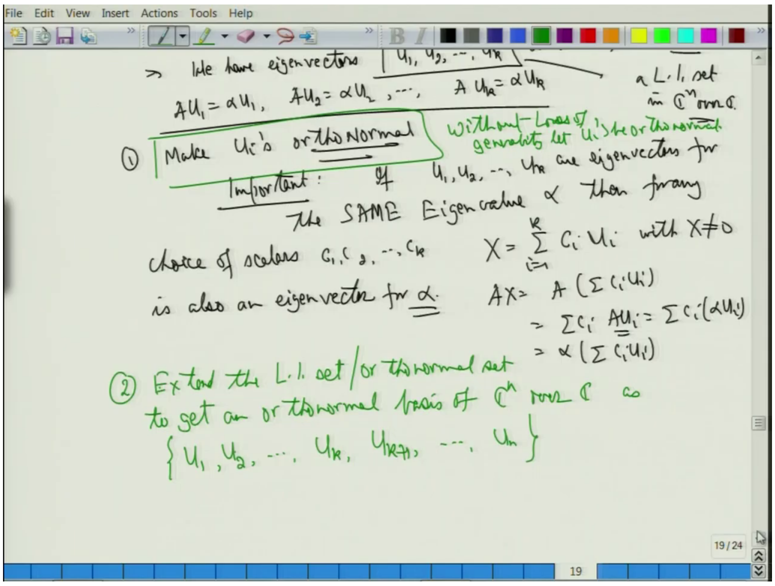
What does it mean? It means that, dimension of null space of alpha I minus A is k; implies we have eigen vectors. So, whenever we say eigen vectors, they are linearly independent; eigen vectors u 1 and u 2 u k a basis of null space of alpha I minus A fine, is that ok. So, what we have is A of u 1 is equal to alpha u 1, A of u 2 is equal to alpha u 2, so on till A of u k is equal to alpha u k, fine. I have this, fine.

Now, u 1 to u k is a linearly independent set, a linearly independent set in C n over C, alright. So, we can extend it to form a basis. So, extend it. So, extend it, alright. So, before extending it, let me first make this. So, first thing first, make u i's orthonormal; I do not need it alright, I need the other part to be orthonormal or orthogonal at least.

So, make u as orthonormal, important; I did not say it directly, but indirectly I said in the very beginning because of linear independence that important that, if u 1 and u 2 are eigen vectors, then their linear combination is also eigen vector. So, if u 1, u 2, u k are eigen vectors for the same eigen value alpha; then for any choice of scalars c 1, c 2, c k, this vector X which is summation c i u i i is equal to 1 to k alright with X not equal to 0 is also an eigen vector for alpha, is that ok.

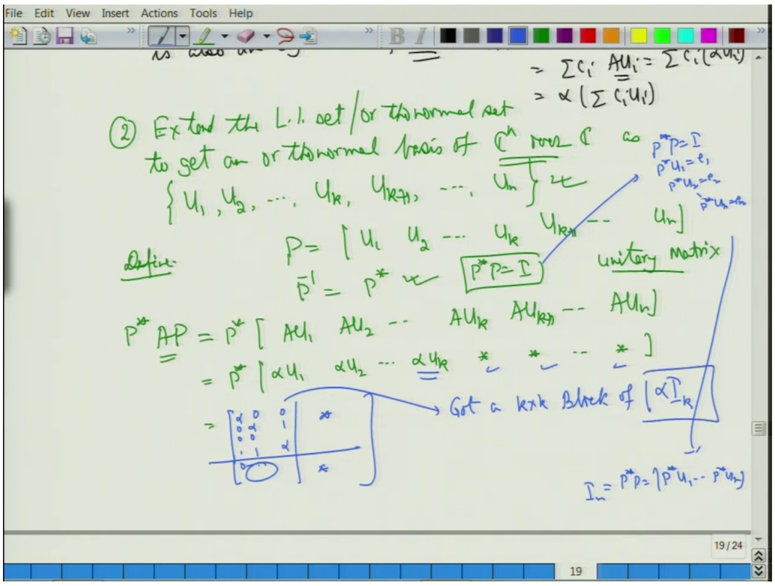
That you can check A X is equal to A of c i u i, which will be equal to c i A of u i, which will be equal to c i alpha u i, which is same as alpha times summation c i u i, alright. So, I can get I can make u i's to be orthonormal alright, that is the first thing source. So, what we have shown is that I can make u i's orthonormal.

