Linear Algebra
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Lecture – 47
Orthogonal Projections

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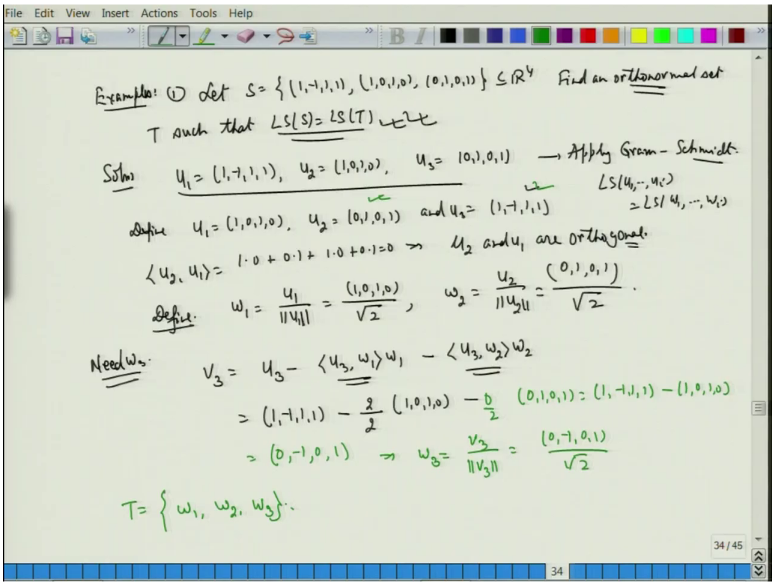

So, let us look at this example. In the last class, what we had done was that given any linearly independent set where this Gram-Schmidt's orthonormalization process which gave us something in the sense that we got a set of linearly independent vectors which were orthonormal and the linear span was maintained at each stage fine.

Now, here we are asking a question let S be a this set, this is linearly independent that you can check fine and then, I want something such that the linear span is maintained, but the T set is orthonormal. So, there are two ways of doing thing. So, let us look at the solution of this fine.

What is that you take u 1 as 1, minus 1, 1, 1, u 2 as 1, 0, 1, 0 and u 3 as 0, 1, 0, 1 and apply Gram-Schmidt's.

So, but the my question here is that I just want LS of S to be equal to be LS of T. At no place we are saying that I have to maintain the linear span at each stage. In the Gram-Schmidt's process, there was this linear span of u 1 to u i was same as linear span of w 1 to w i. I am not asking such a question here alright.

So, therefore, I can change the order. So, I will change the order. So, define for me u 1 to be equal to 1, 0, 1, 0, u 2 to be 0, 1, 0, 1 and u 3 to be equal to 1, minus 1, 1, 1 I want to write this alright. So, now if I look at inner product of u 2 and u 1 is nothing but 1 into 0 plus 0 into 1 plus 1 into 0 plus 0 into 1 which is already 0 and therefore, u 2 and u 1 are orthogonal. So, I do not have to worry about finding the projections and so on.

So, I can just define my w also define w 1 as u 1 upon length of u 1 which is nothing but 1, 0, 1, 0 divided by root 2, similarly, w 2 you can define as u 2 upon length of u 2 which is nothing but 0, 1, 0, 1 divided by root 2 fine. So, I have got these two vectors which are perpendicular, and they are of length 1 fine. I am not writing them as column vectors because I have problem in the space here. So, therefore, everything I am writing in terms of rows, but basically they are column vectors alright.

Now, I have to look at what is w 3. So, need w 3. So, we look at what is called v 3. So, what was v 3? v 3 was u 3 minus the components in the different directions. So, this is the components direction of w 1 similarly I have u 3, w 2, w 2. So, remove the components in the direction w 1 and w 2 alright. So, u 3 is given to be 1, minus 1, 1, 1 minus we have take the dot product of u 3 with w 1; u 3 with w 1 so it is 1, 1 so, it is 2 I hope.

