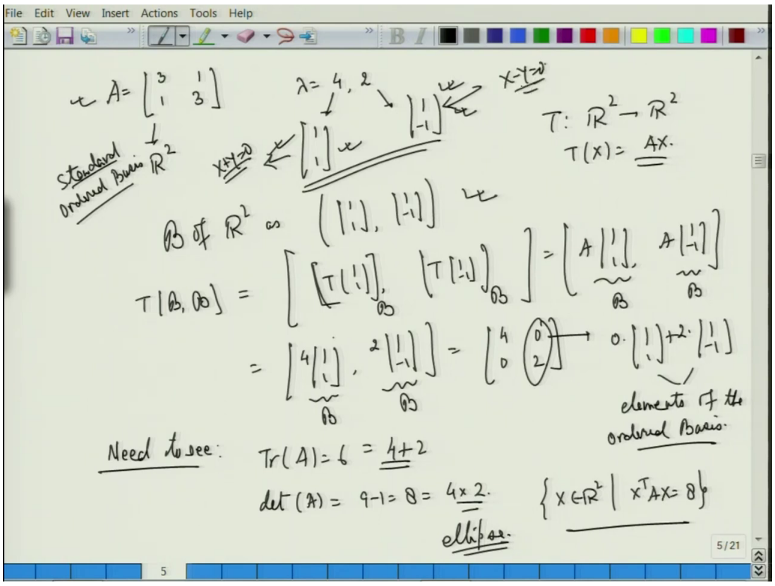
Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 52
Examples and Introduction to Eigenvalues and Eigenvectors

(Refer Slide Time: 00:17)



So, in the previous class, we had this symmetric matrix A and for that we have found out the eigenvalues as 4 and 2; I am not yet given actual definitions, we will give it afterwards. And then there was this corresponding things that I have here 1 1 and 1, minus 1. I would like you to understand that, I have this matrix A with me and this was with respect to the a standard basis of R 2. So, this matrix when we write the standard ordered basis.

So, whenever we write a matrix, generally it is in terms of a standard ordered basis. Now, I want to take my ordered basis B as B of R 2 as 1 1 and 1 minus 1. If I take this as my ordered basis, then this matrix I can define my operated T from R 2 to R 2 by T of A is equal to A X.

And now if I want to look at the matrix of T with respect to this basis B here and B here, then recall that I need to compute T of 1, 1, I need to compute T of 1 minus 1, I need to compute these two, and this has to be evaluated with respect to the ordered basis B. So, this is my matrix T B, B, fine. So, this is same as T of 1, 1 alright; look at T of 1 1 it is A times 1 1 and T of 1, minus 1 is A times 1, minus 1 which is nothing, but 4 times 1 1.

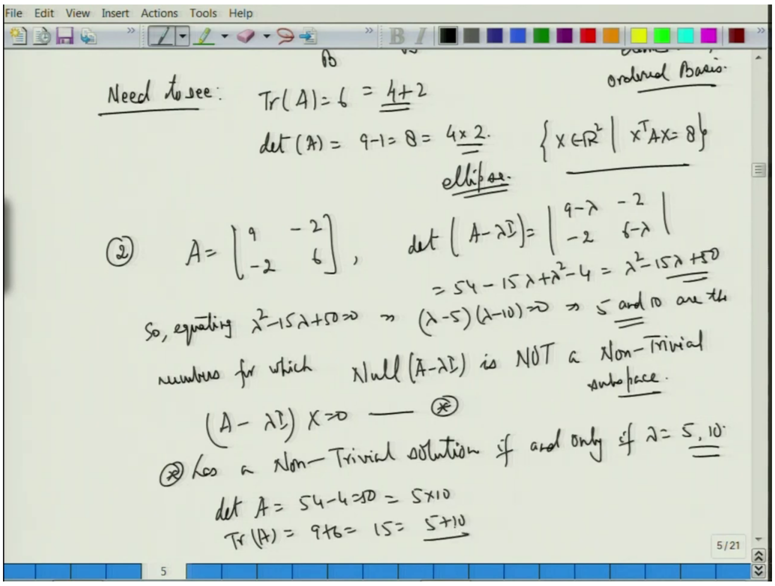
And so, I have to evaluate with respect to, these have to be evaluated with respect to B and that I will do afterwards. And this is nothing but 2 times 1, minus 1 and this has to be evaluated with respect to B for me. So, when I evaluate with respect to B, what I get here is 4, 0; because this is 4 times the first vector and 0 times the second vector and this is 0, 2, because it is 0 times.

