Linear Algebra

Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 53
Results on Eigenvalues and Eigenvectors

So, let us start the class fine. So, what we learnt was there was a notion of eigen values, eigen vectors, then there was a notion of what are called spectral radius, a spectrum alright.

The notation for a spectrum was sigma of a, and characteristic roots, characteristic values, and characteristic equations and so on alright, characteristic polynomial. So, recall all of them. So, now what we will do is that we will try to look at some of the remarks, discuss on the definitions so remarks.

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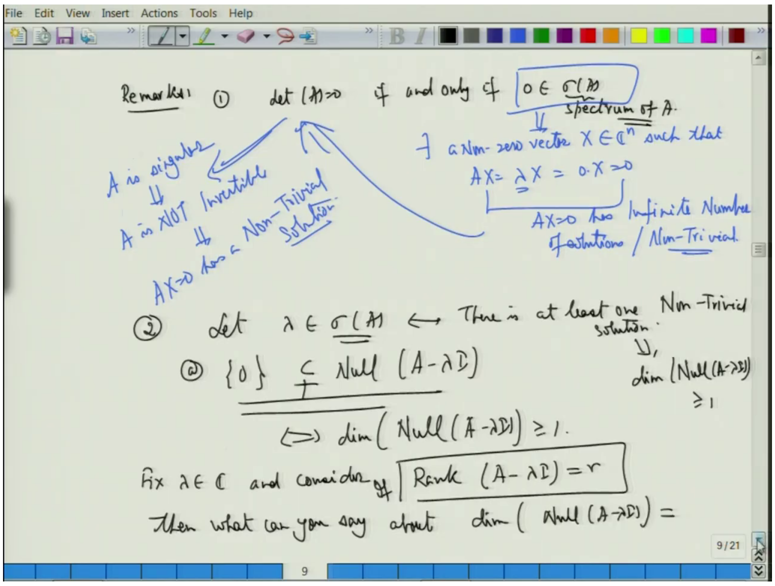

So, 1, now we are saying that we have not yet proved it, but anyway. So, we would like to say that determinant of A is 0 if and only if 0 belongs to a spectrum of A. So, recall this was the spectrum of A alright. So, we are saying that 0 belongs to a spectrum of A.

What do you mean by saying that 0 belongs to spectrum of A? 0 belongs to a spectrum of A basically means this part implies that there exist a nonzero vector X in C n such that A X is equal to lambda X, but here lambda is 0. So, you have 0 times X which is 0.

So, what we are saying is that the system A X is equal to 0 has a non-trivial solution or an infinite number of solution, number of solutions or whatever you want to say a non-trivial solution alright has a non-trivial solution whatever you want to say. And therefore, we know that the matrix A must be non invertible. So, it must be singular and has determinant of A must be 0 fine.

Now, how do I prove that determinant of A equal to 0 will implies that sigma A 0 will belong to that. So, again determinant of A 0 implies. So, determinant of A equal to 0 implies A is singular. A is singular imply that A is not invertible. And this will imply that the system A X is equal to 0 has a non-trivial solution alright. So, you can see that all of them they come together one after the other just as it is alright, nothing special to be done fine.

Remark 2 alright. So, what we know is that if I look at any, so let lambda belong to sigma of A, sigma of A means a spectrum of A alright, then what happens is the first choice for us first is that this vector 0 is properly contained in null space of A minus lambda I fine, because there is at least one. So, there is at least one non-trivial solution. And this will imply that dimension of null space of A minus lambda I will be greater than or equal to 1 alright. This is what it says here fine.

So, therefore, I say that this is true if and only if dimension of null space of A minus lambda I is greater than equal to 1 fine. In general I can have whatever the rank and accordingly I can proceed fine. So, if so fix lambda belonging to C and consider rank of A minus lambda I consider rank of this. If the rank of this is r, if rank of this is r, then what can you say about dimension of null space of this fine.