So, 2 upon root 2, root 2 comes twice 2 here fine and it is 1, 0, 1, 0 minus u 3 with w 2 so, u 3 with w 2 this is u 3 with w 2 which is already 0, 0 times this 1, minus 1, 0, 1 so, it is already 0 here also alright. So, this is 0 upon 2 times 0, 1, 0, 1. So, therefore, I just have 1, minus 1, 1, 1 minus 2, 2 cancels out it is 1, 0, 1, 0 I hope I am doing it correctly, yeah it is correct. So, this minus this is 1 minus 1 is 0, minus 1 minus 0 is minus 1, 1 minus 1 is 0 and this minus this is 1 so, I get this.

So, this implies w 3 is equal to v 3 upon length of v 3 which is nothing but 0, minus 1, 0, 1 divided by root 2 alright. So, for me the set T is nothing but w 1, w 2, w 3 is that. So, I have took an easy example.

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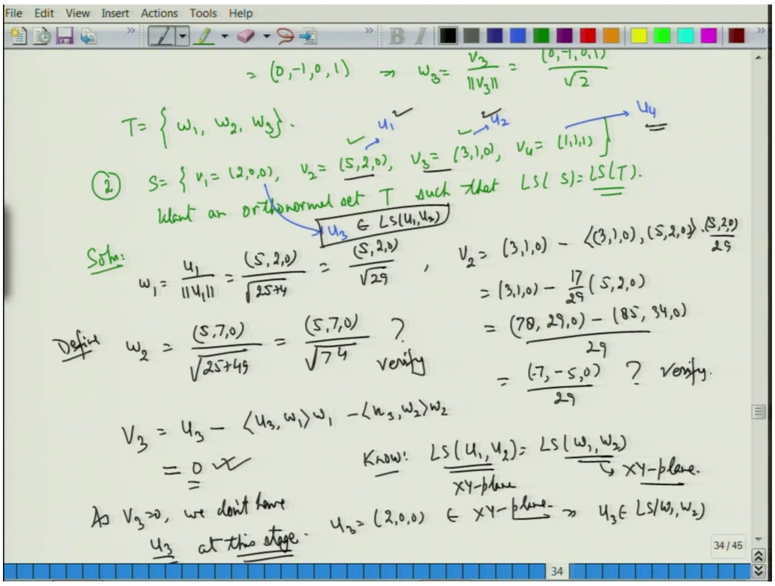

Let me take another example second example. So, I am giving S as v 1 which is 2, 0, 0, v 2 as 5, 2, 0, v 3 as 3, 1, 0, v 4 as 1, 1, 1 fine. I want the same thing that want an orthonormal set T such that LS of S is same as LS of T alright. So, let us try to do that again solution fine. So, let me do with v 2, v 3 a complicated one and then build up ideas alright.

So, I was starting with this so, for me this is my u 1, this is my u 2, this is my u 3 and this is my u 4 alright. I am trying to look at that part right. So, for me since u 1 is this so, I define my w 1 as u 1 upon length of u 1 which is equal to 5, 2, 0 divided by 25 plus 4 square root 25 plus 4 this which is same as 5, 2, 0 divided by root under 29 I hope fine.

Now, I have to define v 2. So, v 2 is u 2 which is 3, 1, 0 minus dot product of 3, 1, 0 with 5, 2, 0 into 5, 2, 0 divided by 29 alright because unit vectors root 29 root 29 will come twice I will get this part. So, this will be equal to 3, 1, 0 minus 3 into 5 is 15 [FL] 17 upon 29 of 5, 2, 0.

So, this will be equal to whatever it is 29 into 3 is 78 I hope and then it is 29, 0 minus 17 into 5 is 85, 34, 0 divided by 29 which is same as 78 minus 85 I think is 7 with a minus sign, then 29 minus 34 will be I think minus 5 or minus 5, 0 divided by 29. I hope it is correct please check it out question mark fine whatever it is.

So, I define my w 2, define w 2 is equal to 5, 7, 0 divided by 25 plus 7 into 49 square root which is 5, 7, 0 divided by [FL] 74 alright so, please check alright verify that is your question that you please verify it yourself verify alright.

So, you can define w and w 2 do it yourself. Now, let us look at w 3. W 3 is u 3 minus u 3, w 1, w 1 minus u 3, w 2, w 2 fine. So, I am claiming that this is 0 why this is 0? Because look at v 2 and v 3 alright. So, what we know linear span of u 1, u 2 is same as linear span of w 1, w 2 fine.

