Linear Algebra
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Lecture – 05

Alright. So, in the last class, we learnt some more ideas about matrix multiplication and we also saw that the matrix product is associative. And it nicely behaves with transpose, conjugate transpose, the scalar multiplication and matrix sum, alright. So, let us look at examples to have a better understanding of things, some questions, alright.

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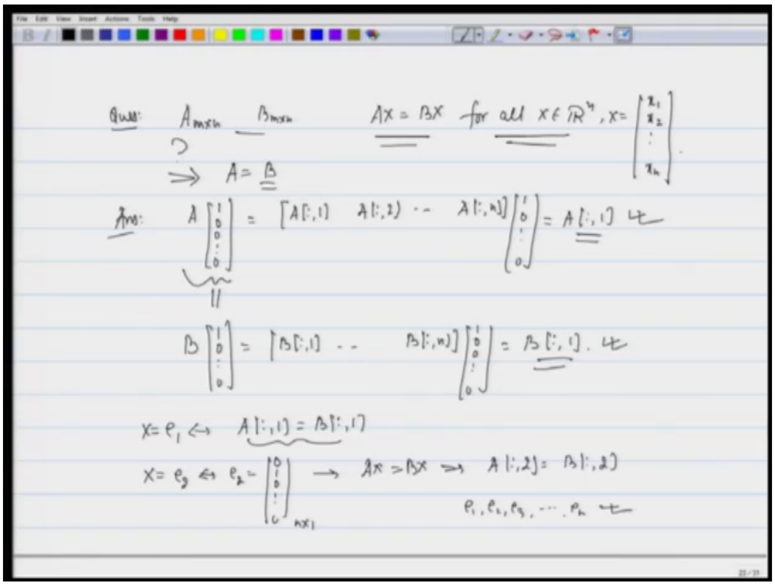

So, questions; suppose I have been given that so, given so I have got to so, A and B are m cross n matrices and you have been given that A times X is same as B times X for all X belonging to R n. So, R n means it has n components. So, X looks like x 1, x 2, x n. So, I been given this, fine.

So, what you saying that AX is equal to BX for all X belonging to R n does this imply that A is equal to B that is the question? So, the answer to this is answer, now what we know is that if I look at a times this vector 1 0 0 0, alright then this is same as.

So, if you want to write it, write it as a column. So, because that is the way we need to write here second column the last column times 1 0 0. And therefore, this turns out to be 1 times the first column, 0 times the second column so, until 0 times the last column. So, it is nothing, but the first column itself, fine.

Now, we have been given that A times X is also equal to B times X. So, in this example X is our 1 0 0 vector. So, again if I write B as in terms of columns, alright and we are multiplying it by the same vector 1 0 0. Again it is same as 1 times the first column 2 0 times the second column and so on. So, I do get back the first column alright.

So, what we see is that these to the first column of a is same as the second column of B fine just by multiplying replacing X by the vector 1 0 0 0 which we as I said it is nothing, but even for us. So, we wrote X as even here to get the first column of A is same as the second column of the first column of B. Similarly, if I replace X by e 2 we can see that so, what is e 2? Recall e 2 is 0 1 0 so on size n cross 1, so that matrix multiplication make sense.

So, this will imply that so, given condition that AX is equal to BX alright so, will imply that the second column of A is same as second column of B. And therefore, we can keep replacing e 1 by e 2, e 3 go on till e n and get that all the columns of A are equal to it all the columns of corresponding columns of B and therefore, the two matrices are equal, fine.

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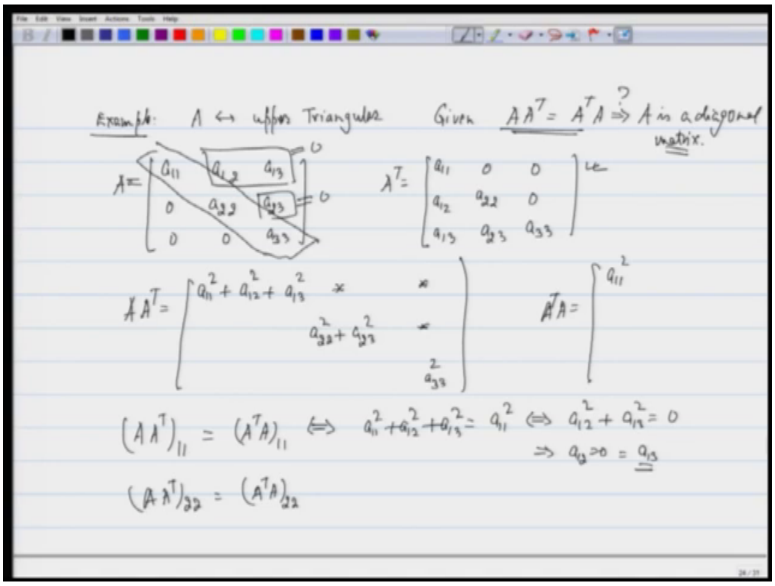

So, this I wanted you to see the use of this matrix product there is a clarity between things, alright. Then I would like you to look at this example or the next example again an example which will be used quite frequently that suppose I have a matrix A which is upper triangular and B alright so, let it be this and given that given A times A transpose is same as A transpose A. From there can I conclude that A as. So, given that this is equal to this can I conclude from here that A is A diagonal matrix alright.