So, if I look at this part, it is 0 times 1, 1 plus 2 times 1, minus 1 and these are the elements of the ordered basis, fine. So, what we see is that ok, we have got an ordered basis which has come from some calculation; we will understand those calculations, alright and things have been nice. That is one thing that we understand that, I am able to get matrix A in terms of a diagonal matrix with respect to some ordered basis, alright. That is one thing that we see.

Another thing that we need to see here is that ok, need to see; look at the trace of the matrix, trace of the matrix A is 6 which is same as 4 plus 2, alright. And let us look at the determinant of A; determinant of A is 9 minus 1 which is 8, which is same 4 times 2, fine. And we also saw that, since A is symmetric alright; we could talk of x transpose A X being 1. So, we could also talk of all X belonging to R 2; X belonging to R 2, such that X transpose A X is 8 or something I wrote, I do not remember it exactly what I wrote.

So, this gave me an ellipse. And the minor and major axis of this ellipse came from these two vectors 1 upon 1 and 1 and 1 minus 1, alright. So, they it gave me the minor axis as one of the axis as x plus y is 0; this gave me one of the axis as x minus y is 0, alright. So, the question is; it is just the chance factor or is there some theory which gives you all these things, fine?

(Refer Slide Time: 04:18)



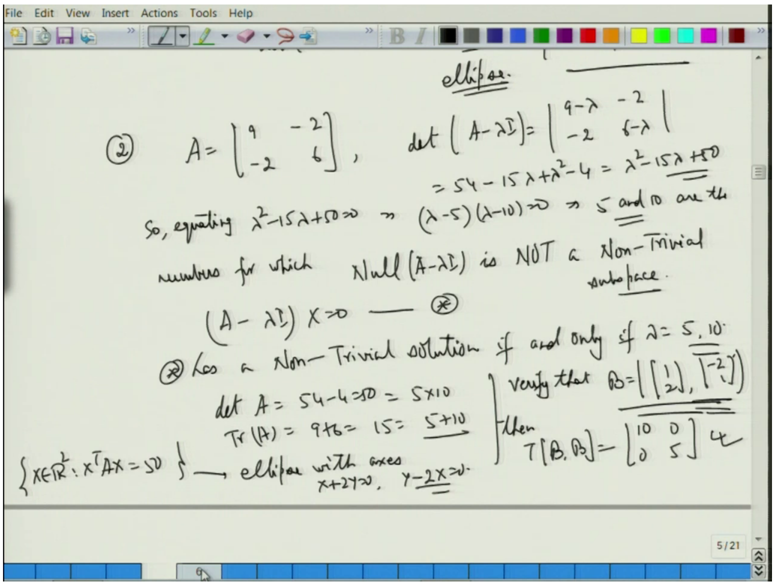
So, let us look at another example, example 2. So, this time I take the matrix A as 9, minus 2, minus 2, 6. So, as done previously would like to compute determinant of A minus lambda I, which is same as 9 minus lambda, minus 2, minus 2, 6 minus lambda.

So, this is same as 54 minus 15 lambda plus lambda square minus 4, which is same as lambda square minus 15 lambda plus 50. So, equating lambda square minus 15 lambda plus 50 is equal to 0 gives me lambda minus 5 into lambda minus 10 is equal to 0, alright fine.

So, this implies 5 and 10; 5 and 10 are the number that we need to look at, are the numbers for which null space of A minus lambda I is NOT a non trivial subspace. Or if you want to say what we had looked at is that ok, if I want to look at A minus lambda I, this is equal to 0.

Then this star has a non trivial solution, if and only if lambda is equal to 5 comma 10. These are the only two choices for which it will have a non trivial solution star; otherwise it will have a trivial solution, fine. Here again I want you to note that, look at determinant of A. Determinant of a is 54 minus 4 which is 50, which is same as 5 into 10 and trace of A is equal to 9 plus 6 which is 15, which is same as 5 plus 10, fine.

