Linear Algebra
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Lecture – 48
Gram-Schmidt Process: Applications

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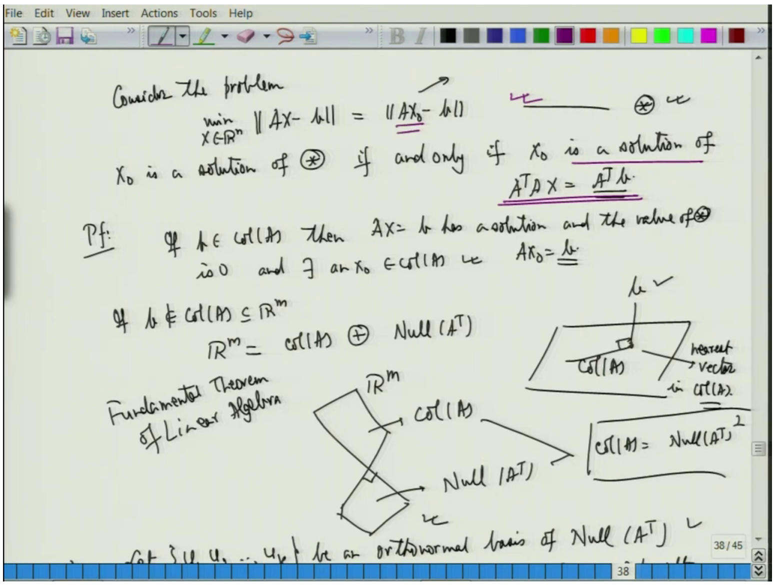

Alright, so, let us look at this problem consider the problem, minimization problem that we want to minimize the distance between AX and b. So, if b is already in the column space, then the distance between them is 0. So, I can solve the system AX is equal to b there no problem.

So, if b belongs to column space of A, then AX is equal to b has a solution and the value of a star; the value of a star is 0 and there exist an X naught belonging to column space of A fine, which gives us AX naught is equal to b fine so, I will get something.

Now, if b does not belong to the column space of A. So, b does not belong to the column space of this is my column space of A, b is outside I am supposed to look at something which is perpendicular here and then proceed fine. So, I am trying to find out the nearest vector in column space of A, alright.

So, now that we have proved the gram Schmidt process and so on. So, let us do that part. So, what I know is that this is a subset of R m. So, we know that R m can be written as column space of A direct sum null space of A transpose recall, the idea that I had R m here so, fundamental theorem of linear algebra; of linear algebra. So, I had this line with me this was 90 degree, this was null space of A transpose I think and this was the column space of A fine; so, I can take a basis of this.

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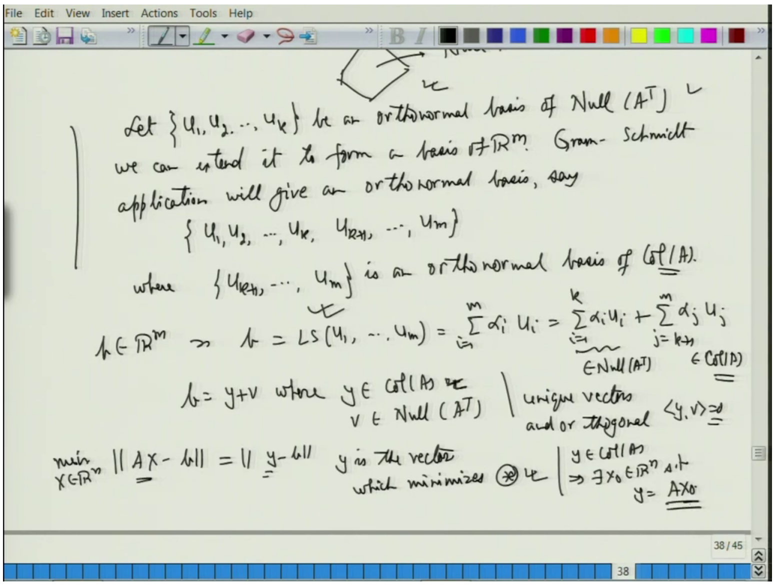

So, let u 1, u 2, u k be an orthonormal basis of null space of A transpose fine, we can extend it, we can extend it to form a basis of R m. Further apply Gram Schmidt application, will give an orthonormal basis say u 1, u 2, u k, then u k plus 1 till u m where, u k plus 1 to u m is an orthonormal basis of column space of A.

