Linear Algebra
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Lecture – 54
Results on Eigenvalues and Eigenvectors

So, now let us try to understand what are called properties of eigenvalues.

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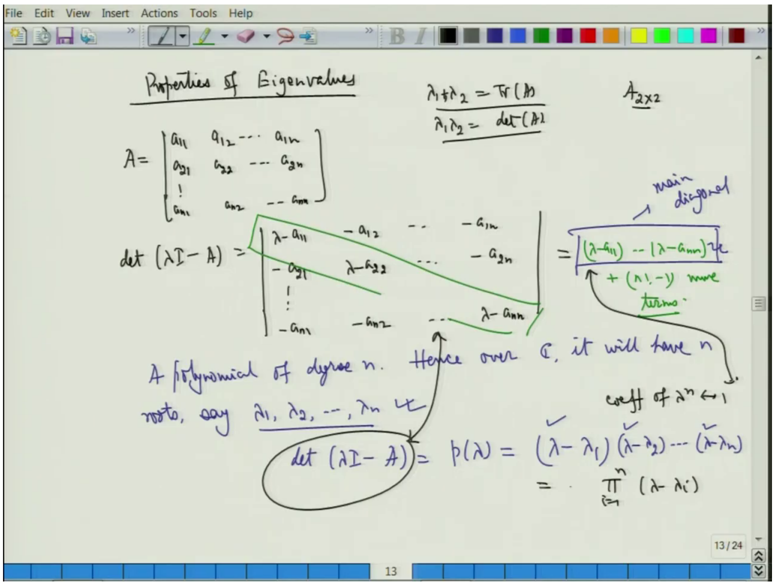

In the example that we had done till now look at remember those things. Where there we had shown that if lambda 1 and lambda 2 are eigenvalues it was for 2 cross 2 matrices if you remember then lambda 1 plus lambda 2 was nothing, but trace of A and their product was nothing, but the determinant of A alright.

Now, I would like to prove this result for the general setup that indeed it is true, fine. So, let us look at A whatever we have. So, A is sum a 11 a 12 a 1n a 21 a 22 a 2n a n1 a n2 a nn. Let us come to determinant of lambda I minus A. So, what is the determinant of lambda I minus A? It is nothing, but determinant of lambda minus a 11 minus a 12 minus a1n minus a 21 lambda minus a 22 minus a 2n minus a n1 minus a n2 and lambda minus a nn. So, this is what I have.

What we know is that this is a polynomial. So, look at just look at the diagonal here the main diagonal. So, look at the main diagonal lambda appears n times and therefore, it is a polynomial in lambda of degree n; that we had said earlier also. I did not do it, I just made a new statement. So, you can see here that lambda see if I look at this it corresponds to looking at lambda minus a 11 times lambda minus a nn plus n factorial minus 1 more terms alright.

So, when we compute the determinant there are n factorial terms. Out of those n factorial terms I have 1 term which is this which corresponds to the main diagonal and then there are other terms also which are playing a role here as far as the determinant is concerned.

So, since it is lambda minus a 11 lambda and lambda minus a nn this itself is a polynomial of degree n and therefore, I will have degree n polynomial, fine. So, a polynomial of degree n. Hence over complex numbers hence over c it will have n roots say lambda 1, lambda 2, lambda n, fine.

They may be distinct they may not be distinct that I am not bothered about, but they will have n roots. Once I have n roots what you are saying is that if I look at determinant of lambda I minus A, this was nothing, but the characteristic polynomial fine, there was a notation for characteristic polynomial.

So, I can write this as see these are the roots I can write this as lambda minus lambda i or lambda minus lambda 1 into lambda minus lambda 2 till lambda minus lambda n I can write like this because those are the roots and look at the coefficient of lambda to the power n coefficient of lambda to the power n.

Coefficient of lambda to the power n here is 1 and the same thing is true here also that coefficient of lambda to the power n is 1 and hence everything makes sense. There is no problem here. We generally write it in terms of product lambda minus lambda i i is equal to 1 to n, in my notes or everywhere else this is the way we write.

