Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 51
Motivation on Eigenvalues and Eigenvectors

Alright. So, now, we want to start what are called the study of eigen values and eigen vectors fine. So, before I do that, in the last class, we had seen that we had the system A X is equal to b. We tried to solve it; we could not solve it because of certain restrictions. And therefore, we had to go to the approximate solution or the best solution or the nearest solution, all these things we could do.

Now, the problem that we have is that we are saying that we can solve it, but the data that we get the matrix that we get may be a very huge size, it could be 100 by 100, 200 by 200 size, million by million depending on what applications we are looking at that is one thing.

And the data may be incomplete it may not have all the entries there. The entries even if it is all of them is there, there could be some ambiguities in it, in the sense that they are not exact the way it should be have come some person have put some entry and so on, there is always data error and so on that comes into play fine.

Further, your computing power may not be so good to take care of the things that are coming one after the other, the your calculations may not be so fast as you want because things are changing very fast. Whether you are in the stock market or you are trying to catch a thief who is running away or anything like that, where things are happening at a very fast pace is very difficult to keep track of things. So, what you want is that, you want to keep track of only few things which are important.

Another example will be when you look at health each of one of have some health profile we have some ailments and so on. So, whenever you go to a doctor, he or she prescribed such that do these tests certain tests are done. And based on that test, the doctor decides that what ailment you have, sometime the doctor is successful, sometimes he is not fine. Similarly, there are psychological study there other things to find out what are the issue, what are not the issues and so on, fine.

So, at each stage, we have some data points which comes either from say your blood test, or from your health, previous health status and so on. Or if you are if there is a financial company who wants to try to lure you into buying certain things, then he or she will try to look at your expenditure profile; in the sense that which are the shops, which are the hotels, which are the malls that you have gone and spent how much money.

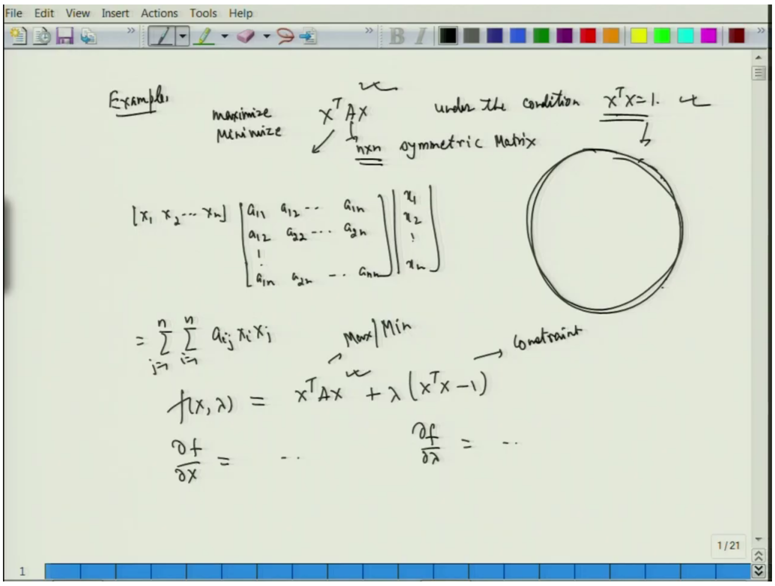
They may not know actually what exactly you bought, but they will know that you have a spent so much of money in certain food items, or at certain places, or in certain malls where these things are available and so on. And then they can try to correlate those ideas. And based on that, they will try to lure you into going things.

So, what I am trying to say is that a person may have full information about you a computer or it is, but it may not like to look at everything it may just like to concentrate on only a few aspects based on which it can lure you into certain things or it can diagnose your heath issues and so on alright. So, all the data that I have may not be useful, only few things may be useful to give me approximate ideas alright or solutions. And hence I may not need the full matrix.