Why you can do it? Basically because look at this what we are saying is that column space of A is perpendicular to null space of A transpose alright. This perp is there therefore, perpendicularity comes and therefore, you can do things alright. So, this is important for us.

So, what we are saying is that, I have a vector b, I decompose be in terms of column space some element of column space and something with the null space is that ok so, I am doing that. So, because of this part now I can say that there exist a unique. So, b belongs to R m implies b is equal to linear span of u 1 to u m.

So, this will be equal to sum alpha i u i, i is equal to 1 to m, I can write it as i equal to 1 to k m alpha i u i plus summation j equal to k plus 1 to m alpha j u j. So, this part belongs to null space of A transpose and this belongs to column space of A alright and this is unique that is important alright. So, I am writing b as fine as some y plus v, where y belongs to column space of A and v belongs to null space of A transpose and this vectors are unique; unique vectors and orthogonal, alright y and v are orthogonal.

This is the way the graph is and that is what the fundamental theorem of linear algebra tells us that they are orthogonal fine. So, therefore when I am looking at this thing so, if I want to look at minimum over this AX minus b minimum over X belonging to R n, then this is attained by alright, I have take something from X, so, I would take y belonging to the column space.

So, this is attained by y minus b alright. So, y is the vector which minimizes it, minimizes the star alright. But what is y? y is element of column space. So, it is nothing but this so if we I can write in terms of ordered basis also, but we are just saying here is that or in this problem is that since y belongs to column space of A, implies there exist X naught belonging to R n such that y is equal to AX naught alright this is what it is fine.

So, now, we already know that. So, what we are saying here is that I have a solution of a star here which is X naught, I have to prove that A transpose AX naught is A transpose b alright.

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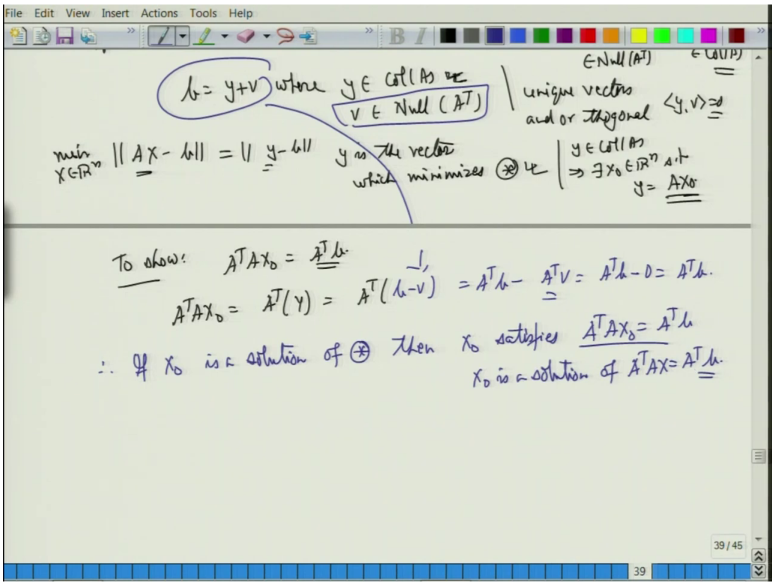

So, let us try to prove that to show A transpose AX naught is equal to A transpose, let us just do that part. So, let us look at what is A transpose AX naught. A transpose of AX naught, we just saw it was alright AX naught is Y. So, it is just Y for me which is same as A transpose of what is Y. So, y if I look at from here fine.

So, y from here is nothing but b minus v which is same as A transpose of b minus A transpose of v, which is same as A transpose of b minus what is A transpose of v? v belongs to null space of A transpose. And therefore, A transpose of v is 0. So, I get it as A transpose of b.

Therefore, if X naught is a solution of a star, then X naught satisfies A transpose AX naught is equal to A transpose of b or X naught is a solution of A transpose AX is equal to A transpose b.