So, the idea is to understand the two things the left hand side is this expression which is quite long the right hand side is just a polynomial which has been factored into linear factor which has been factored. So, there are two terms and from there I want to build up my ideas, alright.

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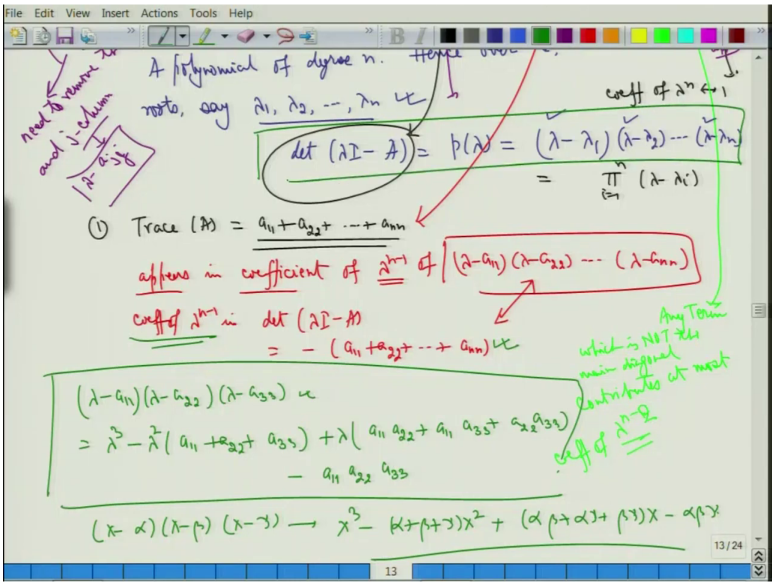

So, let us try to do that part let us try to understand. So, first thing is what about trace of A? How do I get trace of A? So, what we know is that trace of A is nothing, but a 11 plus a 22 plus a nn alright. Now, where does that term appear? So, if I look at this thing appears at this place.

Look at the main diagonal here look at the main diagonal, fine, look at coefficient of lambda to the power n minus 1. So, this appears in coefficient of lambda to the power n minus 1 of lambda minus a 11 lambda minus a 22 so on till lambda minus a nn alright, fine. And, there is no other term of lambda to the power n minus 1.

Why? Because if I do not want a term of lambda to the power so, if I look at this part the main diagonal main diagonal gives me lambda to the power n lambda to the power n minus 1 lambda to the power n minus 2 and so on, fine.

If I do not want to look at the main diagonal, if I want to look at any other thing alright so, as soon as I look at any other term here for example, if I want to look at any term here, then this tells me that I will have to remove this row and I will have to remove this column this is what the determinant was.

Determinant means if I am looking at there is a term a ij i not equal to j; it means that need to remove the i-th row and j-th column. Now, if I were to remove i-th row and j-th column it means that j-th column means this will give me lambda minus a jj will get cancelled out and from here I will get lambda minus a ii cancelled out alright. So, one of them will always get cancelled out. This what you have to understand.

So, when I want to compute the determinant; determinant has n factorial terms, out of that there are what are called the main diagonal and the rest are n factorial minus 1 terms. If I look at this n factorial minus 1 terms lambda to the power n minus 1 does not appear.

So, lambda to the power n minus 1 does not appear. Why? basically because I will get some a ij and that a ij and there I know that i is not equal to so there will be at least one term a ij which is i is not equal to j and that will get rid of lambda minus a ii as well as lambda minus a jj.

And, therefore, there will be only lambda to the power n minus 2 terms at most lambda to the power n minus 2 terms. So, any other non-diagonal elements. So, any other term here. So, this term so, any term which is not the main diagonal contributes at most lambda to the power n minus 2 coefficient of lambda to the power n minus 2 it would not compute give me lambda to the power n or lambda to the power n minus 1 is that.

So, let us go back now. So, I am looking at trace; trace of A is this. So, this is same thing as looking at appears in the coefficient of lambda to the power n minus 1 which is this and how does it come com? So, if we look at lambda to the power n minus 1, the term here is coefficient of lambda to the power n minus 1 in determinant of a minus lambda I minus A I wrote lambda I minus A is nothing, but minus of a 11 plus a 22 plus a nn.