(Refer Slide Time: 20:38)



Now 2, so without loss of generality, without loss of generality; let u i's be orthonormal, alright. 2, extend the linearly independent set oblique orthonormal set to get an orthonormal basis of C n over C as u 1, u 2, u k, u k plus 1 till u n, is that ok? So, I have got this basis now. And now I want to multiply things and see what is happening.

(Refer Slide Time: 21:44)



So, define the matrix P as u 1, u 2, u k, u k plus 1 till u n. Let me write this, now this is an orthonormal basis. So, p inverse is same as p star fine; because I do not know whether the u i's are real numbers or complex numbers, I do not have anything, but I know there are over C n. So, I can talk of unitary matrix.

So, p is a unitary matrix, because we have taken this as orthonormal, fine. So, let us compute P star A P, fine. So, what is P star A P? P star is P inverse itself. So, let me write P star here. And what is A P 1? So, A P will be equal to A of u 1, A of u 2, A of u k, A of u k plus 1 till A of u n, fine. So, this is equal to P star, look at A u 1; A u 1 is nothing, but alpha u 1, A u 2 has alpha u 2, alpha times u k, fine.

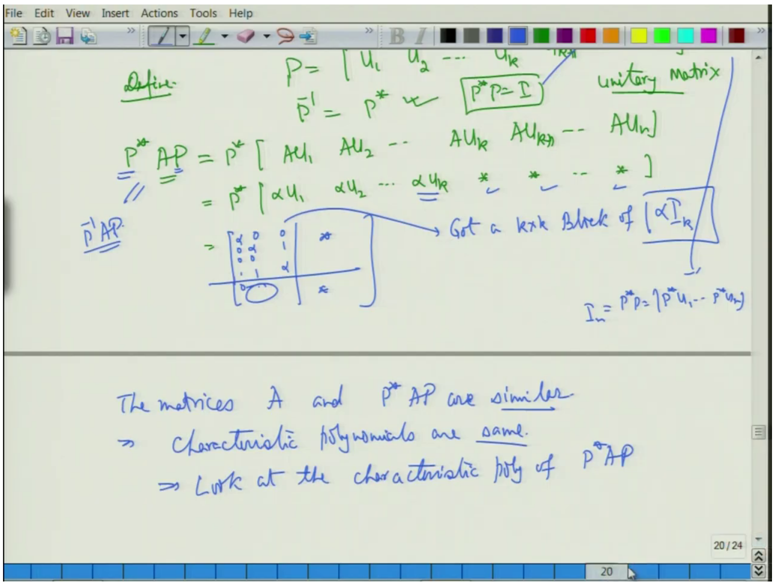
I do not know what they are these parts; I do not have any handle on those parts, fine. But I knew that u 1, u 2, u k they were eigen vectors corresponding to alpha. So, therefore, I will get alpha here, is that? Now when you multiply P star with alpha u 1; what do I get? P star if you look at this, we are saying that, since it is a unitary matrix; P star P is identity fine, it means that u 1 multiplied to P star is nothing, but 1, 0, 0, 0, fine.

So, look at the matrix multiplication again here recall. So, P star P is identity means that, where we multiplied P star with u 1 will give me e 1, P a star with u 2 will give me e 2 and so on p star of u n will give me e n alright, that is the way you multiply, fine. Because P star will go inside; so recall here P star P will give me p star u 1 till p star u n and this is my identity, alright. So, this is the way it is.