Linear span of u 1, u 2 if you look at 5, 2, 0 and 3, 1, 0 it gives me the XY-plane so, this will also give me the XY-plane itself fine and u 3, u 3 is nothing but 2, 0, 0 this is my u 3 which belongs to the XY-plane and therefore, implies that u 3 belongs to linear span of w 1, w 2 and therefore, so, this was my v 3, v 3 was this so, I get that v 3 is 0 alright and therefore, I do not have u 3. So, as v 3 is 0, we do not have u 3 is that u 3 at this stage fine.

This is what you have to be careful about that when you are doing things; if the earlier set was linearly dependent you will get a corresponding dependence at that stage itself. So, u 1 and u 2 are independent. So, you got w 1, w 2. U 3 was dependent on u 1 so, u 3 belong to the linear span of u 1, u 2 and therefore, the dot product or the component will be 0 here for example, we had this 0 and this is what is going to play the role that understanding is very important fine.

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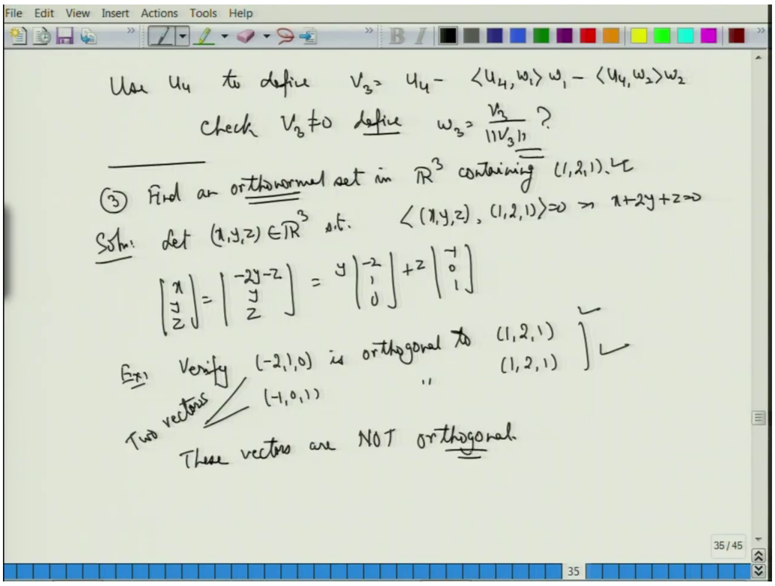

And now, you can compute u 4. You can use your u 4. So, now, use u 4 to define v 3 as u 4 minus u 4, w 1, w 1 minus u 4, w 2, w 2 check v 3 is not 0 define your w 3 is equal to v 3 upon norm of v 3 alright. So, I leave it for you to do all these calculations yourself is that ok. So, this is what you need to understand that you can do things on your own, you are good at it so, no problem, please complete it out yourself alright.

The next example that I am going to look at is this question. Find an orthonormal set; orthonormal set in R 3 containing 1, 2, 1 alright. Solution: so, let x, y, z belong to R 3 such that inner product of x, y, z with 1, 2, 1 is 0 I mean orthonormal set. So, I want everything perpendicular to each other.

So, it has to be perpendicular to 1, 2, 1 also. So, I am looking that condition. So, this gives me that x plus 2y plus z is 0 or therefore, what I get is if I write the vector x, y, z this is same as x is same as minus 2y minus z, y and z which is same as y times minus 2, 1, 0 plus z times minus 1, 0, 1 fine.

So, check that. So, exercise : verify minus 2, 1, 0 is orthogonal to 1, 2, 1 unless I have done a mistake so, it is ok, yeah its correct minus 1, 0, 1 is orthogonal to 1, 2, 1 that is also true no problem everything thing is fine.

But so, what I had done is that I have got these vector. Using that vector, I have got two vectors here. So, two vectors which are perpendicular to these the given vector alright, but these two vectors themselves two vectors, these vectors are not orthogonal alright. So, I will have to make them orthonormal alright, orthogonal I have to make them. So, I will have to do some work.