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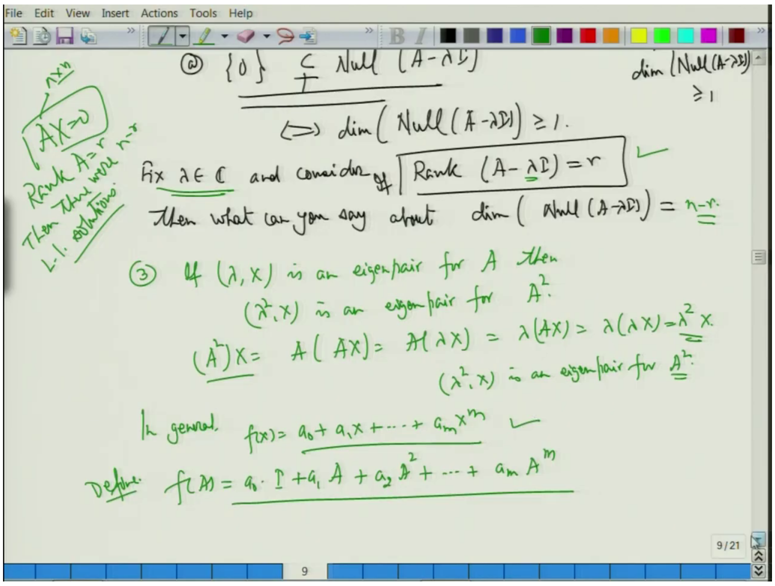

So, recall that when I had the system A X is equal to 0 rank of A was r, then there were n minus r, so A is n cross n, n minus r linearly independent solutions. You can use either the rank nullity theorem alright or you can look at the solution space in the linear system of equations, or you can look at what are called fundamental theorem of linear algebra from wherever you want you can look at it. So, say the rank of this I am fixing lambda, lambda is fixed.

For a fixed lambda if it is r, then this is what I have the dimension of this will be n minus r. Is that ok? Linear independence dependence I have already talked about, I will not talk about it. Third one, fine. So, if lambda is so if lambda comma X is an eigen pair lambda comma X is an eigen pair for A, then look at A square then lambda square comma X is an eigen pair for A square alright. So, let us try to prove this part and from there build up a idea.

So, if I want to look at A square of X A square of X is A of A X which is same as A of lambda X which is same as lambda times A of X which is lambda times lambda of X which is lambda square X alright. So, we can see here that lambda square comma X is an eigen pair for A square fine. So in general, suppose I have got a polynomial F of X which is say a 0 plus a 1 X plus a n X to the power n suppose I have this, so let me a m X to the power n.

Suppose I have this I compute f of A what I define f of A as to define f of A as a 0 times identity plus a 1 of A plus a 2 A square plus a m A to the power m. So, define f of A as this. So, given a polynomial f of X, I am defining what is called f of A fine.

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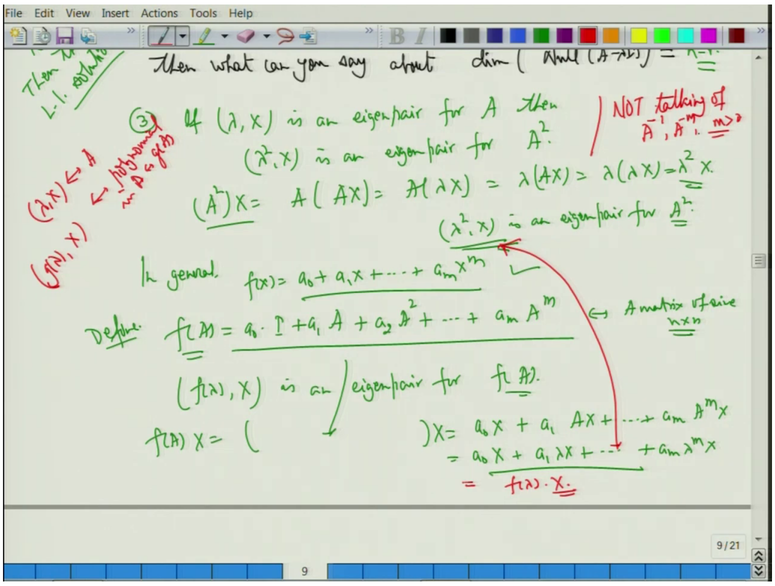