Look at here, fine? Why it is happening? So, let me write that nicely maybe there will be some clarity. So, I want to look at lambda minus a 11 lambda minus a 22 and lambda minus a 33 then I would like you to see that this is nothing, but lambda cube minus lambda square times a 11 plus a 22 plus a 33 plus lambda times a 11 a 22 plus a 11 a 33 plus a 22 a 33 alright minus a 11 a 22 a 33, alright. This is what we have.

So, basically what we are saying is that any polynomial if these are the roots a 11 alpha beta gamma are the roots. So, if alpha beta gamma are the roots if. So, I should be written X X minus alpha X minus beta X minus gamma then this corresponds to X cube minus alpha plus beta plus gamma times X square plus alpha beta plus alpha gamma plus beta gamma times X minus alpha beta gamma. This is what we are writing here nothing else, fine.

So, what we see is that I have got this part coefficient of lambda to the power n is this, but what we are saying here is understand carefully there should not be a confusion that I am writing determinant also as like this, fine. So, I think I should go to the next page for clarity.

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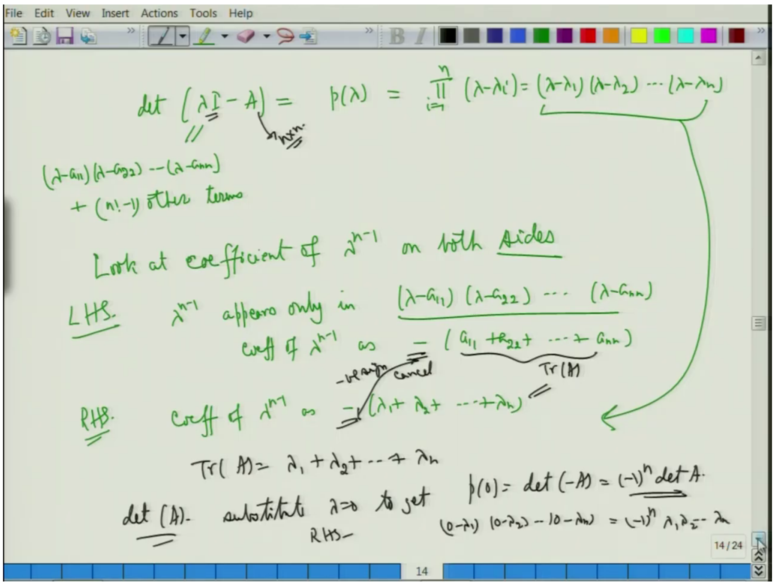

So, what we have taken is determinant of lambda I minus A. One way this is nothing, but the characteristic polynomial we wrote it as product i going from 1 to n lambda minus lambda i which is same as lambda minus lambda 1, lambda minus lambda 2 lambda minus lambda n. This was one term of this was lambda minus a 11 a 22 lambda minus a nn plus n factorial minus 1 other terms. This is what we had alright.

So, in this I am looking at so, look at coefficient of lambda to the power n minus 1 on both the sides on both sides. So, on the left hand side if I look at left hand side as I said lambda to the power n minus 1. So, lambda to the power n minus 1 appears only in lambda minus a 11 lambda minus a 22 lambda minus a nn and this gives me coefficient of lambda to the power n minus 1 as minus of a 11 plus a 22 plus a nn, alright.

If I look at the right hand side; right hand side is basically this itself and therefore, this will give me coefficient of lambda to the power n minus 1 as minus of lambda 1 plus lambda 2 plus lambda n, fine and therefore, I know that this is nothing, but A. So, this will be same as this, this minus sign and this minus sign they cancel out alright negative sign cancels fine.

So, therefore, what I get is that trace of A is nothing, but some of the eigenvalues fine. What about the determinant let us look at determinant of A. So, if I want to get the determinant of A, I need to substitute lambda equal to 0 here, alright.