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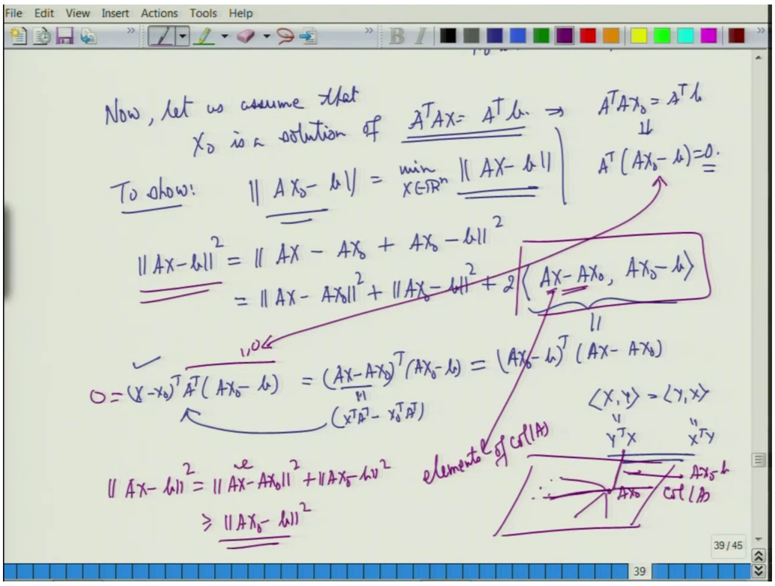

So, I have proved one way. Now, let us prove the other way around, now to show. Now, let us assume that X naught is a solution of A transpose AX is equal to A transpose b, to show AX naught minus b this is equal to minimum over all X belonging to R n norm of AX minus b, this is what I have to show fine. So, let us try that out.

So, what does it mean to say that X naught is a solution here, and what does it mean to do things here alright. So, this part implies that look at this part nicely, it says that A transpose AX naught is equal to A transpose b, I wrote inverse A transpose b. And therefore, I look at A transpose of AX naught minus b is 0, fine. Now, from here I want to relate these ideas here, how do I do that? Fine. So, I am looking at minimum of AX minus b.

So, let us look at norm of AX minus b this whole square be careful. So, we are going to write it as norm of AX minus AX naught plus AX naught minus b whole square. So, the way we have been decomposing what we will get here is this will be equal to norm of AX minus AX naught, whole square plus norm of AX naught minus b whole square plus, since everything is real here should be 2 times inner product of AX minus AX naught comma AX naught minus b. This is what we will have fine.

That is the way we would be decomposing that u and v are there, then if I write u plus b comma u plus v the norm of u square plus norm of v square plus 2 times inner product between u and v alright, that is what we have been doing. So, let us look at what is this. So, this is equal to if I want to look at is by definition is AX naught minus b whole transpose AX minus AX naught fine, which is nothing but or I think I have written it the other way round here.

So, let me write the other way round alright. So, this is same as AX minus AX naught transpose AX naught minus b, because X Y is same as Y X for us because this is same as Y transpose X, which is same as X transpose Y, everything is real. If it was complex then you can do at complex conjugate will get the same thing.

So, this is same as just look at this part it is X naught. So, this is equal to X minus X naught transpose A transpose into AX naught minus b, alright. So, this is equal to if you look at it is X transpose A transpose minus X naught transpose A transpose. So, I am taking A transpose on the right and then writing here fine.

So, this is what I have? X transpose A transpose minus X naught transpose A transpose X naught transpose, X naught I have taken it as separately here fine and then A transpose. Now, we have been given that this thing is 0 look at here, it is given that this is 0 and therefore, this is 0. So, what we are saying here is that this part is 0. Now, when I say that this part is 0 what does this imply? It means that look at this part AX means elements of column space of A, fine.

So, any column any vector here minus this. So, I am saying that any vector so, this if I look at so, I have a plane here which is the column space of A. So, AX is any vector here that I am not bothered about AX naught is some fixed vector here look at this look at this. This is my vector AX naught minus b then we are saying that this vector is perpendicular to everything here, fine.

And what will that imply? If it is perpendicular to everything it means that that must be 0 as such, fine. So, therefore, what we are saying here is that this is greater than equal to, this is 0. So, norm of this is this, so, norm of this is greater than equal to the norm of this. So, therefore, what I get is that norm of AX minus b whole square is equal to norm of AX minus AX naught a square plus norm of AX naught minus b whole square. So, which is greater than equal to norm of AX naught minus b whole square.