So, the eigen values and eigen vector idea is related with those ideas. In the sense that as far as we are concerned, we will study all of eigen values and eigen vectors how to compute them, but it will be only few eigen values and eigen vectors which will be helpful in future. And not only that when you do computational work it turns out we need only few of them to do all the work for us alright.

So, there is a notion of what is called the norm of the matrix alright, you can divide it then get eigen value 1 and then proceed and do lot of work with that alright. So, let us try to understand some, so that is all about the motivation part that why do we need it.

(Refer Slide Time: 04:16)



Some more examples where we need such things, so examples, so my question will be I want to maximize or minimize, maximize or minimize X transpose A X, A is n cross n symmetric matrix under the condition under the constraint or condition X transpose X is 1 alright. So, what I am doing here is that I have this, a sphere with me. So, the X that I am taking at points only on this circle alright fine. So, I am looking at only points on the circle; and from there, I want to see what is the maximum and minimum value of this

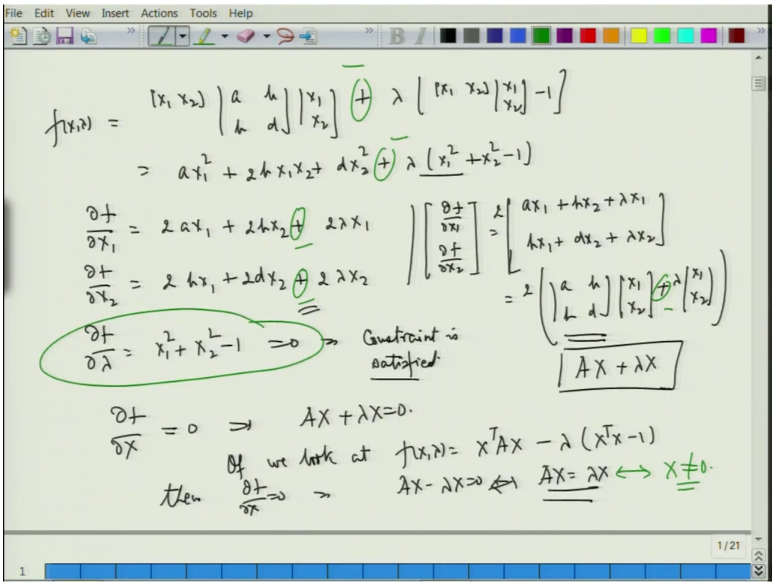
If you look at these function alright, so this is nothing but x 1, x 2, x n times a 1 1 a 1 2 a 1 n, and I am assuming it is symmetrical. So, again it will be a 1 2 a 2 2 a 2 n a 1 n a 2 n a n n fine times x 1 x 2 x n. So, this will turn out to be equal to if you look at it will be summation of a i j X i s X j i going from 1 to n j going from 1 to n alright, this is what it is going to look like fine. And we would need this condition that we have.

So, if I want to maximize or minimize over this thing, then if you recall there was an notion of what is called Lagrange multiplier method alright in our calculus several variable. So, what we do is that we form another one. So, we form a function f of X and lambda and look at X transpose A X plus lambda times or minus lambda times X transpose X minus 1.

This is coming from the constraint side, and this is coming from the maximal and minimal objective part max min, and this is the constraint that we are looking at. And then we talk of del f by del X we compute this del f by del lambda, and then proceed. We compute these things and proceed alright.

So, let us try to understand, what do I mean by differentiation of this X is a vector. So, I am looking at grad of this. So, let us see what it is. So, I will do it for only 2 cross 2 matrix. And from there, I would like you to understand it.

(Refer Slide Time: 07:01)



So, I am looking at X 1 X 2 it is a b b d X 1 X 2 plus lambda times I am looking at X transpose, so it is X 1 X 2 X 1 X 2 minus 1 alright. I am looking at this part. So, I want to look at this function which is f of X and lambda for me, is that ok? So, this I can rewrite it as a times X 1 square plus 2 b X 1 X 2 plus d X 2 square plus lambda times X 1 square plus X 2 square minus 1 fine.