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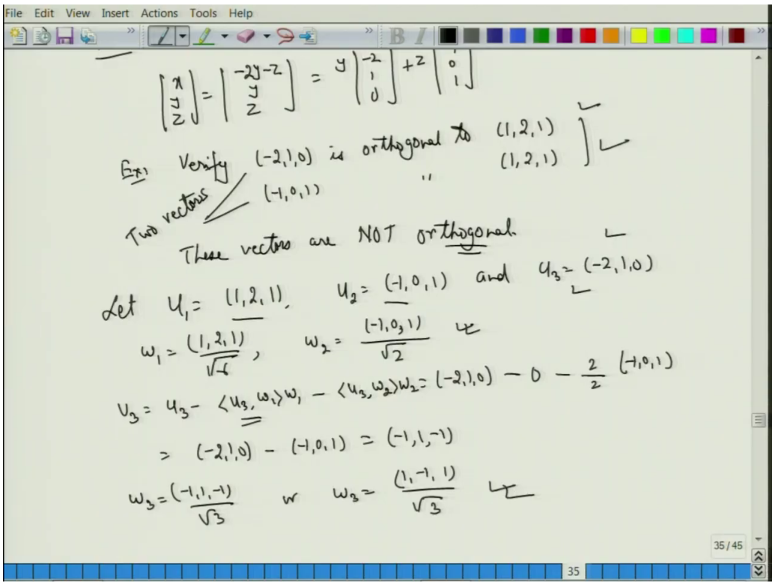

So, let me define things now alright. So, I have something. So, I can define is to define. So, let u 1 is equal to 1, 2, 1, u 2 is equal to minus 1, 0, 1 and u 3 as minus 2, 1, 0. I am using this. So, I already know that these two are perpendicular so, I define w 1 as 1, 2, 1 divided by root 6 the length, w 2 as minus 1, 0, 1 divided by root 2, let us compute v 3.

So, v 3 is u 3 minus u 3, w 1, w 1 minus u 3, w 2, w 2 which is same as minus 2, 1, 0 minus u 3 with w 1 so, this with w 1 which is already 0 for us there is no problem just look at it, that is the way we have constructed so, that part is 0, minus 0 minus u 3 with w 2 alright this is the one that is creating the problem for us. So, let us look at this part.

This root 2 will give you twice so, it will become 1 upon 2, dot product this will be minus 2 into minus 1 is 2 and this is 0 so, it is 2 upon 2 of minus 1, 0, 1 so, this is same as minus 2, 1, 0 minus of minus 1, 0, 1 so, minus 2 so, this is equal to minus 2 plus 1 is minus 1, 1 minus 1 is 1 here and 0 minus 1 is minus 1 alright fine. So, therefore, w 3 is equal to perpendicular.

So, you can write w 3 as minus 1, 1, minus 1 divided by root 3 or w 3 as 1, minus 1, 1 divided by root 3 whatever you want you can do that is that ok. So, that is the way we get these things fine. So, I want you to understand this is very very important thing that you can do this calculations fine.

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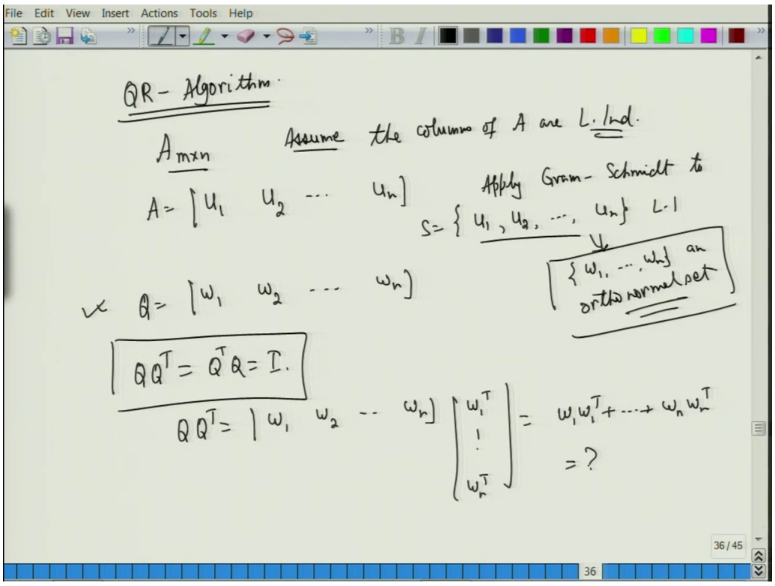

Now, I want to go to what is called an QR algorithm. This was one of the milestones for us QR algorithm. So, we had done what was called the rank factorization, LU decomposition, the next idea was QR algorithm so, we are doing the QR algorithm.