So, if I substitute lambda equal to 0, what I get is. So, substitute lambda is equal to 0 to get p 0 is equal to determinant of minus A which is same as minus 1 to the power n into determinant of A. Why minus 1 to the power n? Because this matrix is of size n cross n alright.

So, I multiplying by a scalar minus 1. So, that gets multiplied fine, what about the right hand side? Look at the right hand side; right hand side right hand side is nothing, but 0 minus lambda 1 0 minus lambda 2 0 minus lambda n which is same as minus 1 to the power n lambda 1 lambda 2 lambda n.

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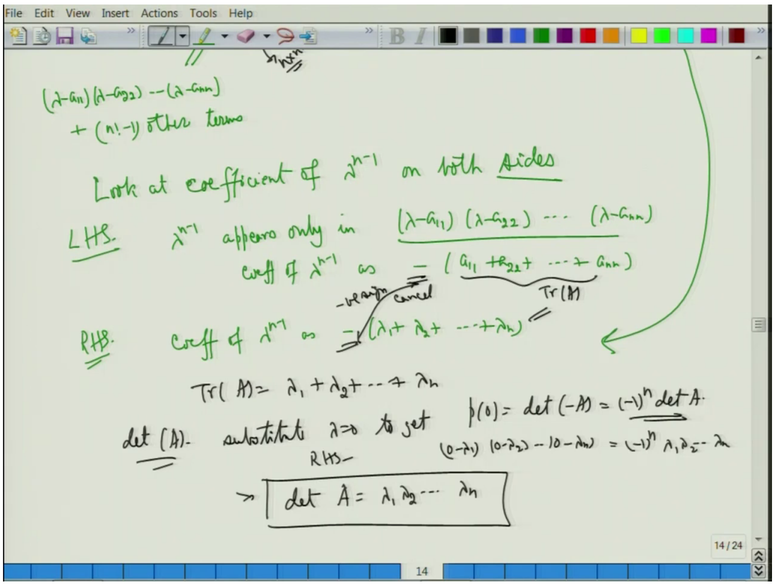

And, therefore, I get that minus 1 to the power n minus 1 to the power n cancels out I get determinant of A is nothing, but lambda 1 lambda 2 lambda n, fine. So, we have shown that trace of A is same is same as some of the eigenvalues and determinant is nothing, but the product of eigenvalues it is very important idea it helps us at lot of places.

There are other things also you can go for 2 by 2, 3 by 3 and so on, but that is not in our syllabus. Our syllabus is only about trace and determinant. So, you need to keep track of that, fine.

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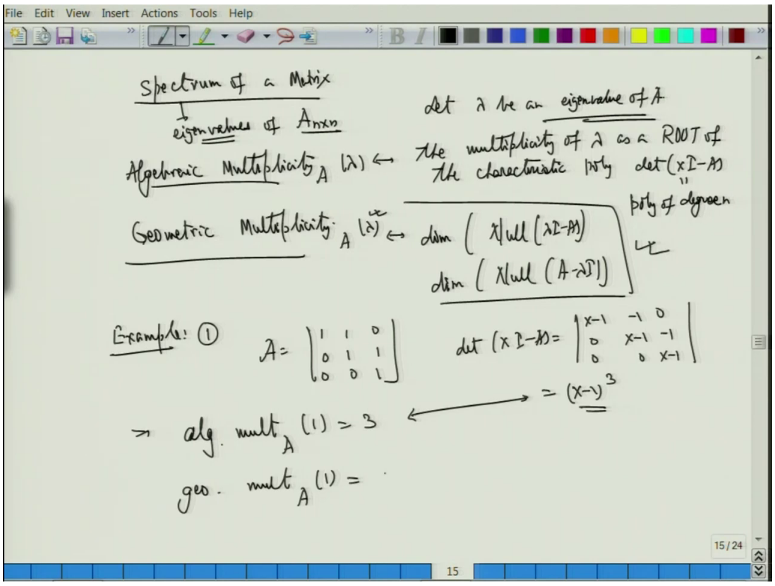

Now, let us look at what are called algebraic multiplicity and things like that. So, let me go into the next idea what. So, let us concentrate on a spectrum of matrix. So, spectrum means eigenvalues so, of A, fine. Then there is a notion of what are called now algebraic multiplicity and there is a notion of what is called geometric multiplicity.