So, the minimum is attained at X naught itself alright, whatever you do because if you change any X you get something extra here fine. So, this is what we wanted to prove look at the thing here it said that, if X naught is a solution of A transpose A equal to this, then X naught is a solution of a star. So, X naught is the one which gives you the minimum, is that ok. So, we have been able to show this also. So, what I would like you to understand now is there are two things here.

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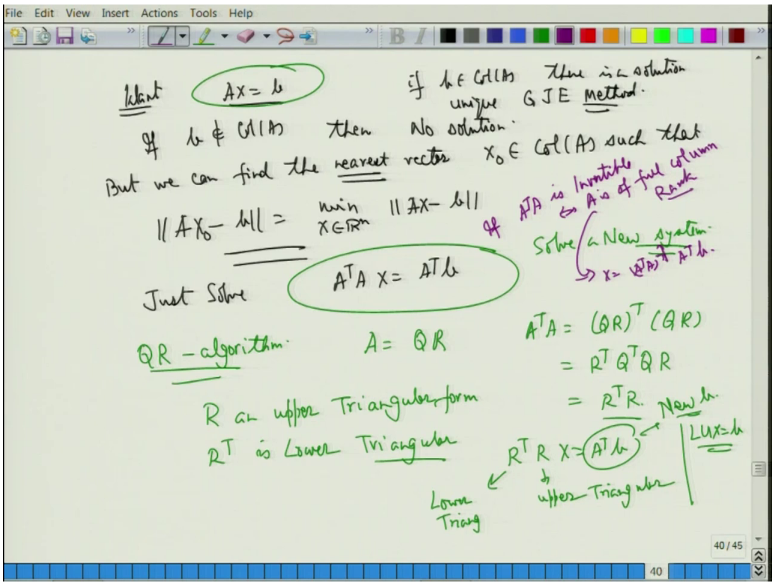

One is that if you want; so, want 1: if you want to solve AX is equal to b, if b belongs to column space of A there is a solution using Gauss Jordan Elimination method alright, I already have a solution fine. If b does not belong to column space of A, then no solution.

But we can find the nearest vector, X naught belonging to column space of A, such that nearest vector such that norm of AX naught minus b is minimum over X belonging to R n norm of AX minus b. And how do we do that? And to get this what we are saying is that, just solve A transpose AX is equal to A transpose b. So, in place of solving the system AX is equal to b I need to solve this system to solve a new system alright fine.

Now, what the QR algorithm says? QR algorithm, this we are solving it because if you so, what we had was, A was equal to QR fine. So, if we want to compute A transpose A is nothing but QR transpose QR, which is same as R transpose Q transpose QR, which is R transpose R alright; R was an upper triangular form R an upper triangular form. So, R transpose is lower triangular fine.

So, I just have to solve the system R transpose R is equal to A transpose b, I just have to solve it. So, again it is in some sense I am looking at a lower triangular and an upper triangular, I am looking at this. And I am trying to solve the system this and we have already know how to solve the system L u of X is equal to b. So, we are doing the same thing this is a new b and I just have to solve this system as we did earlier fine.

So, there are lot of ways of doing it, understand it, if A transpose A, if A is full rank. If A transpose a is invertible, which is same thing as saying that A is of full column rank, then I can compute the A transpose inverse, because in that case Q transpose Q is invertible everything is nice.

So, if A transpose invertible, then this will imply that this will imply X is equal to A transpose A inverse A transpose b fine. If it is not invertible, then we can solve it using what is called a Pseudo inverse or the Moore Penrose inverse, but that is not in our syllabus alright. For us the syllabus is that only to pick those A's for which A transpose A is invertible, alright.