So, I have got a matrix A which is m cross n fine. So, first let us assume that the rows are assume so, assume the columns of A are linearly independent alright. So, they are linearly independent. So, I have got A as say u 1, u 2, u n. I can apply Gram-Schmidt to S which is u 1, u 2, u n if I do this what happens? This will give me a set of vector so; I will get a vector Q which will be w 1, w 2, w n alright.

So, Gram-Schmidt said that I get start with S which is linearly independent will imply me a set which is w 1 to w n an orthonormal set. This is what the Gram-Schmidt process gives. So, whatever that orthonormal set I have, I take Q as that fine. Since, this is my orthonormal set so, therefore, if I want to look at QQ transpose and Q times Q Q transpose Q, this will turn out be an identity matrix.

Why? Can you think about it? So, if you compute Q times Q transpose, look at this you get w 1, w 2, w n these are the vectors column vectors. And, I have got now w 1 transpose till w n transpose so, this in one side gives me w 1 w 1 transpose plus so on plus w n w n transpose. I do not know what this is, I cannot make anything head or tail of it at present fine.

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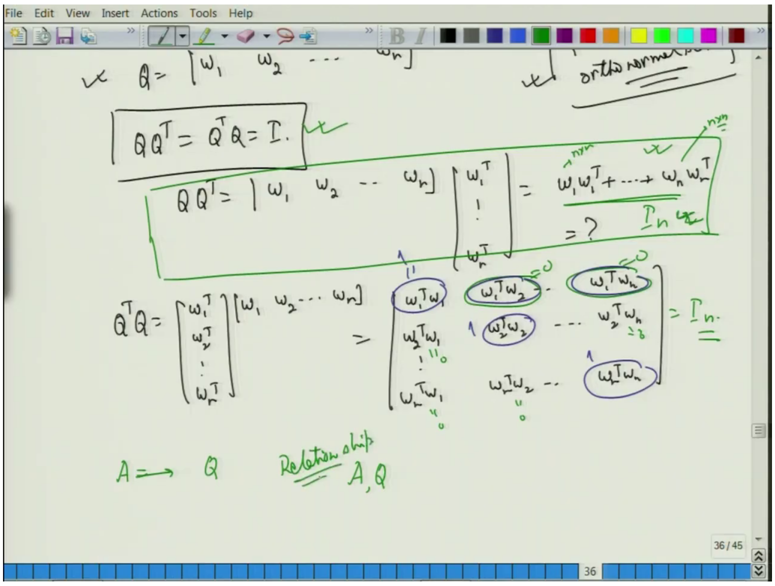

But what you can do is that if I look at Q transpose Q fine so, Q transpose Q will be w 1 transpose, w 2 transpose, w n transpose w 1, w 2, w n. So, this will be look at the first column first row it is w 1 transpose w 1, w 1 transpose w 2, w 1 transpose w n, then it is w 1 transpose w 1, w 2 transpose w 2, w 2 transpose w n and so on till w n transpose w n, w n transpose w 2 fine.

Now, we have been saying that this set w 1 to w n is orthonormal it means what? It means that length of this vector is 1 fine, length of this vector is 1; length of this vector is 1 alright. So, this dot product w 1 transpose w 1 is 1, w 2 transpose w 2 is 1, w n transpose w n is 1 fine and what happens to these vectors?

If I look at these vectors or these scalars, w 1 transpose w 2 is 0 because the dot product of w 1 and w 2 is 0 so, this is 0, this is 0, this is 0, this is 0, this is 0 and so on. So, I do get back identity with me, is that ok.