So, algebraic geometric multiplicity of what? Eigenvalues of this, alright. So, let lambda be an eigenvalue of A. Recall, we are doing everything over complex numbers. So, let lambda be an eigenvalue of A.

This is the assumption then there is a notion algebraic multiplicity of lambda as an eigenvalue of A. So, this is the multiplicity of lambda as a root of the characteristic polynomial determinant of XI minus A. So, this is a polynomial of degree n polynomial of degree n, fine.

Now what is a geometric multiplicity? This is the corresponds to dimension of null space of lambda I am looking at. So, it is as you can write lambda I minus A null space of this or same as dimension of null space of A minus lambda I, whichever you are comfortable with you are supposed to look at this.

So, geometric multiplicity is the dimension, fine. So, let us take an example to understand it, so that I can proceed further. So, so example. So, let me take one example. I define A as 1 1 0 0 1 1 0 0 1 I look at this alright, fine. So, I want to compute determinant of XI minus A which is same as determinant of X minus 1 minus 1 0 0 X minus 1 minus 1 0 0 X minus 1. So, this is an upper triangular matrix. So, determinant of this is X minus 1 to the power 3.

So, this implies algebraic multiplicity of 1 as an eigenvalue of A is 3 fine because 1 is repeated thrice, fine. What about the geometric multiplicity? For the geometric multiplicity I need to compute this null space. So, let us compute the dimension of the null space. So, we want to compute. So, we want to look at geometric multiplicity of 1 here alright. So, this is equal to what? So, let us do that out.

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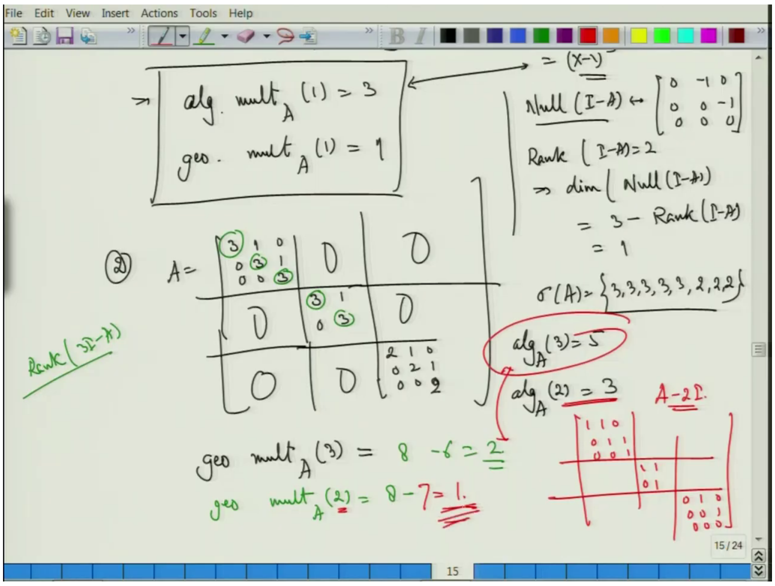

So, I need to look at null space of I lambda is 1 here. So, I am supposed to look at I minus A null space of this. So, this corresponds to looking at just have a look at it 1 minus A. So, this corresponds to the matrix replace X by 1 here. So, if I do that 1 minus 1 is 0 minus 1 0 0 0 minus 1 0 0 0 alright.

So, this is the matrix that I am looking at. Now, the rank of this matrix is 2 rank of I minus A is 2. This implies dimension of null space of A will be equal to 3 minus the rank of I minus A by the rank nullity theorem which is 1 alright. So, the geometric multiplicity is 1 is that try that out yourself.

Another example so, another form of this was already given to you earlier. I will write it now with 3 here 1 here 0 0 3 1 0 0 3 I look at this, fine and again I look at say 3 1 0 3 2 1 0 0 2 1 0

0 0 or 0 0 2 suppose I look at this matrix huge matrix for us fine. So, all these entries are 0 for us fine. So, I would like you to see that this is an upper triangular matrix.