So, now if I want to differentiate with respect to X 1, I get it as 2 A X 1 plus 2 b X 2 plus look at here 2 lambda X 1; del f by del X 2 if I want to look at, it is 2 b X 2 X 1 2 b X 1 X 2 goes off plus 2 d X 2 plus 2 lambda X 2; and del f by del lambda is X 1 square plus X 2 square minus 1. So, this part substitute to be 0 alright. If I substitute this to 0 imply that constraint is satisfied fine.

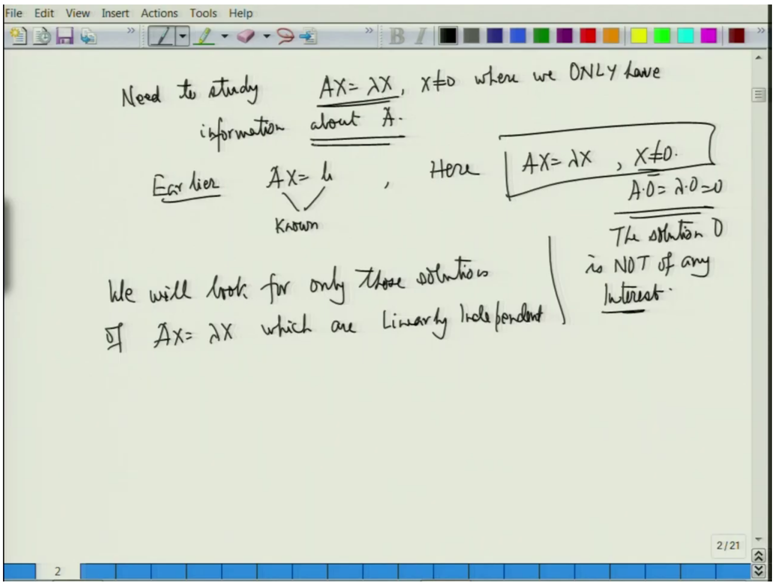
If you want to write this in terms of matrix because everything is matrix here, you can see that look at it nicely I can write it as. So, if I want to look at del f by del X 1 and del f by del X 2 as a matrix fine, I can write it as just look at this nicely, so 2 f comes outside A X 1 plus b X 2 plus lambda X 1 alright, then I get here b X 1 plus d X 2 plus lambda X 2. So, I can write it as two times a b b d times X 1 X 2 fine plus lambda times X 1 X 2, I am looking at this fine.

Or if I change it slightly, if I write it minus here, there will be a minus here, minus here, minus here, a minus here, alright, if I am doing that or even if I am not doing it. What we are looking at is that this corresponds to looking at A X plus lambda X, I am looking at this part fine. So, when I am looking at del f by del X; in some sense equal to 0 this gives me, I need to look at A X plus lambda X is 0 alright fine. So, I put in minus here, so that there is a clarity, because that is the way we are going follow.

So, if I put minus, so if we look at f X, lambda as X transpose A X minus lambda times X transpose X minus 1. Then del f by del X is equal to 0 implies A X minus lambda X should be 0 or which is same thing as saying that A X is equal to lambda X fine. Now, what is more important for us is that look at this constraint, this constraint says that this point X 1, X 2 has to be on the circle.

Since X 1 and X 2 are on the circle, therefore, X cannot be 0. So, with the condition that, X cannot be the 0 vector alright. So, I need to study equations of the type A X is equal to lambda X, X is not equal to 0 fine.