So, since I get identity so, Q is invertible. So, Q transpose is invertible means I get this part and therefore, this is also identity, is that ok. So, this is what is important that you are able to write identity in terms of these vectors w 1 w 1 transpose is a matrix n cross n matrix each of them is a n cross n matrix alright. And, since w 1 to w n are independent, these matrices each one of them have rank 1 and I have written I n identity as sum of n rank 1 matrices fine.

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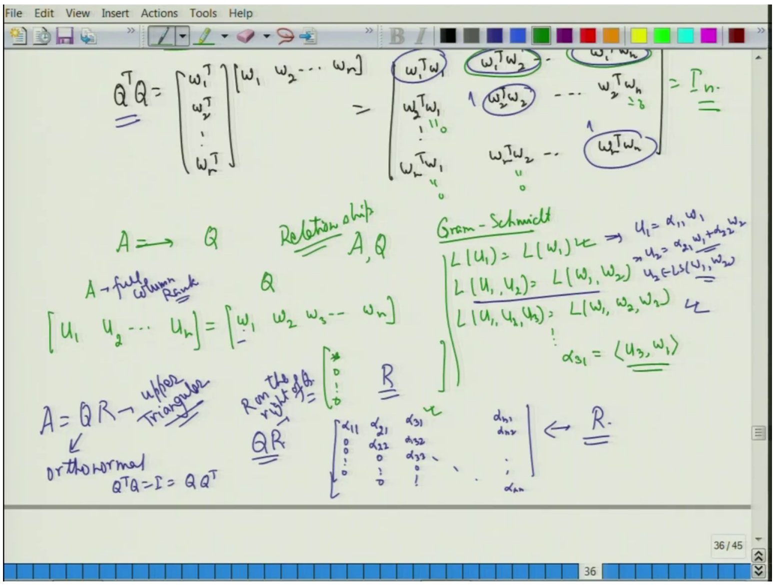

So, let us look at this. So, what we are saying is that I am looking at these matrix and trying to understand now what it is. So, what we have done is that I have got a matrix A. I have written it as from here I have got what is called Q. I want to relate A and Q. I want to relate A and Q as such relationship alright.

So, let us understand this very very important thing what we have so, if I look at Gram-Schmidt's it tells me that linear span of u 1 is same as linear span of w 1. Linear span of u 1, u 2 is same as linear span of w 1, w 2. Linear span of u 1, u 2, u 3 will be equal to linear span of w 1, w 2, w 3 and so on this is what it says. I have the matrix A as so, A is u 1, u 2, u n to me fine and the matrix Q for me is w 1, w 2, w 3, till w n to me fine.

So, since it says that linear span of u 1 is same as linear span of w 1 if I multiply this on the right by a matrix so, let me multiply this matrix here, I will get something here it is 0, 0, 0 if I look at this matrix so, I am looking at I am trying to write this matrix as R, I am looking at Q times R alright.

Now, since I am multiplying R on the right; R on the right of Q so, I am doing column transformations. So, it means that I am multiplying a star to w 1, 0 to w 2, 0 to w 3 and so on fine. Since, I know that linear span of u 1 is w 1, this implies that u 1 is equal to some alpha 11 times w 1 alright fine. So, what I get here is that this matrix, R matrix that I am looking at has alpha 1 here fine, 0, 0, 0 fine.

Now, this part tells me that fine that u 1 is already linear combination above, u 2 is a linear combination of u 2 is some alpha 21 times w 1 plus alpha 22 times w 2. So, u 2 belongs to linear span of w 1, w 2. So, I can find alpha 21 and alpha 22; so, that this will happen fine. So, therefore, when I look at matrix product again, I can write here alpha 21 here, alpha 22 here, 0, 0, 0 because then I can say that it is alpha 21 times w 1 and alpha 22 times w 2.

Similarly, when I go to u 3, I will get some alpha 31, alpha 32, alpha 33 and 0 here. In general, at the last stage, I will get alpha nn and alpha n1, alpha n2 and so on fine. So, this is the way the matrix R is to be obtained fine. So, you can see that I can get so, what I have here is that I have matrix A, I have got QR, Q is orthonormal; orthonormal.