I would like you to see that fine that f of A is a matrix of size n cross n. So, again it will have eigen values and eigen vectors. And I can talk of them. So, I would like you to see that look at this f of lambda comma X this is an eigen pair for f of A. Why? Because just compute f A of X will be equal to this part times X which is a 0 of X plus a 1 times A X plus a m A to the power m X which will give you a 0 X a 0 of X plus a 1 times lambda of X plus a m lambda to the power m X alright.

See we have computed this lambda I square fine. So, just to use this to get this part that everywhere it gets multiplied alright. And therefore, I get that this is equal to a 0, a 1 lambda, a m lambda to the power m which is nothing but f of lambda times X. Is that ok? So, X remains the same and you get all of them with you fine.

So, whenever I have, so whenever I have lambda comma X and eigen pair for A, I can look at any polynomial in A and then corresponding I will get eigen values and eigen vectors. So, if I am taking polynomial g of A, I will have g of lambda and X as an eigen pair fine.

I am not talking of reciprocal; I am not talking of A inverse here that is important, not talking of A to the power minus 1 or A to the power minus, m greater than 0, I am not talking of that part fine. I can talk of them only when A is invertible that I have not yet come, but you can look at it yourself and do it fine.

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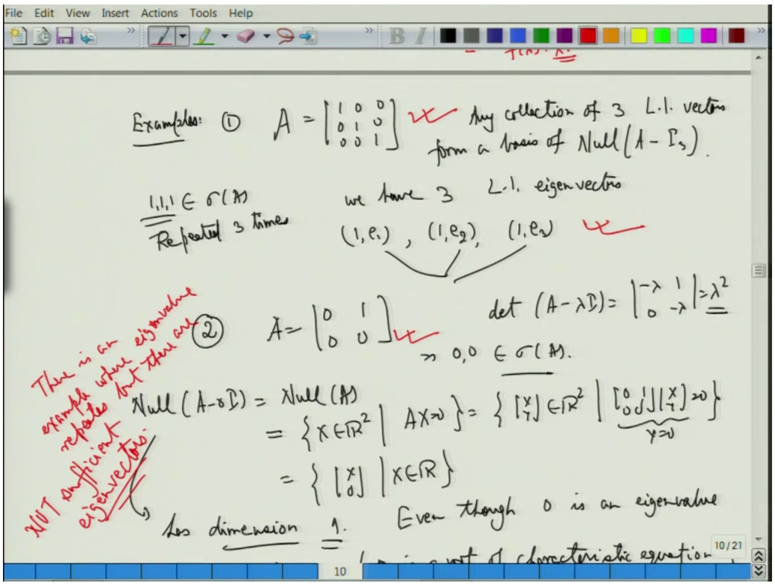

So, let us look at some more examples to go further. So, another examples, examples. So, one example that is very important is what is called an identity matrix. Suppose, I have got I am looking at the identity matrix 1 0 0 0 1 0 0 0 1 fine. If I have this, then you can see that any collection of 3 linearly independent vectors form a basis of null space of so I n here I n minus, so I should write it nicely otherwise you will have a confusion here.

So, let me write A here I think, A minus I sub 3 alright is a basis of this fine. So, what you see here is that 1 1 1 belongs to sigma of A that is 1 is the only eigenvalue. It is repeated 3 times, repeated 3 times, and we have 3 linearly independent eigen vectors. One collection you can take it as 1 e 1 – this is one pair, another pair is this, and the third pair is this, these are standard basis of r 3, or you can take any three, and then you can do it fine.

2, let us look at this matrix A which is 0 1 0 0 alright fine. So, you can see here that determinant of A minus lambda I will be equal to determinant of minus lambda that means minus lambda 1 0 minus lambda which is lambda square. So, this implies 0 and 0 they belong to sigma of A fine. Now, let us look at the null space. Now, let us try to compute the eigen vectors of this. So, let us look at the null space of A minus 0 I which is same as null space of A fine.