(Refer Slide Time: 06:44)



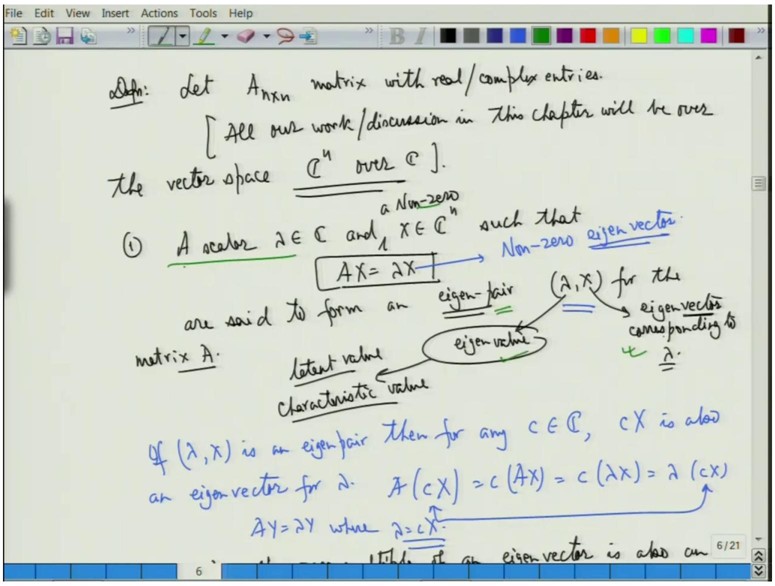
What about getting an ordered basis? So, I hope I have done my calculations correctly. So, I would like you to verify that. So, verify that, if you take B as. So, B as 1, 2 and minus 2 comma 1; if I take this, this is again have symmetric matrix you can see here, alright. If I verify that, if I take B as this; then T the way we defined earlier T of B, B will be equal to 10. So, it will be 10 here, 0 here, 0 here, 5 here, alright.

So, this is the way it will look like, just try that out yourself. And again you can talk in terms of it will turn out to be an ellipse. So, if you want to look at X transpose A X is equal to some number say 50, X belonging to R 2 such that this happens. So, then it will be an ellipse with axes.

So, find out the principal axes; one of them will be x plus 2 y is equal to 0 and the other x, other will be y minus 2 x is equal to 0, alright. So, just look at depending on this parts that I have got the ordered basis; the axes comes from there, is that ok, fine.

So, I will not look at the ones which are slightly different, in the sense that which do not which are not symmetric; but for symmetric matrices also, non symmetric matrices also you can get such things, alright. So, let me now go to the theory. We have seen certain ideas, certain things here and now I want to understand things, fine. So, let us go with definitions and show on motivation is enough, so definition.

(Refer Slide Time: 08:31)



So, let A be an n cross n matrix with real entries or complex entries, with real oblique complex entries. I would like you to emphasize one thing here that, all our work; all our work or discussion in this chapter alright, will be over the vector space C n over C, fine.

So, we are not going to look at R n over r; because there will be lot of problems that will come into play, the theory has to change. So, we will look at C n over C, so that you can say that, if I have a polynomial of degree n; I will have exactly n roots, alright. I cannot talk of that when I am looking at real numbers; for real numbers, for example, the polynomial x square plus 1 does not have any root, fine. It is only complex numbers that comes into play.

So, therefore, we will be looking at only complex number, even though our examples almost, all the examples will be real numbers; but we will need C n over C to do our work alright, otherwise you cannot build up the ideas, fine. So, for us we need C n over C. So, recapitulate yourself and then come back alright to understand things.

So, definition, so this is the assumption with definition 1, alright. A scalar lambda belonging to C and X belonging to and X belonging to C n lambda and a non-zero is belonging to C n, such that A X is equal to lambda X the scalar lambda and these are said to form an eigen pair lambda comma X for the matrix A, alright.

So, I am saying eigen pair, fine. So, lambda is called an eigen value and X is called an eigen vector, vector. So, it corresponds to something so corresponding to lambda, alright. So, you have to set together, eigen pair when you say, then lambda is an eigen value and x is the corresponding eigen vector, is that ok.