So, let us compute the 11 entry on both the sides. So, A for me is going to look like. So, let me do it for 2 cross or say 3 cross 3 matrix a 11 a 12 a 13 0 a 22 a 23 0 0 a 33, this is your A. So, A transpose will be a lower triangular matrix a 11 a 12 a 13 0 a 22 a 23 0 0 a 33. So, if I multiply AA transpose if I look at AA transpose the 11 entry of that the 11 entry of AA transpose will be a 11 square plus a 12 square plus a 13 square, alright.

The 22 entry will be a 22 square plus a 23 square and the last entry here 33 will be a 33 square. The rest of the entries I am not bothered about for the time being. If I look at A transpose A A transpose A, then it will be the first row of A transpose times the first column of A and therefore, it will be a 11 square itself.

So, if I compare the two sides so, and given that A times A transpose is same as A transpose A. So, that 11 entry are also the same. So, the entry of 11 entry of this is same as 11 entry of this because they are same supposed to be equal. So, therefore, what I get is here is that a 11 square plus a 12 square plus a 13 square is same as a 11 square and which is nothing, but which tells me that a 12 square plus a 13 square will be 0.

And, when is the sum of two positive numbers 0? When both of them are 0 or some of two non-negative numbers. Recall that a ij I have taken it to be a real number. So, a square of a real number is either 0 or positive and therefore, this implies that and gets implied that a 12 is 0 is equal to a 13. So, therefore, these two entries of a are 0, fine.

Similarly, if I look at the A 22 entry of AA transpose 22 entry on both the sides I would like you to see that that will imply that a 23 is 0. And therefore, what we are left out to is a matrix which is a diagonal matrix alright. Now, let us look at another idea what is called how do I construct certain matrices with certain property.

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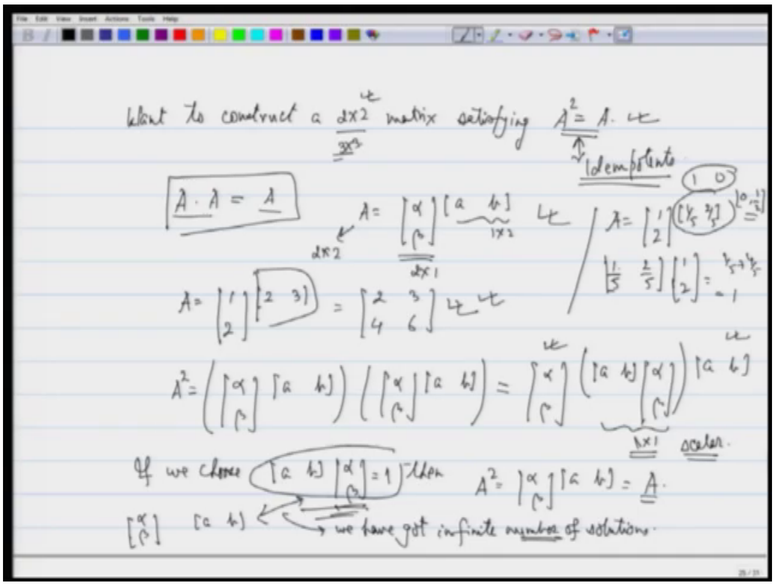

So, suppose I want to construct a 2 cross 2 matrix satisfying A square is A, alright. So, I want such a thing, want to construct a 2 cross 2 matrix I can replace 2 cross 2 by 3 cross 3 or anything, fine and I would like to this. We are not saying here that I want to have some nice properties that it is symmetric, skew symmetric and so on. I just want A square is a and such matrix they have a special name they are called idempotents, alright.

So, what we are doing here is I have got A here, I multiplying A with A and I want A, fine. I want A which is 2 cross 2 I can think of A as coming from a vector alpha beta times a b. So, if we look at this this matrix is of size 2 cross 1, this is of size 1 cross 2. So, A is now 2 cross 2, fine.

So, for example, I can take A as say 1 2, 2 3, fine. So, this gives me a matrix which is, so, if I look at this this is 2 3 times 4 and 6. So, I have got this matrix A for us fine, but what I need is

I need A square equal to A. So, if I write A square here so, for this matrix if I write A square the general setup A square will be alpha beta times a b times alpha beta a b.

So, now if I look at the associativity of matrix product I have got alpha beta here times a b into alpha beta times a b. So, if I look at this this is a 1 cross 1 matrix. So, this a scalar for us. Since this a scalar I can take it out, fine. So, if I see now this part the first part and the last part they are nothing, but they are giving me A itself. So, if I can choose so, if we choose this product a b times alpha beta to be equal to 1, then A square will be same as alpha beta times a b which will be A, fine.