So, the eigenvalues a spectrum of A here consists of 3 3 3 3 3 5 times 3 plus 3 3 plus 2 and 2 appears 3 times. So, this is an spectrum of A, fine. Algebraic multiplicity of 3 as an eigenvalue of A is 3 plus 2 is 5 algebraic multiplicity of 2 as an eigenvalue of A is 3.

What about the geometric multiplicity? Check that the geometric multiplicity of 5 of 3 as an eigenvalue of as an eigenvector here will be equal to. So, this will give me a look at the previous example. If we use the previous example the rank of this if I look at this whole part A, I want to look at the rank of that part. So, I want to compute rank of 3I minus A if I look at that part what will happen is, this will become 0, this will become 0, this will become 0.

So, the top one gives me only rank 2 this is 0, this is 0. So, it gives you one more rank. So, rank is 3 1 2 3 rank is 3. What about here? Here I will get 3 minus I so it will give 3 minus 2 is 1, again 3 minus 2 is 1, again 1 here. So, this will give me a full rank 3. So, rank is 3 plus 1 is 4, 5 and 6. So, rank of this is 6. This matrix is of size what is the size of this matrix 3 plus 3 6 plus 2, 8. So, 8 minus 6 which is 2 alright.

So, the geometric multiplicity of this is 2. If you look at geometric multiplicity of 2 here will be equal to 8 minus, now I need to remove. So, I need to look at now 2 here 2. So, 2 minus that is 2 minus 2 minus 3 will be minus 1 fine, while or let me look at 3 minus. So, let me look at A minus 2I; if I look at A minus 2I. I will look at 3 minus 2 is 1, 1 here and 0 1 1 0 0 1 this is the first part. Second block will give me 1 1 0 1 and the last part will give me 0 1 0 0 0 1 0 0 0 alright, fine.

So, therefore, if I see the rank of this matrix is 3 plus 2 5 plus 1 6 plus 1 7. So, 8 minus 7 which is 1, fine. So, I have certain issues that you can see here that this matrix is not a diagonal matrix and what we see here is that look at the previous examples also our matrix whenever it was not a diagonal matrix, fine so, this number is not same as this number.

Algebraic multiplicity is 5, geometric multiplicity is 2; algebraic multiplicity is 3, geometric multiplicity is 1.

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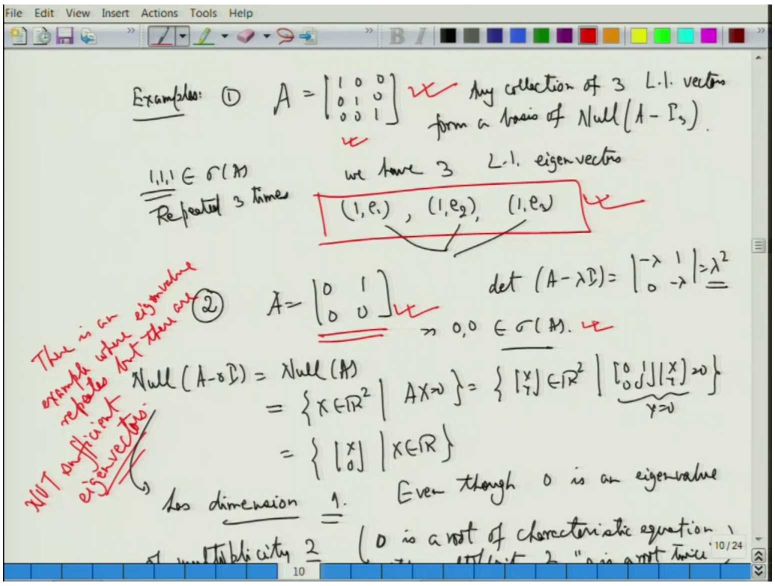

Look at previous examples also wherever we have looked at, fine. Look at this here it is a diagonal matrix. So, algebraic multiplicity is same as the geometric multiplicity because there 1 was repeated 3 and I had three linearly independent eigenvectors. Look at here, fine. Here and I have only one eigenvector fine and this is not an example of a diagonal matrix, fine.