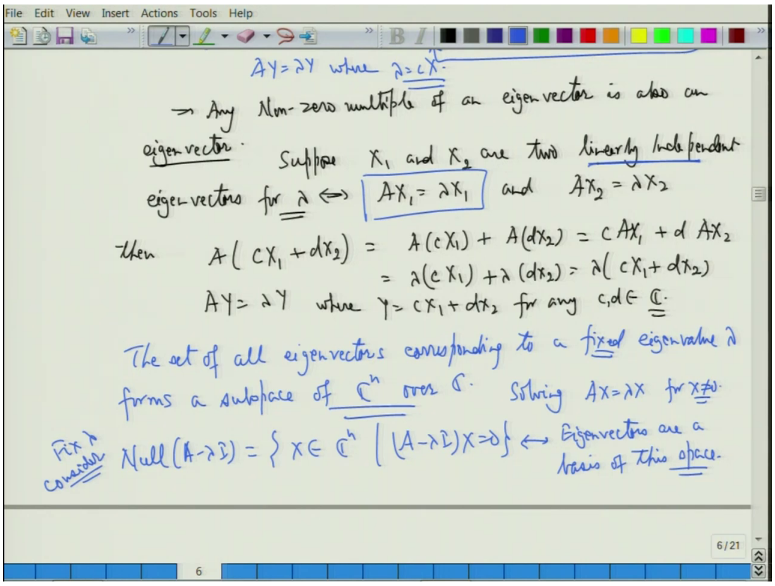
So, that is important for me; further you need to understand something which are important that, many books they use the word in place of eigen value, they use a word what is called latent value alright, or they may also use the word characteristic value.

And then other names also, I will not get into those parts; but find out yourself, alright. So, latent root, latent value, characteristic root, characteristic value and so on; there are so many names that they have, but the idea is simple that, you are looking at the system A X is equal to lambda X and you want lambda to be a scalar, complex number and X to be a non-zero eigen vector, x has to be non-zero, this X non zero eigen vector it is very important.

Zero is not allowed, fine. Not only that, now suppose that I have this; lambda comma X is an eigen pair. So, if lambda comma X is an eigen pair; then for any c belonging to complex number c times X is also an eigen vector for lambda, alright.

Verify it A times lambda X, A times c X [noise is equal to c times A X, which is c times, A X is nothing, but lambda X, which is same as lambda times c X, alright. So, you can see that you get here A Y is equal to lambda Y, where lambda is equal to c times X, fine. So, any multiple of an eigen vector is also an eigen vector, alright.

(Refer Slide Time: 13:42)



So, what it means is that ok, this implies any non-zero multiple of an eigen vector is also an eigen vector, fine. Further if I suppose to, suppose. So, suppose x 1 and x 2 are two linearly independent eigen vectors for lambda, lambda is the same, alright.

So, then what we are saying is that ok, A X 1 is equal to lambda X 1 and A X 2 is equal to lambda X 2; then A times c X 1 plus d X 2 is same as A of c X 1 plus A of d X 2 which is same as c times A X 1 plus d times A X 2, which is same as again lambda will come outside, lambda times c X 1 plus lambda times d X 2 is same as lambda times c X 1 plus d X 2.

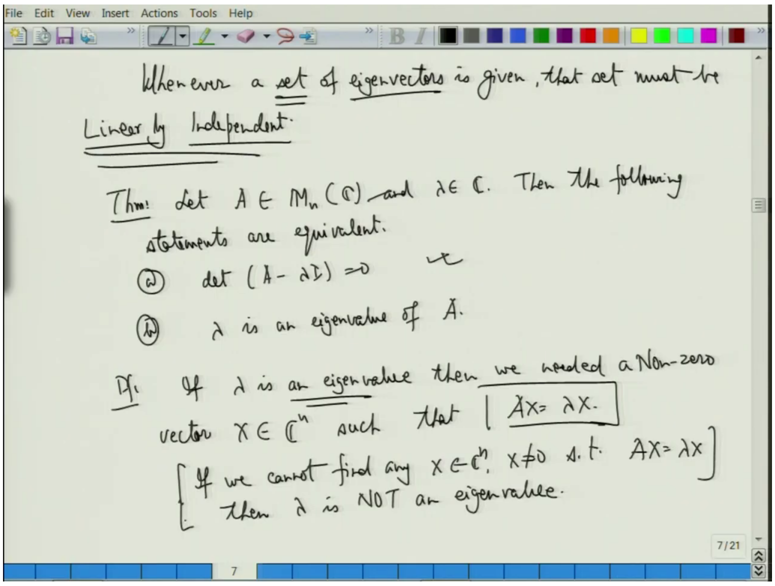
So, you can see that A of Y is equal to lambda Y, where Y is c X 1 plus d X 2 for any c comma d belonging to the complex number, alright fine. So, what we are saying here is that ok, if I have a eigen vector X; then a non-zero multiple of this is also an eigen vector. I need a non-zero basically, because an eigen vector is suppose to be non-zero vector, alright.