(Refer Slide Time: 11:27)

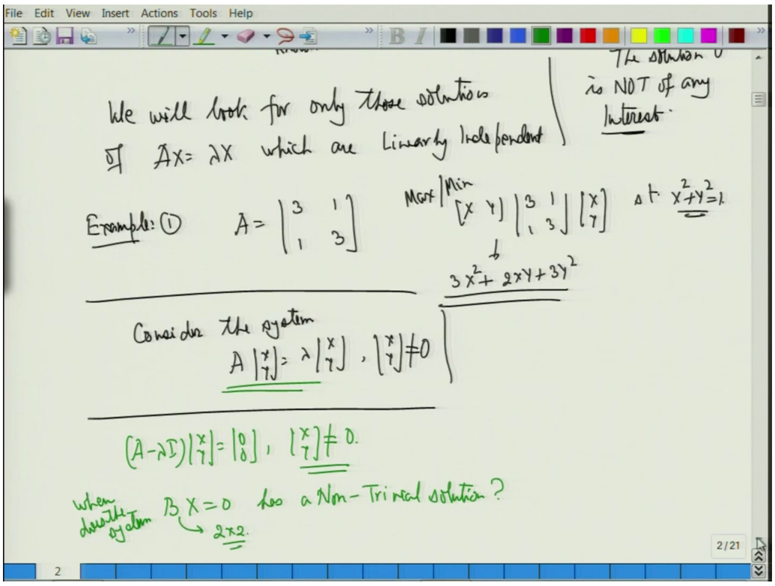


So, need to study, A X is equal to lambda X, X not equal to 0 where we only have information about A alright. So, earlier we had looked at, earlier we looked at A X is equal to b, A and b both where known in some sense for us fine. We got exact solutions approximate solutions and so on, but there A and b was known.

Here it turns out here I want to look at A X is equal to lambda X, and X not equal to 0 If you see 0 is already a solution here A times 0 is lambda times 0 is 0 alright. So, X is equal to 0 or the solution 0 is not; the solution 0 is not of any interest fine that is important that I want solutions only which are non-zero alright.

Finally, we will see that we want only solutions which are linearly independent. So, we will see afterwards we will look for only those solutions of A X is equal to lambda X which are linearly independent fine, this is what we look at. So, I hope you have understood what I am trying to say I am trying to study this part A X is equal to lambda X, and try to understand what do I mean by that.

(Refer Slide Time: 13:35)



So, let us take an example to proceed further example. So, the first example that I take is for me A is the matrix 3 1 1 3, suppose I look at this matrix fine. So, as usual I want to maximize X Y 3 1 1 3, X Y maximize this maximize or minimize subject to X square plus Y square is equal to 1. So, this part, so if I want to write this, this is nothing but 3 times X square plus 2 X Y plus 3 Y square I want to maximize or minimize this fine.

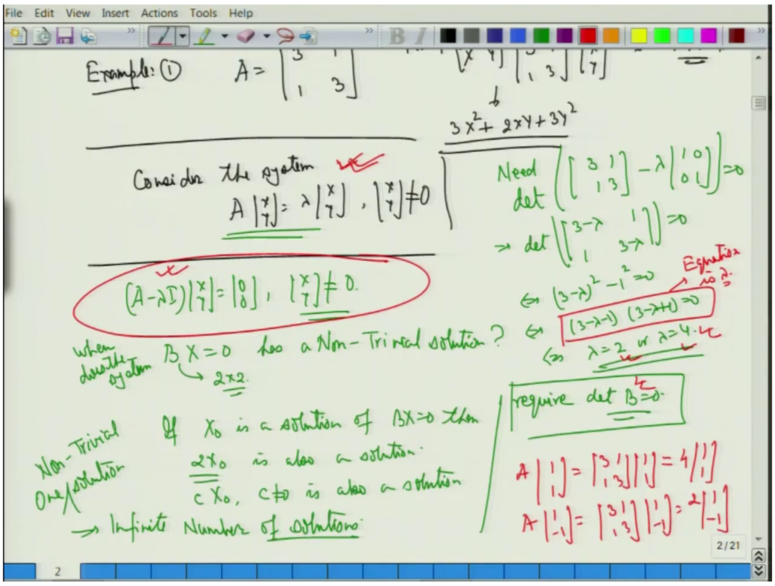
So, let us try to solve this part for us. So, I want to look at, so consider the system A of X Y is equal to lambda of X Y, X Y not 0 alright fine. Now, I do not know anything how do I proceed. The question is how do I proceed here? Fine, I have a some issues, in the sense that earlier A is equal to b was there. So, I had the coefficient matrix, I had the augmented matrix, I could do it. Here I do not know the value of lambda, I do not know X Y, how do I proceed?