So, we want to compute this, I would like you to see that. So, this is all X belonging to R 2 or say C 2 whatever you want to say such that A X is 0 which is same as X Y belonging to R 2 such that 0 1 0 0 times X Y is 0. Now, this gives you Y 0 fine. So, you can see that this is just nothing but all vector of the type X comma 0, X belonging to R alright fine. So, this space this has dimension 1 fine.

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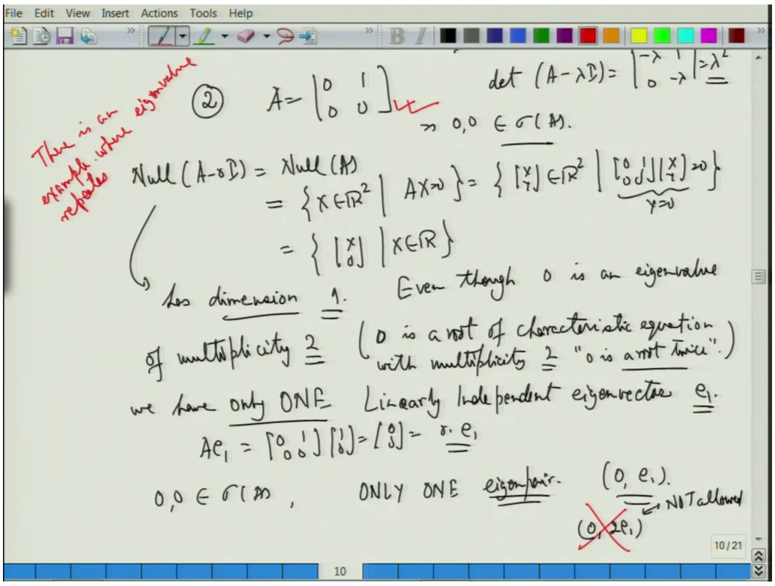

So, even if so even though understand even though 0 is an eigen value; eigen value of multiplicity 2, multiplicity 2 means 0 is an 0 is a root 0 is a root of characteristic equation like characteristic root or characteristic whatever you want to say of equation characteristic root of equation. So, 0 is the root of characteristic equation with multiplicity 2; with multiplicity 2 means 0 is a root twice. So, basically 0 is a root twice alright, this is what I am saying.

We have only one nonzero, or I do not want to say only one nonzero you will get confused, only one linearly independent eigen vector. And you can take it as e 1 itself. So, you can see that A times e 1 is equal to so 0 1 0 0 times e 1 which is 1 0 which is 0 0 which is 0 times e 1 itself alright. So, I have got only one eigen vectors. So, even though I have 0, so even though I have got 0 comma 0 belonging to sigma A, we have only one, so only one eigen pair only one eigen pair 0 comma e 1.

See, so recall I had said that whenever I want to look at eigen vectors fine, they have to be linearly independent. I cannot say that I have this and 0 comma 2 e 1 not allowed alright. So, if I am writing such a thing, then this is wrong. I am allowed only one not more than one is that ok. So, you have to be careful when you make these statements.

So, what we are seeing here is that I have an example here where the eigen value 0 appears twice, but there is only one eigen vector. So, there is an example where eigen value repeats fine. So, there was a repetition in the previous example also which was identity for us alright. There was a repetition there also, but there we had all the linearly independent eigen vectors.

We had 3 linearly independent eigen vectors here, but in this case I do not have 2 linearly independent eigen vectors alright. There is an example of eigen values repeats, but there are not sufficient eigen vectors. Is that ok? So, this is what is important for us. We need to understand this part fine.