So, we will come across all these ideas in the next class, fine. So, I will like you to understand them which are very important. Now, so, let me just do something now at the last part of this problem; let me do this what are called similar matrix definition.

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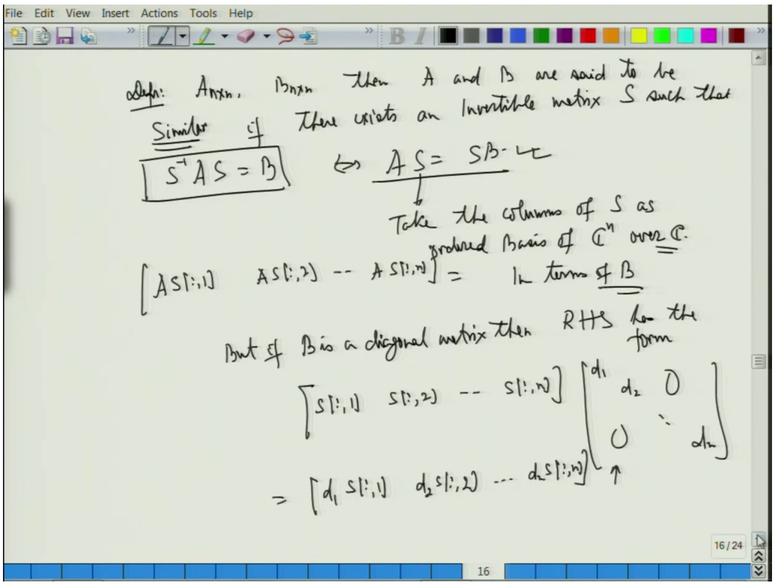

Definition. So, A is n cross n, B is n cross n then A and B said to be similar if there exists an invertible matrix S such that SAS inverse is B alright or recall that I can write this as which is same thing as writing or I should have written S inverse here I think and S here AS is equal to SB is the same thing because you are just changing them.

We are saying that A of S is same as S of B think of S. So, take the columns of S as ordered basis of C n over C, fine. Then what we are saying is that I have got a here I am applying. So, S is there, so, I am applying A to the first column, A to the second column and so on A to the n-th column. I am looking at this and then I am saying that this is same as S times B.

So, now I am looking at the other way around the column wise alright. So, you can write again like this itself in terms of B; in terms of B fine, this is what it is, but if B is a diagonal matrix, then RHS has a nice form RSS has the form alright. So, S I am looking at the first column, the second column, the nth column whether I am multiplying by diagonal matrix say d 1, d 2, d n 0 here 0 here then d 1.

So, this first column is getting multiplied to the first one, second to the second and so on. So, this corresponds to d 1 times S of this due to time S of this and d n time S of this fine. So, look at this in general if I am looking at just any B, I do not get anything, alright. It is just an expression AS is equal to SB and that appeared at the time of looking at matrix of the linear transform composition of matrices and we said that we need to understand that to talk of similarity operation, alright.

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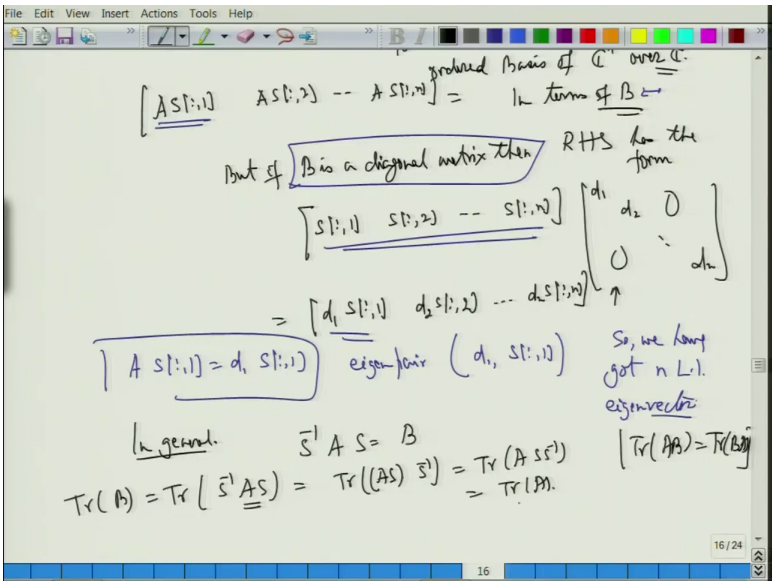