So, again system of equations comes to our hand, and here what we do is that we rewrite this as a minus lambda I of X Y as 0 0 alright. We take everything on the left hand side. And we want what we want we want X Y to be not 0. So, do you remember some theorem which talks about when do I have a system this, when does the system B X is equal to 0 has a non-trivial solution. Do you remember something about it?

Here B is a 2 cross 2 matrix for us; it is a square matrix alright. In this chapter, every matrix, all our matrices will be a square matrices, I forgot to say that. So, everything here is a square matrix that I am looking at fine. I have started with the symmetric matrix, so that square matrix that I am looking at alright. So, my question is do you remember something about when does the system this has a non-trivial solution?

So, there was some comments about it in the sense that the system A X is equal to b, where A is a square matrix has a unique solution if and only if A was invertible alright. So, now here I want that it has a non-trivial solution means it should have infinite number of solutions, fine.

(Refer Slide Time: 16:30)



So, if X naught is a solution of B X is equal to 0, then 2 X naught is also a solution. Forget about 2 X naught. any C times X naught C naught equal to 0 is also a solution alright. So, we have got infinite number of solutions. So, one solution, one non-trivial solution, solution implies infinite number of solutions fine.

And so let us go back where this that A X is equal to b has a unique solution if and only if A was invertible or A was singular determinant of A was not 0. Here we have infinite number of solutions; therefore, what you require is the determinant of B must be 0. So, require determinant of B should be 0 fine.

So, what do I get here? So, if I look at this example, I need alright A was 3 1 1 3 this minus lambda times I for identity, so it is 1 0 0 1. Determinant of this should be 0 this what we need fine. So, which is same thing as looking at determinant of this matrix 3 minus lambda 1 1 3 minus lambda determinant of this should be 0.

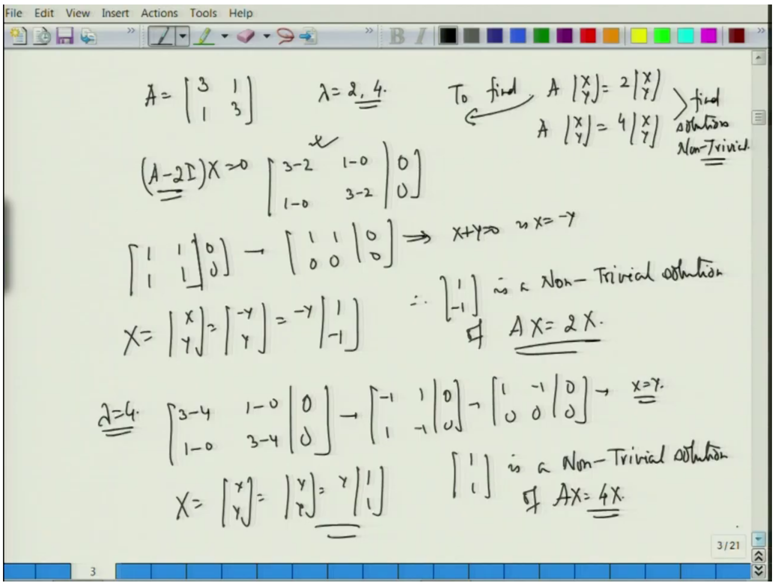
So, determinant of this is nothing but 3 minus lambda whole square minus 1 is 0 which is same as the square of this which is same as 3 minus lambda minus 1 into 3 minus lambda plus 1 is 0, which is same as lambda is equal to 2 or lambda is equal to 4 alright fine.