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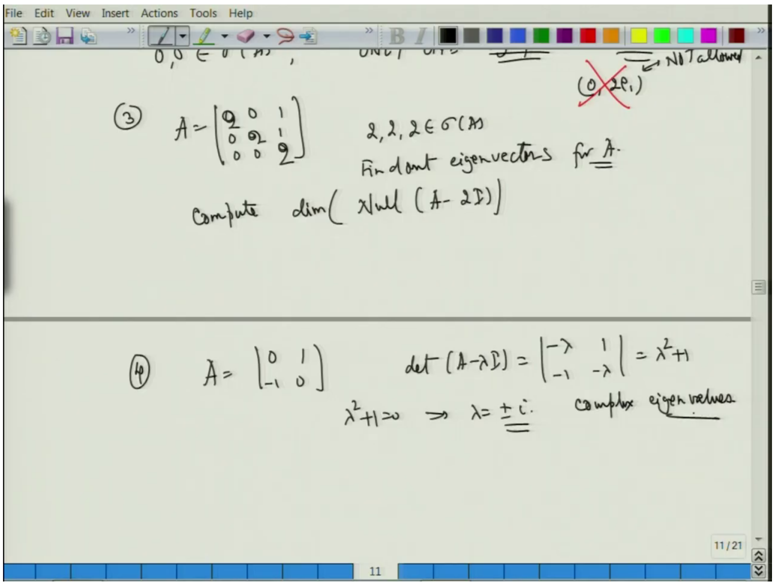

Some more example, alright, example 3 fine. You can look at A as 0 0 0 1 0 0 1 0 0 0. Look at this or in place of this just try to put 2 here, 2 here and 2 here, then you can see that 2 2 2 belongs to sigma of A find out eigen vectors for A or compute dimension of null space of A minus 2 I. Compute this out and check that how many eigen vectors do you get is that ok. So, try that out yourself.

Fourth look at this matrix A which is 0 1 minus 1 0 alright fine. So, if I look at this, so everything here is real entries, let us compute determinant of A minus lambda I. So, A minus lambda I determinant of this will be 0 minus lambda 1 here alright, minus 1 here and minus lambda here.

So, this is same as lambda square plus 1. So, if I want to get the roots eigen values of this, so eigen values of this will be you have to write lambda square plus 1 is 0, and this will imply that lambda is equal to plus and minus I fine. So, we have complex eigen values.

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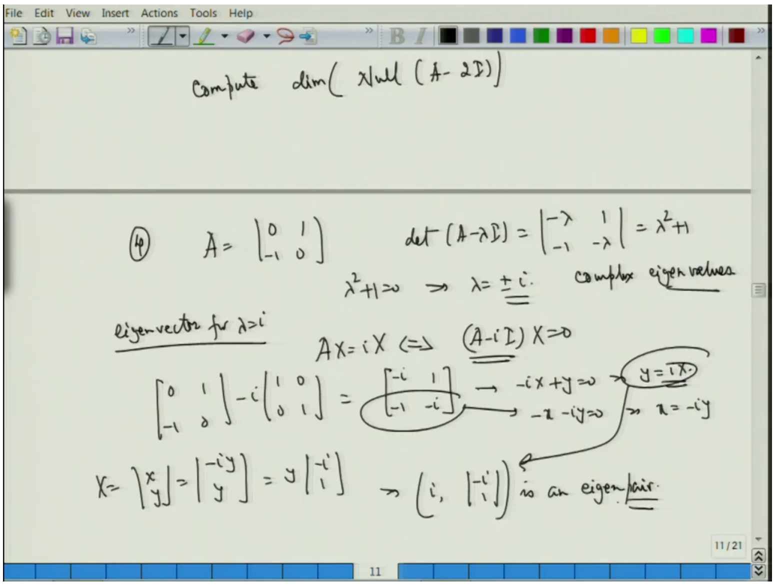

What about the eigenvectors? Let us look at the eigen vectors for this. Eigen vector for lambda is equal to i fine. So, now, looking at lambda equal to i, I have to solve A X is equal to i X which is same as solving for A minus i times I of X is equal to 0. So, let us solve it for A minus this. So, A is this 0 1 minus 1 0 minus i times 1 0 0 1 which is same as minus i 1 minus 1 minus i fine.