So, in this example that I looked at in place of choosing 23, I can choose it in such a way that this product is one. So, for example, I can choose my a to be is equal to, so, let me take the first one as it is I want to choose it in such a way that their product is 1. So, how do I choose it? There are lot of ways of choosing it; I can take it as 1 upon 5, 2 upon 5 here for example.

So, for example, if I look at 1 upon 5, 2 upon 5 times 1 2 so, it will be 1 upon 5 plus 4 upon 5 which is 1 or I can take it as 1 0, I could have taken it as 0 comma half. So, there are lot of ways I can choose these things, alright. So, what we are saying is that given alpha beta to me alright I have. So, if I want to choose a b such that this is 1 means that we have got infinite number of solutions, alright.

So, these ideas that I can construct matrices which have A square equal to A alright, idempotents. Finally, we will see that we will need also some symmetric part of that, that will come afterwards, but even if you do not want symmetric you can see that there are infinite number of ways you can construct some matrices whether it is 2 cross 2, 3 cross 3; the basic idea being that some product has to be 1 that is all, nothing more than that alright.

So, you have to understand these are small – small ideas when we want to construct matrices. When you want to deal with matrices that, it is the each row or each column of the matrix which plays very very important role for us, alright.

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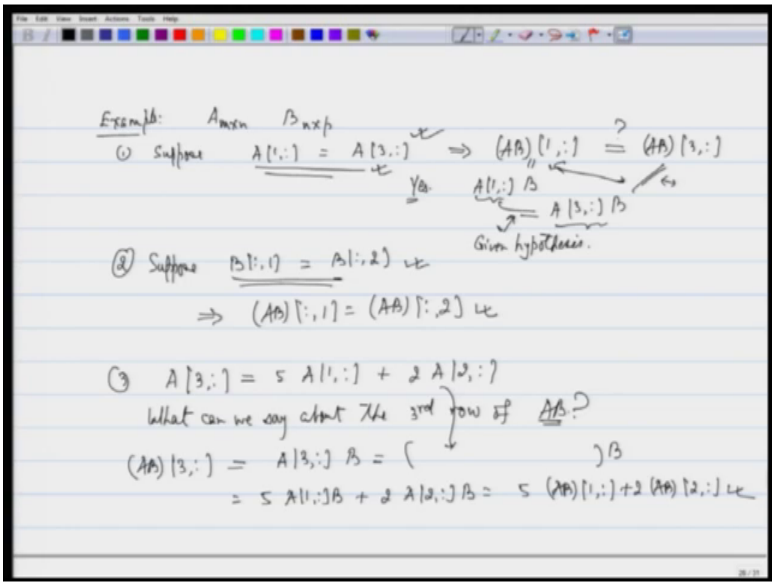

So, the last part about this matrix product that I would like you to understand now is this example alright, example again. So, I have a matrix A and I have a matrix B, matrix product is meaningful for me so, n cross p, suppose. So, I want to ask this question suppose I know that the first row of A is same as the third row of A; suppose they are equal, then can I say that from here that look at AB; the first row of AB will this be equal to the third row of AB? That is the question, fine.

The answer is yes. Answer is yes basically because, if you remember this is nothing, but the first row of AB is first row of A times B. And we have been given that the first row of A is same as the third row of A that is the given condition and which is this and again by matrix product we get this. So, this part is given hypothesis and this part is and this part they are coming from matrix product, alright.

Similarly, if I have been given column suppose that; so, this was 1 I have been given that suppose the first column of B is same as the second column of B. So, this is I would like you to try that that the first column of AB will be equal to the second column of AB. Again, so, important thing is when I want to look at the rows it is the matrix A which is on the left which is playing the role and if I want to look at the columns it is the matrix B which is playing the role which appears on the right.

So, matrix product means if I want to play with the rows alright, I need to look at doing something on A, so that I can play with the rows of B, fine. And if I want to play understand columns I have to look at columns of A to is to be played and I have to multiply B on the right fine. The last example here will be suppose I have been given that the third row of A is same as 5 times the first row of A plus 2 times the second row of A, alright. What can I say about the third row of AB? What can we say about the third row of AB? Alright.

Again, so, let us recall if I want to look at the third row of AB, then by matrix product this is nothing, but the third row of A times B and therefore, I get this part coming into play here times B. So, I will get it as 5 times this plus 2 times this which is nothing, but 5 times the first row of AB plus 2 times the second row of AB, alright.

So, similarly we will have corresponding to columns. So, what we are trying to say here is that, if by chance if by looking at it we see that some row of a is combination of certain things of some of certain other rows of A itself, then there is a relationship between the final product as per AB is concerned, alright.

So, if I know that there are columns of B which are nice properties that I can see that they have 0's or there is some sort of combination between them that I can get 1 from the others, then in AB also the same property will hold, alright.

So, that is all for today as per matrix multiplication is concerned. In the next class, we will look at invertibility of a matrix, how do I compute the inverse of a matrix.

Thank you.