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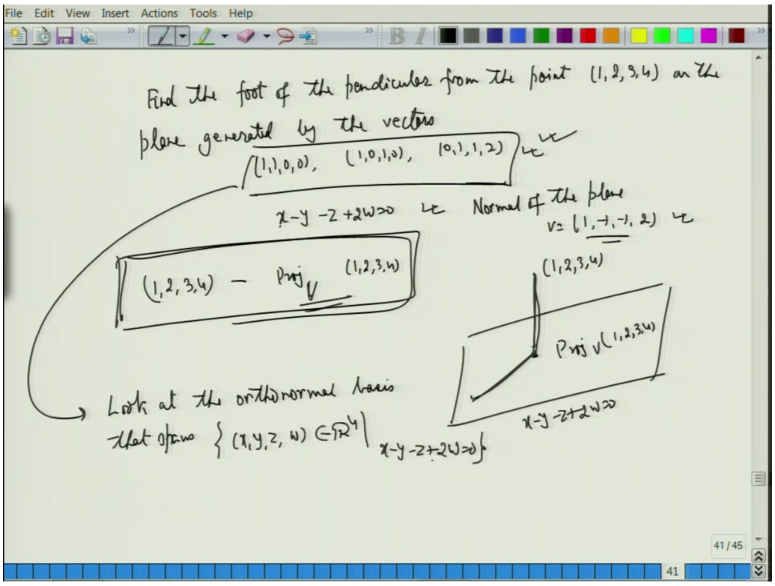

So, let us go to the one of the first example that we looked at. I do not remember the exact question there, but the idea was the following that we have to find, the foot of the perpendicular from the point 1, 2, 3, 4 on the plane generated by the vectors. I think it was 1 1 0 0, 1 0 1 0, I think 0 1 1 2. I think this is what it was alright. So, we had got it by one method fine.

So, plane generated by the vectors this. So, we had got it using the idea that this thing is generated by the vector by the x minus y minus z plus 2 w was 0. I think let me check it. 1 1 is 0 1 0 1 is 0 1 1 and minus 2 yeah 1 1 minus 2 alright. So, we looked at this from there we got the normal of the vector, normal of the plane as 1, minus 1, minus 1, 2 using this, what we did was we looked at 1, 2, 3, 4 minus projection of 1, 2, 3, 4 on this vector, this vector whatever it is.

So, V I write it as this on this. So, this gave me. So, what we are doing is look at this? So, again I have a plane which is generated by x minus y minus z plus 2 w is equal to 0, I have the vector 1, 2, 3, 4 here. I drop a projection here; this is my projection vector projection of 1, 2, 3, 4 on this fine. And we looked at this minus this. So, this was the vector, this is the projection and this is the one that I am computing so, I computed this.

So, once I computed this I could get this point which was on the plane alright, the other way will be to look at this itself and proceed fine. So, the other way will be look at the orthonormal basis that expands x, y, z, w belonging to R 4. So, that x minus y minus z plus 2 w is 0 alright.

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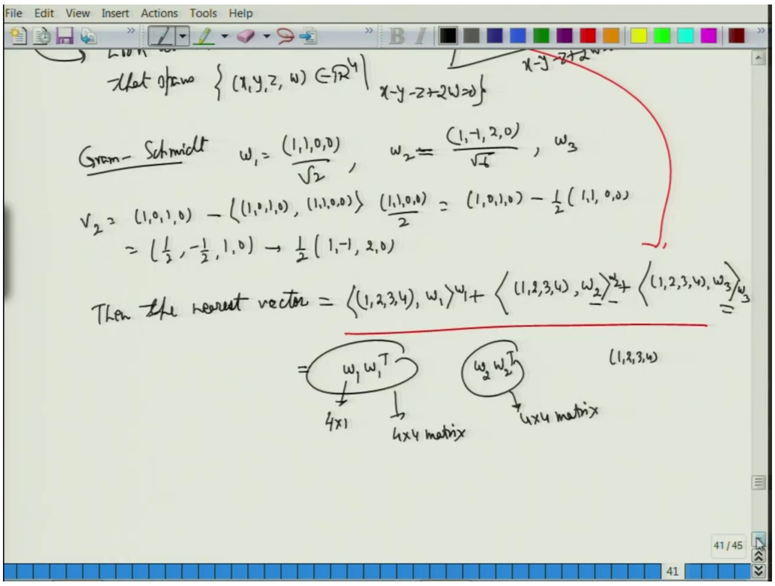

So, to do that I can apply the Gram Schmidt process; Gram Schmidt, so, I can take my w 1 to be is equal to 1, 1, 0, 0 divided by root 2, I can take v 2 is equal to 1, 0, 1, 0 minus 1, 0, 1, 0 inner product with 1, 1, 0, 0; 1, 1, 0, 0 divided by 2, which is same as 1, 0, 1, 0 minus 1 upon 2, 1, 1, 0, 0 which is same as 1 minus half is half 0 so, minus half. So, this is 1 and 0.