So, therefore, what I will get here is, this will give me alpha, 0, 0, 0; this will give me 0, alpha, 0 like this, for alpha k again I will get 0, 0, 0 and alpha here. So, this part will be 0, fine. I do not know. So, let us look at these parts; I do not know what they are. So, there will be some things here, is that ok? So, at this stage I only know that, I have got a, got a k cross k block of alpha times identity I k, is that I have got this, fine.

And therefore, from there I want to make statements about the whole thing. So, what I have seen here is that, P star and P they are invertible matrix and P star.

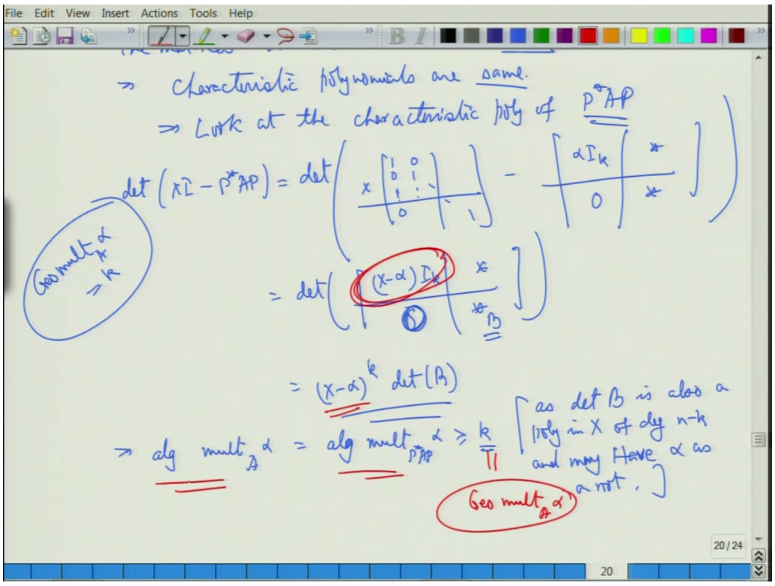
(Refer Slide Time: 25:04)



So, this is same as p inverse A P, fine. Since they are the same; so I am saying that, the matrix A and this they are similar. So, the matrices A and P star A P are similar, fine. Since they are similar; it means that, they have the characteristic polynomial; algebraic multiplicity is the same, geometric multiplicity is the same and so on, fine.

So, what we know is that, they are similar. So, this implies algebraic multiplicity; this implies characteristic polynomials are same. So, this will imply that look at it. So, now, look at the characteristic polynomial of P star A P.

(Refer Slide Time: 26:15)



So, the characteristic polynomial of P star A P will be determinant of X I minus P star A P, which will be equal to determinant of X times 1, 0, 0; 0, 1. So, this is 1 here on the main diagonal minus P star A P; P star A P and alpha I here, something here 0 here and something here, this is what I had, fine. So, this will give me a determinant of just look at this part also, I can just break up like this.

So, I will get here X minus alpha times I k; I will have something here, 0 here and something here, fine. So, this determinant of this part now, I can look at is as X minus alpha I k is there, there is a 0 matrix here. So, determinant of this should be determinant of this part into determinant of this part. So, if I write this as B, then I will get it as X minus alpha to the power k into determinant of B, is that fine.

So, therefore, from here we see that, this implies algebraic multiplicity of alpha as A which will be same as algebraic multiplicity of alpha for P star A P, which is greater than equal to k; because as determinant of B is also a polynomial in X of degree n minus k and may have alpha as a root, alright.

I am not sure whether what is B; I do not understand, I cannot understand what the B is, what are the entries of the B. But at least I know that, that since geometric multiplicity. So, since geometric multiplicity of alpha of A was, this was this number was k; I could get u 1, u 2, u k which were orthonormal, fine.

And then extend it to the get the whole basis, I could do that. And therefore, I could get this matrix. And because of this matrix I could say that, I have A minus alpha to the power k coming into play and therefore, I could show that algebraic multiplicity is greater than equal to k. But what was k? k was nothing, but the geometric multiplicity of alpha this, alright.

So, we have shown that algebraic multiplicity is greater than equal to geometric multiplicity, alright. So, that is all for now.

Thank you.