So, I have to solve this system. So, I will have to do the RREF of this. I would like you to see that this gives me minus i times x plus y is 0 implies y is equal to ix. We can also write in terms of x, using this you can see here that I can write this as minus x minus iy is 0 will imply that x is equal to just look at this x is equal to minus iy fine. So, whatever way you want you can do that.

So, if I use this part, what I get here is and just multiply minus i will get back the old one alright. So, you are looking at X which is x y. So, this is same as I want to write alright minus i y and y, the other way which is same as y times minus i comma 1 alright. So, this implies i and minus i comma 1, this is an eigen pair.

Let me check whether I have done it correctly or not, because I am good at mistakes. So, with i it is I wrote the other way around I think alright. So, we are looking at this. And let us see that this is equal to I can write this also as, so as I said if lambda X is an eigen pair then lambda comma C X is also an eigen pair for c not equal to 0.

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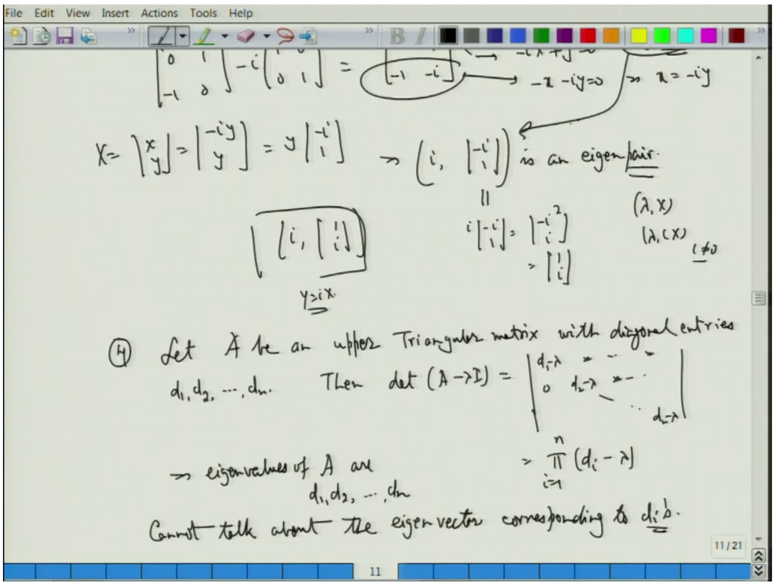

So, I can also multiply this by i, if I multiply this by i, so i times minus i comma 1 will be equal to minus i square comma i and minus i square is 1. So, it is 1 comma i. So, i comma 1 i is also an eigen pair alright. And this is the one which corresponds to this part y equal to ix, this corresponds to y is equal to ix fine. Another example, let A be an upper triangular matrix with diagonal entries with diagonal entries d 1 d 2 d n.

Then what we know is that then determinant of A minus lambda I will be equal to, so I am looking at d 1 minus lambda, then there will be somethings here similarly 0 d 2 minus lambda and something here. Similarly, finally, it will be d n minus lambda. And therefore, since it is an upper triangular matrix determinant of this is nothing but product i is equal to 1 to n di minus lambda and this will imply that eigen values of A are d 1, d 2, d n.

At this stage, I cannot talk about cannot talk about the eigen vectors corresponding to d i’s. Why I cannot talk, because I do not know how many of them are repeated, how many of them are not repeated, and how they behave. You do not have any information, so you cannot make any statements alright.