That is one things. If X 1 and X 2 are two linearly independent eigen vectors; if they are two linearly independent eigen vectors, then their linear combination is also an eigen vector. So, what we are saying is that ok, the set of all eigen vectors corresponding to a fixed eigen value lambda, lambda forms a subspace of C n over C, alright.

This is important; it forms a vector subspace, fine. And if you see basically what we are doing is, at each stage we are solving the system A X is equal to lambda X. So, basically what we are solving is solving A X is equal to lambda X, for X not equal to 0, alright fine. So, what we are doing here is, we are looking at all X belonging to C n, such that A X or such that A minus lambda I of X is equal to 0 vector.

So, this is called the null space of a minus lambda I. So, fix lambda and consider this, consider this, alright. So, all the eigen vectors they come from here. So, eigen vectors, vectors are a basis of this space. So, important thing that, you need to understand here that, when I am looking at eigen vectors; so when I want to give you a set of eigen vector, those eigen vectors must be linearly independent, alright.

(Refer Slide Time: 17:48)



So, whenever you give, whenever a set of eigen vectors is given, a set consisting of eigen vectors is given that, set must be linearly independent. So, if you give a set which has, which you say is linearly dependent at the same time it is an eigen vector, then the two things do not make sense, alright.

That is the statement which is vacuous, which does not make sense. So, linear dependence and eigen vectors they do not go together; when you are saying that you have given a set which is set of eigen vectors, that set must be linearly independent fine, that is very important.

So, linear independence is very very important; because you want a basis of this null space of A minus lambda I, you do not want anything else, fine. So, I given you the idea of what is called the characteristic vector, what an eigen vector, null space, latent root or characteristic root or eigen values and so on, fine.

So, let us try to now prove results for these ideas, alright. So, theorem, fine. Let A belong to M n of C. So, as I said most of our examples should be real numbers, but will be working over complex numbers only, alright fine and lambda belong to c, then the following statements are equivalent.

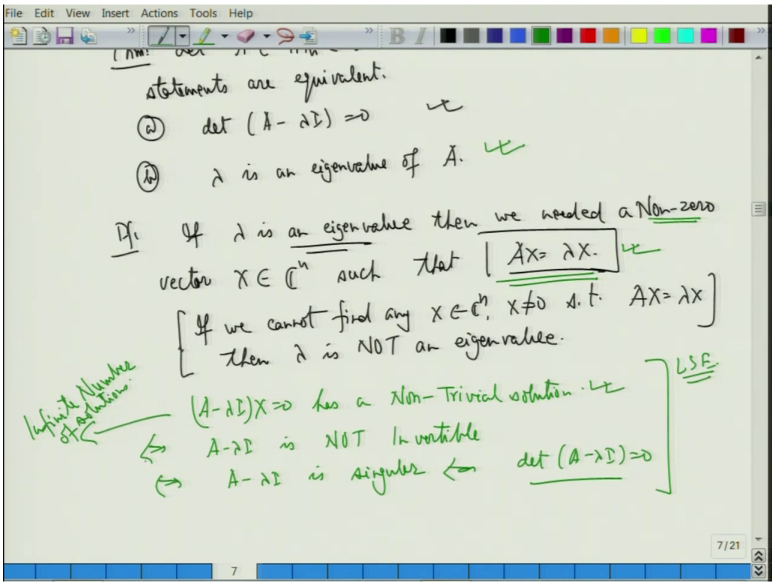
A determinant of A minus lambda I is 0, B; oh I wrote the other way around, alpha I lambda is an eigenvalue, eigenvalue of A, fine. So, let us try to prove this, what do I mean by saying this part that, the two are equivalent; means whenever this is there, then I am saying it is an eigen value. But the idea of eigenvalue and eigen vector had nothing to do with the determinant.