Here what we are saying here is that in this case if B is a diagonal matrix if B is diagonal then what we are seeing is that look at this part. This part, it says that the first column is nothing, but an eigenvector. So, this gives me eigen-pair d 1 and this recall that any invertible matrix cannot have a zero vector inside it. So, S is a nonzero vector. Not only that since S is invertible all these columns are linearly independent.

So, we have got n linearly independent eigenvectors. So, we have got n linearly independent eigen vectors alright. So, I would like you to keep track of that. Further in general, what we are saying here is that in general we are saying that S inverse AS is B. So, let us look at the trace. What is trace of S inverse AS which is same as trace of B alright. So, this is same as trace of I can interchange. So, what we know is that trace of AB is same as trace of BA, fine.

That we had seen earlier. So, this is same as trace of AS into S inverse which is same as trace of ASS inverse, alright, fine which is same as trace of A. So, when I do similarity transformation the trace does not change, fine.

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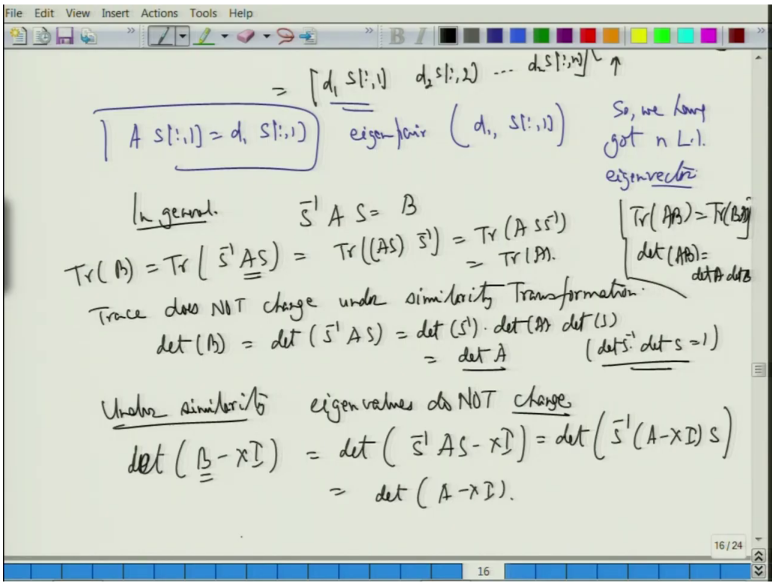

So, trace does not change under similarity transformation. What about the determinant? We have computed the determinant also. So, what is determinant of B? Determinant of B is same as determinant of S inverse AS. I know that determinant of AB same as determinant of A into determinant of B. Determinant of AB is same as determinant of A into determinant of B.

So, let us do that. So, determinant of S inverse into determinant of A into determinant of S which is same as determinant of A, fine because determinant of S inverse S is 1 alright fine. So, you can see that under similarity transformation the trace and determinant does not change can we say about eigenvalues also that the under eigenvalues things will be the same, alright?

So, I would like you to check that under determinant under the similarity. So, under similarity eigenvalues do not change. Why does not change? Because we compute determinant of why because to compute we need to compute determinant of B minus XI which is same as determinant of B is S inverse as minus XI which is same as determinant of S inverse A minus XI into S.

Just multiply X inverse as minus X inverse again. So, you will get back this which by the same argument as above it gives me determinant of A minus XI alright. So, therefore, the eigenvalues will remain the same, eigenvectors will change.

We will go back to this again in the next class. That is all for now.