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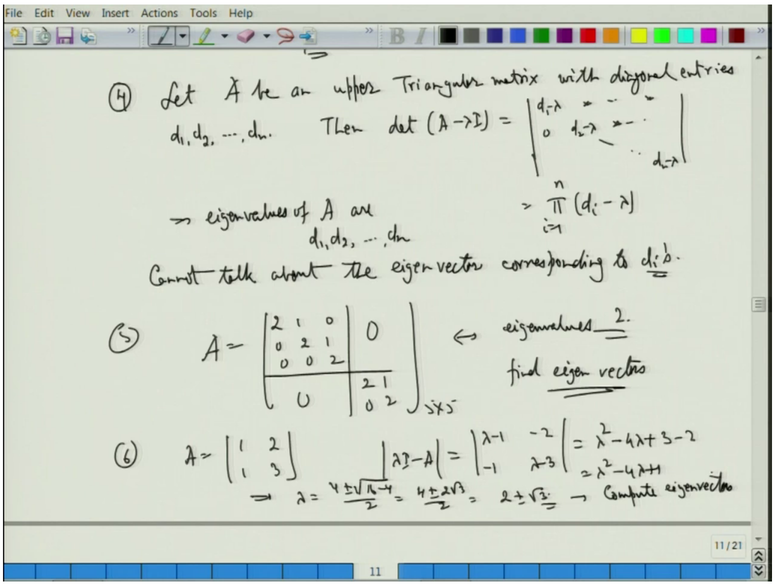

So, to make you slightly better understand this, let us look at this example A as 2 1 0 0 2 0 0 0 2. Let me write this matrix. Let me make it 1 here. I have a 0 here, 0 here, 2 here, 1 here, 0, 2.

So, look at this 5 cross 5 matrix all its eigen values, the diagonal, this is an upper triangular matrix upper trianglular with all its eigen values 2, eigen values 2. Find eigen vector. So, we will try that out find eigen vectors. So, try that out, so that you have a better clarity of what I am trying to say, what are the issues that come into play fine, that is very important for us alright.

I have already given you I one more example I think I should give that I can go from real to complex, now real to complex I have already done, from rationals to irrationals may be alright. So, let us write A as 1 2 1 3. Let us look at this matrix A. Let us compute the eigen values of this matrix. So, I need to compute. So, till now I have been computing A minus lambda I. Let me compute lambda I minus A. So, this is just a polynomial.

So, I could compute in whatever way I like. So, this is same as determinant of lambda minus 1 minus 2 minus 1 lambda minus 3 which is same as lambda square minus 4 lambda plus 3 minus 2 which is lambda square minus 4 lambda plus 1. I hope I have done correctly, I am very bad at calculation as I said. So, this will imply that the roots of this are 4 plus minus b square b square 4 a c upon 2 which is same as 4 plus minus 2 root 3 upon 2 which is same as 2 plus minus root 3.

So, you have eigen values here. You can look at eigen vectors also, you can compute them accordingly. So, compute eigen vectors for yourself alright. So, compute them yourself.

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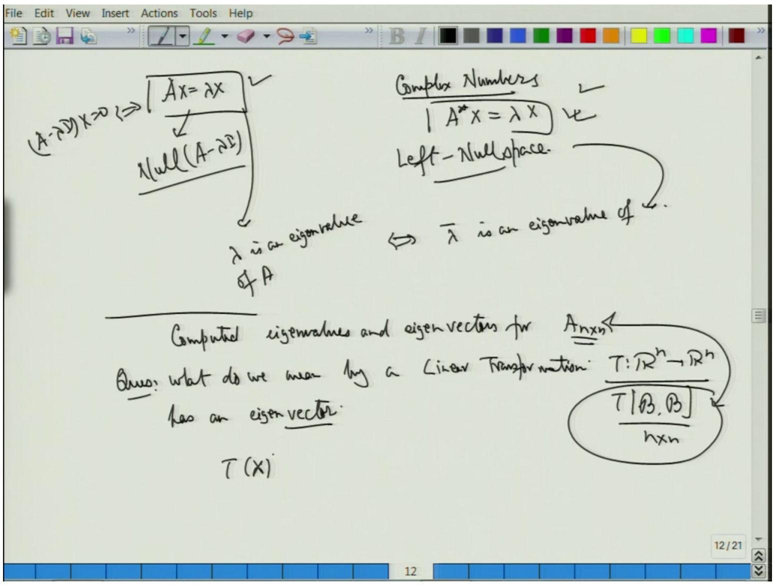

Then there is also a notion of what are called. So, here if I look at what we have done we have looked at A X is equal to lambda X fine. Since I am working over complex numbers, we need to talk of A star of X is equal to lambda of X also.