So, the idea of eigen value and eigen vector was that, if lambda is an eigen value alright; then we needed, then we needed a non zero vector X belonging to C n, such that A X is equal to lambda X, alright. So, if I cannot find. So, if we cannot find any X belonging to C n; X not equal to 0 such that A X is equal to lambda X, then lambda is not an eigen value, alright.

This is what we have to be careful about that, the definition clearly says; look at the definition here that, a scalar lambda belonging to C and the non-zero X, such that this happens are said to form an eigen pair and from that eigen pair, you get the eigen value and then corresponding eigen vectors. If you cannot get an X which is non-zero satisfying A X is equal to lambda X; then you do not have an eigen pair that is very very important, alright. So, that comes with that part.

So, let us try to prove that a and b are equivalent. So, assume that b is true; if b is true means, you have ah A X is equal to lambda X coming in to play. So, X is a non-zero solution.

(Refer Slide Time: 22:06)



And therefore, A minus lambda I of X is equal to 0 has a non-trivial solution and whenever we have that A minus lambda I of X has a non-trivial solution implies that, fine A minus lambda I is not invertible.

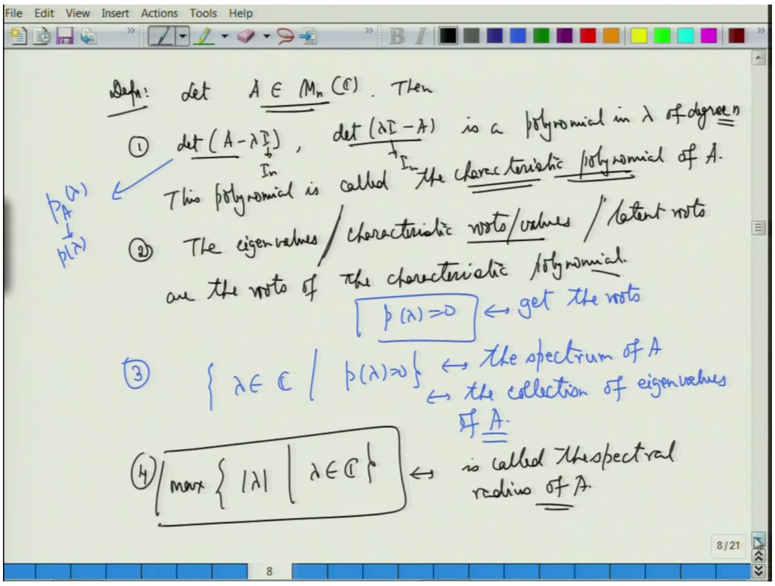
So, recall the 6 or 7 condition, that I had and which implies that, A minus lambda I is singular. And what does singular mean? Determinant of A minus lambda I is 0, alright. So, recall things here very nicely, there were 7 or 8 in the chapter one vector spaces, we had only 6 of them; when we went to determinant, we added that part and things like that fine.

So, linear system of equations and invertibility and so on, you all have these things, alright. So, this was defined only at the end of chapter 2 which was about linear system of equations alright; all these conditions were given to you. So, if lambda is an eigen value of A implies that, A X is equal to lambda X has a solution for some choice of x which is non-zero.

Since x is nonzero; so it has infinite number of solutions, alright. So, this has an infinite number of solutions. Once you have infinite number of solutions; so the matrix must be non-invertible, it should not be invertible. And therefore, it is singular; because they are all equivalent conditions and therefore, your determinant of A minus lambda is 0.

Now, a implies b, suppose determinant of A minus lambda is 0; this will imply that A E minus lambda is singular, that will imply that A minus lambda I is not invertible and therefore, the system will have infinite number of solutions, alright. Since it has infinite number of solutions, I can choose one which is non-zero and get my answer, fine.