So, understand it fine. So, this part tells me that determinant has to be 0 fine. I go back and write look at this part. So, the B matrix here corresponds to A minus lambda I matrix here. So, I look at the A minus lambda I, solve it, I get an equation in lambda fine. So, I get an equation in lambda, it was a quadratic equation here in this example because it was a 2 cross 2 matrix. In general, it could be higher if it is n cross n, there will polynomial of degree n and so on. So, I need to solve that polynomial fine.

So, here my solutions were lambda equal to 2 and lambda equal to 4. I would like you to see that a times this vector 1 comma 1, which is same as 3 1 1 3 times 1 1 is equal to 4 times 1 1. And A times 1 minus 1 is equal to 3 1 1 3 times 1 minus 1 is 2 times 1 minus 1 alright.

So, the one that X Y and Y that we are trying to look for we also given to me, I am able to get X and Y that is because of something else that I have my experience, but let us try to compute how do I get X and Y that is the second thing. But the important thing is that given this 2 and 4, I have a solution here alright. How do I get a solution? Let us go into that part.

(Refer Slide Time: 20:20)



So, I have my A was the matrix 3 1 1 3, lambda was equal to 2. These are the two choices for lambda, fine. So, idea was to find to find A X is equal to 2 X Y, and A of X Y is equal to 4 times X Y. I want to solve find solutions here alright solutions which are non-trivial right that more non-trivial solutions.

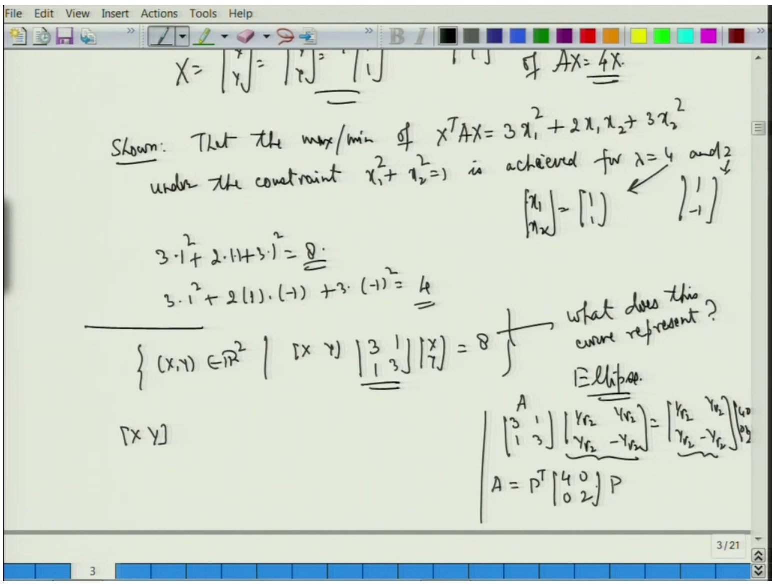
So, the first one takes us to looking at a minus 2 I of X is equal to 0. So, I am looking at this matrix augmented matrix you want to say or coefficient matrix whatever you want to look at, it is 3 minus 2 A minus 2 I means A minus 2 I, this 1 minus 0 1 minus 0 3 minus 2 augmented matrix is basically this. So, I am looking at the matrix 1 1 1 1 fine. So, 0 0 which will give me the r r e f of this is just this itself. This is my r r e f.

And this implies that the solution is X plus Y is 0 implies X is equal to minus y. So, my capital X which is x comma y is minus y comma y. I will take out minus y outside I get 1 minus 1 fine. Therefore, 1 minus 1 is a non-trivial solution of A X is equal to 2 times X fine because I started I used 2 here to get my answers here is that fine.