So, therefore, w 2 can be taken as so, I can take w 2. So, just look at this part I can take out half outside it will be 1, minus 1, 2, 0. So, again w 2 as 1, minus 1, 2, 0 divided by 6 the square root of 6. Similarly, compute w 3 and then thus vector the nearest vector, then the nearest vector will be equal to; so 1, 2, 3, 4, inner product of this with w 1 plus inner product of 1, 2, 3, 4 with w 2 plus inner product of 1, 2, 3, 4, with w 3.

So, again I would like you to check that these three will also give you the same answer as this alright. So, these two answers are going to be the same fine, we will look at we will try to understand them. So, would like to look at this part. So, let us recall some, one theorem that we had. And then from there I want to say that this is the one that I am looking at fine. So, what I need is that I had these are the components that I am looking at.

So, I have to multiply by w 1 here, w 2 here and w 3 here to get the answer alright, the nearest vector will be a vector not a scalar quantity right. So, I have to write that. So, let us try to understand what it is. So, if write it I can think of this as I have a vector here which is 1, 2, 3, 4 alright, I am looking at w 1, w 1 is a for me it is a column vector.

So, it is a 4 cross so, what is this? This is a 4 cross 1 vector w 1 transpose. If I look at this part this together gives me a 4 cross 4 matrix fine. Again If I look at just w 2, w 2 transpose this also gives me a 4 cross 4 matrix. And similarly this part also gives me a 4 cross 4 matrix.

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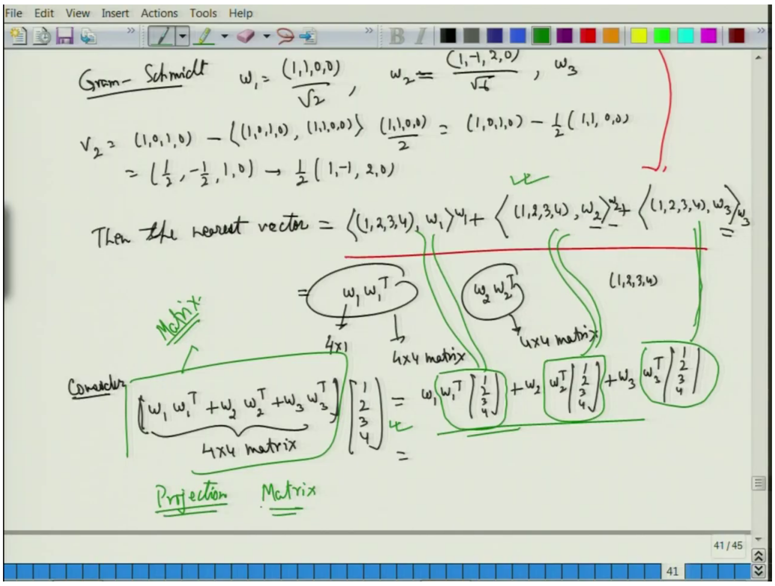

So, therefore, if I want to look at so, consider w 1 w 1 transpose plus w 2 w 2 transpose plus w 3 w 3 transpose, it is a 4 cross 4 matrix fine. Now, this is a 4 cross 4 matrix, I have just write like this. And let us multiply this with 1 2 3 4. What do I get? Fine. If I multiply this what will I get?

I will get it as w 1 w 1 transpose times 1 2 3 4 plus w 2 w 2 transpose of 1 2 3 4 plus w 3 w 3 transpose 1 2 3 4 which is same as alright; which is same as this is a scalar quantity, this is also a scalar quantity, this is also a scalar quantity. And this scalar quantity is nothing, but this, this scalar quantity is this and this is same as this.

So, we have been able to write if I write this as a matrix and then multiply with the given vector I am just saying that this part is nothing but the projection part alright fine. So, I want you to understand this part is very important, what we are trying to say here is that that in place of computing this separately I can just compute the matrix alright. This matrix is called matrix, a projection matrix. So.