See you recall this corresponds to what you looking at the column space of A in some sense, then there is a notion of or you can say that I am looking at if you do not want to have this, then I am looking at null space of A minus lambda I is not it. So, this is equivalent to looking at A minus lambda I of X is equal to 0.

So, I am looking at the null space of this. So, I need to look at what are called left null space also. So, I need to look at this also fine. But the theory will be same other than here I am talking of A, there I am talking of A star, and therefore I am not get going into it alright.

There are some results that if there is an eigen value here, there is an eigen value here, will they be same, will they not be same and things like that can I say that if lambda is an eigen value lambda is an eigen value of A, can I say that this will imply and get implied by that lambda bar is an eigenvalue of this and so on.

So, there are theories that you would need to look at it look at examples and do it yourself alright. When I come to what are called normal matrices we will prove some results there, but at this stage I want you to do them yourself alright. So, we have looked at some examples here. One thing I we need to talk off what are called so we have computed or understood computed eigen values and eigen vectors for matrices alright, n cross n matrices I have computed.

What do I mean by saying that a linear transformation has an eigen vector. So, question what do we mean by a linear transformation T from R n to R n has an eigen vector alright. So, recall that since T is from R n to R n fine I can take an ordered basis of R n and this will give me. So, if I have got an ordered basis here, so this will be an n cross n matrix.

So, once I have an n cross n matrix, it is nothing but A of n cross n for some A, and therefore, I can talk off eigen values and eigen vectors fine. So, I can still talk off in that sense T of X; so, T some linear map.

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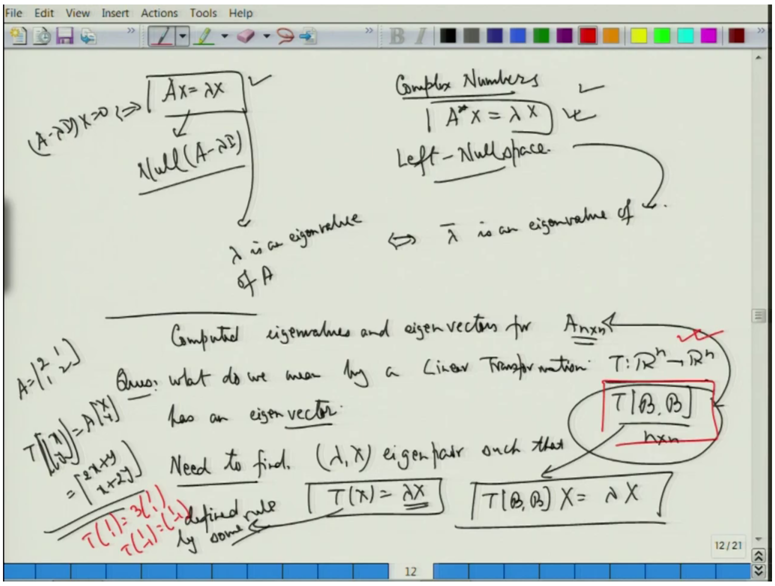

So, we need to find; need to find lambda comma X eigen pair such that T X is equal to lambda X. So, what we are saying here is that with respect to this matrix I have got T of B B with me, I want to find an X such that this is equal to lambda times X. Is that ok? So, you can do to the standard basis also whatever way you want, but T of X is defined. So, you are defining T of defined by some rule, and then you are asking for X all those X for which T of X is equal to lambda X alright.

So, you can look at here say for example, if I take A as 2 1 1 2, then I can define my T of x y as A of x y. So, this will be equal to 2 x plus y and x plus 2 y. So, I define T of X like this then I would like you to check that if for this example T of 1 1; T of 1 1 is equal to 3 times 1 1, and T of 1 minus 1 is 1 minus 1 alright So, I will have such eigen pairs and eigen vectors here also.

So, the idea is that when I want to go to linear transformations from R n to R n and I want a basis of I want a eigen values and eigen vectors of this, then I am supposed to look at T of B B for some ordered basis B get a matrix out of it, and then proceed. Is that ok?

So, that is all for now, we will look at things in the next class.