Similarly, if I want to go for 4, so lambda is equal to 4, if I want to do for lambda equal to 4, I need to look at 3 minus 4 1 minus 0 1 minus 0 3 minus 4 0 0. I need to look at this augmented matrix which is same as looking at 3 minus 1 is minus 1 1 1 minus 1 0 0 which r e r r e f is nothing but 1 minus 1 0 0 0 0. And from here I get that x has to be equal to y, and therefore, capital X which is small x small y is y which is y times 1 1 alright.

So, you can check that 1 1 is a non-trivial solution of A X is equal to 4 X fine. So, I want you to understand this part that is very important for me. So, what exactly we have done is, so just verify it, this is what we had here 1 1 and 1 minus 1 fine. There is no issue with that part fine.

(Refer Slide Time: 23:33)



So, what we have shown? So, shown that the maximum or minimum of X transpose A X which was nothing but 3 times x 1 square plus 2 x 1 x 2 I think plus 3 y 1 3 x 2 square 3 x 2 square under the constraint x 1 square plus x 2 square is equal to 1 is achieved for lambda is equal to 4 and 2 for lambda is equal to 4 and 2. This 4 corresponds to the vector x 1 x 2 is equal to 1 1; and this 2 corresponds to 1 and minus 1.

So, just plug in the value here. So, what you get is that the maximum is 4 3 times 1 square plus 2 times 1 plus 3 times 1 which is same as, so 1 into 1 and 3 square [FL]. So, the maximum is 8. And what is the minimum?

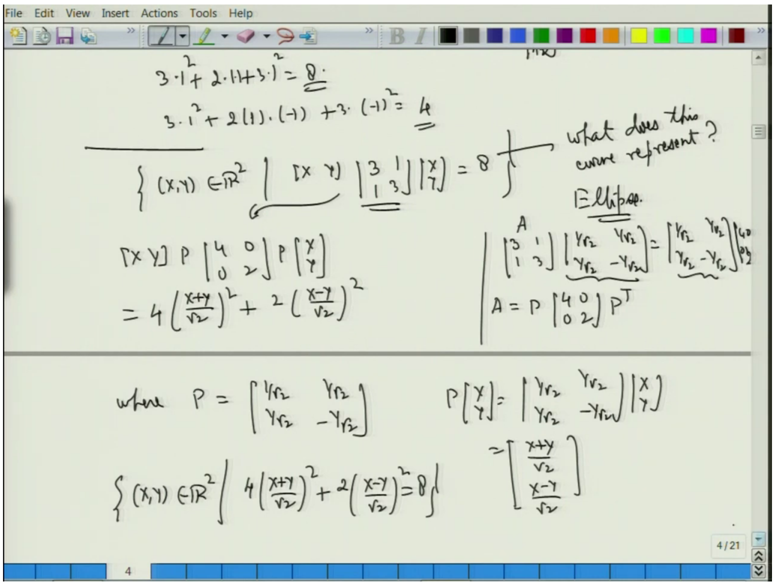
The minimum is 3 times 1 square plus 2 times 1 into minus 1 plus 3 times minus 1 square which is 3 plus 3 6 6 minus 2 which is 4 alright. So, this is the minimum and maximum value that you obtained fine. Something more has also happened. I would like you to go for that part also alright. So, as a small thing, I would just like to impress on that also.

So, let us look at this thing. I want to look at, so consider all x, y belonging to r two such that such that x y into 3 1 1 3 times x y is equal to say 8. I want to solve this system. I want to see what is this figure, What does this curve represent? I want to understand this fine. So, from what I have is you can try to understand that this will turn up to an ellipse alright. So, it is an ellipse.

How do I see it? So, I would like to see it like this that I have this x y with me that I have got. 3 and 1, they have some property that I will write afterwards or let me do it here itself. So, if I look at 3 1 1 3 times this half half half and minus half should have written square root here, but anyway let me write it as 1 upon square root of 2 1 upon square root of 2 1 on 1 square root of 2 minus 1 upon square root of 2.