(Refer Slide Time: 24:16)



So, they are all equivalent, there is nothing special about it, fine. Now, some more definitions; what are called characteristic values and things like that. So, let A belong to M n of C; I may not write again and again this part A belong to M n of C, we will just assume it is. Then one, determinant of A minus lambda I or in many books they look at determinant of lambda I minus A. So, A is n cross n. So, I is also it is I 7 that you have, fine.

Now, what is determinant here? This is a polynomial in lambda of degree n, is that ok. And this polynomial, this polynomial is called the characteristic polynomial, alright. So, this is the reason. So, what we are saying is that ok, I compute the determinant of lambda I minus A; it is a polynomial in lambda of degree n and this polynomial is called characteristic polynomial.

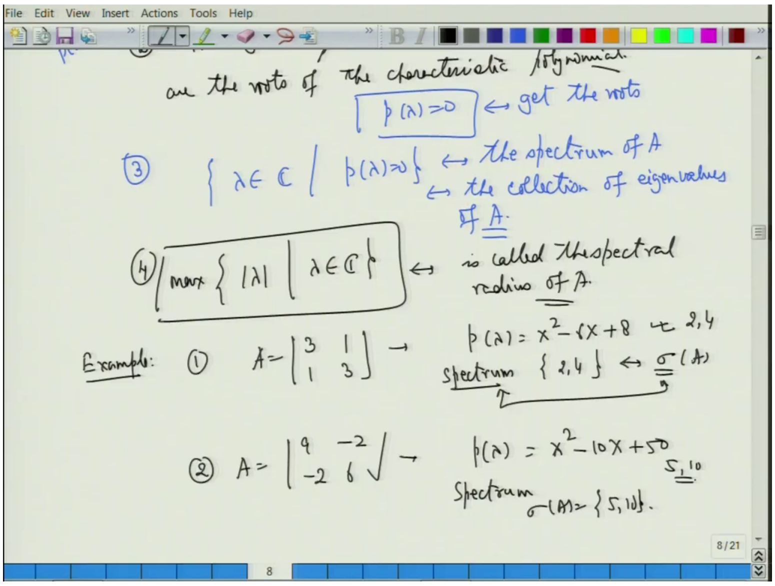
So, the roots of this polynomial are called characteristic roots, fine one. Two, so the eigen values, the eigen values or characteristic roots or values alright or latent roots are the roots of the characteristic polynomial characteristic polynomial, fine.

So, if I write this, if I donate this polynomial as p. So, books have different notations. So, I do not know what I have used p of A; no it should be p of lambda A, p lambda of A or just p lambda, fine. Then we are saying that, just compute p lambda put it to 0 and get the roots, get the roots, fine.

Third, look at all the set lambda belonging to C, such that p lambda is 0, alright. So, this set is called the spectrum of A or which is same thing is saying that, the collection of eigen values of A, is that ok. So, in the first example, if you recall; fine.

So, the in the first example where this notion of or let me give one more definition; look at the set mod lambda such that lambda belonging to C. So, this is a collection of numbers, I want to look at the maximum of this. So, this maximum alright; this is called the spectral radius of A, alright.

(Refer Slide Time: 28:08)



So, they have different definition that you have here. So, example, let us go back to example; the first example that we had was, I took A as I think 3, 1, 1, 3. So, for this matrix p lambda was equal to x square minus 6 x yeah plus 8, fine. Another example I took A as; I do not remember it, so let me go back and see what I had taken, it was 9, minus 2, 9, minus 2, minus 2, 6.

And for this p lambda was x square minus 10 x plus 50, fine. So, look at these the roots of this were 2 and 4, fine. So, therefore, a spectrum is, a spectrum here is 2 comma 4; we write it as a set. And do we write it generally at sigma of A, sigma for a spectrum here, is that ok.

In this case the, we had 5 and 10 as our roots. So, 5 and 10 are the things, a spectrum was. So, a spectrum which is sigma of A is 5 comma 10, is that ok. A spectral radius here is 4 and here the spectral radius is 10, alright. So, that is for now, we will look at something's more in the next class.

Thank you.