And, not only that Q transpose Q is identity is same as QQ transpose in this special case because of Q or being or A being invertible or A is has full rank alright column full column rank so, a has full column rank. I did not say about m, I am talking only about n, full column rank I am talking about fine and R is look at R, R is upper triangular fine this is one thing.

The other thing that you should understand here is that I do not have to solve a system of equation here. If I want to get alpha 31, alpha 31 is nothing, but inner product of u 3 with w 1 because w 1 to w n was an orthonormal set; so, each of these scalars which are in the linear combination, they are nothing but things like this fine. They just come from the inner product so, you can compute them directly, you can put it here. So, nothing is special about it fine.

So, this is about QR algorithm for this set up. But if I want to have a general set up so, let us go to some example where I had an issue. So, there was an issue here yeah. So, here was the vector. So, look at this example here where I have a problem, that if I have something like this where the columns of the matrices that I am going to take they are linearly dependent.

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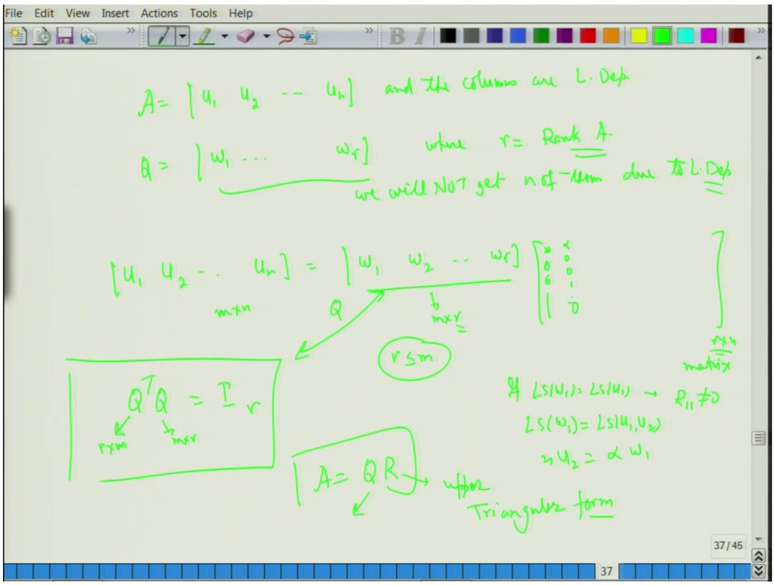

So, if I take a matrix A where the I have got u 1, u 2, u n and the columns are linearly dependent, then what will happen is that the matrix Q that I will get it as it will be w 1 to w r where r is equal to rank of A alright. I will not get n. So, we will not get n of them due to linear dependence alright fine. So, what will happen is that I have got only r of them.

So, I have to maintain everything with that. So, I again write u 1 here, u 2 here, u n here, then again I can write it here as w 1, w 2, w r. Now, this matrix will not be nice, but it will have some property because look at the this is r of them only so, I need only r here as such fine. So, it will be an r cross n matrix alright. So, this is starting with an m cross n matrix. So, this is of size m cross r, r cross n so, that I got m cross this and r here is less than equal to m so, r is less than equal to m fine.

So, depending on so, w 1 if w 1 was already u 1, I have something here if so, if linear span of w 1 was equal to linear span of u 1, then I will get a star so, so Q 11 so, not Q 11 r 11 will be non-zero else it will become 0 alright so, otherwise I will get 0's here. So, if LS of w 1 itself is equal to LS of u 1 and u 2 implies that u 2; u 2 is equal to some alpha times w 1 itself. So, I will just get alpha here, 0 here, 0 here, 0 here alright.

So, it will not be an it will be an upper triangular matrix, but it is not a square matrix, it will be upper triangular form the diagonal entries may be 0 here alright. So, this is what you have to be careful that when you generalize some issues will come, but what is more important is look at this matrix this, this is an m cross r matrix and this is the matrix Q. See, if I look at Q transpose Q, this is r cross m, this is m cross r and you can check the way that I have checked that this is an identity matrix alright fine.

So, we have been able to get a matrix right A as QR, R has an upper triangular form; triangular form and Q is orthogonal with Q transpose Q is I r fine. Now, why do we need orthogonal matrices basically? Because the norm does not change alright; so, that is all for now.

Thank you.