So, let us look at this part. So, this, this matrix is A. A times this is nothing but 4 times this itself alright. So, I would like you to see that this is 1 upon root 2 1 upon root 2 root 2 times the matrix 4 0 0 2 alright. This is what this matrix is. Or I am writing a as alright, let me see this is one matrix, this is the same matrix. Here it transfer their transpose. So, I would like to write it is transpose. So, I got a matrix P transpose diagonal matrix 4 0 0 2 times P, I have writing like this alright.

(Refer Slide Time: 27:32)

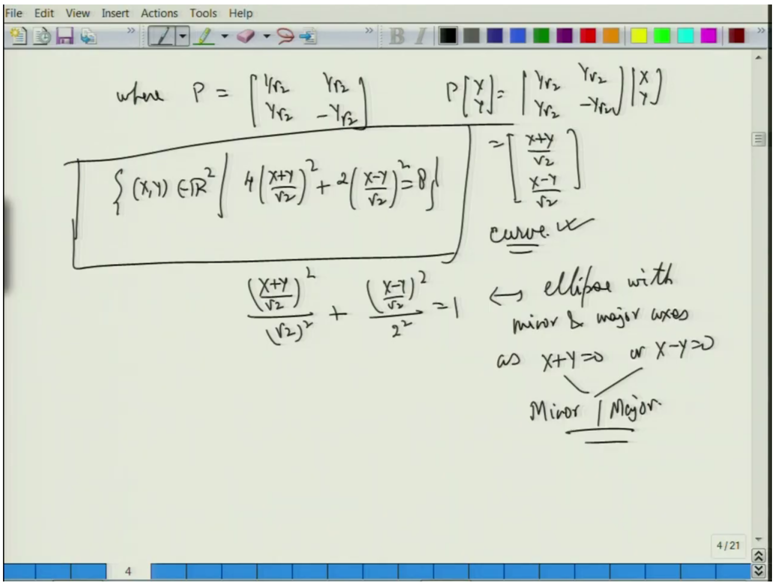


Or let me write this way P transpose P where P is equal to 1 upon root 2 1 upon root 2 1 upon root 2 minus 1 upon root. Let me write like this alright fine. So, in that case this corresponds to looking at this time 3 1 1 3 that is A is P times. So, it is P here alright, P 4 0 0 2 P times X Y alright.

So, what is P times X Y? P times X Y is 1 upon root 2 1 upon root 2 1 upon root 2 minus 1 upon root 2 times X Y which is same as. So, this into this gives me X plus Y upon root 2, and the next one gives me X Y upon root 2, this what it gives me fine. So, I would like you to see that this is nothing but 4 times X plus Y upon root 2 whole square plus 2 times X minus Y upon root 2 whole square alright. Just plug in the values here this is what you get fine.

So, from here what I want to say is that I am looking at all X, Y belonging to R 2 such that 4 times X plus Y upon root 2 whole square plus 2 times X minus Y upon root 2 whole square is 8. I am looking at this system. Is that ok?

(Refer Slide Time: 28:59)



So, I am looking at this curve now. So, the curve that I was looking at is this curve itself alright. So, you can see here now it is nothing but X plus Y upon root 2 whole square upon divide by 8 here. So, 8 upon 4 I think root 2 whole square plus X minus Y upon root 2 whole square divided by 8 upon 2 that is 2 a square is equal to 1.

So, this is the ellipse that you are looking at ellipse with minor and major axes as what is it X plus Y equal to 0, or X minus Y is equal to 0, which is minor which is major? Alright. So, what I am trying to say here is that somehow the study of this eigen values is eigen vector has also resulted in the study of what are called curves. I am able to talk about curves, and we will look at those things in the next class alright.

So, that is all for now.

Thank you.