Linear Algebra
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Lecture – 46
Gram-Schmidt Orthonormalization Process

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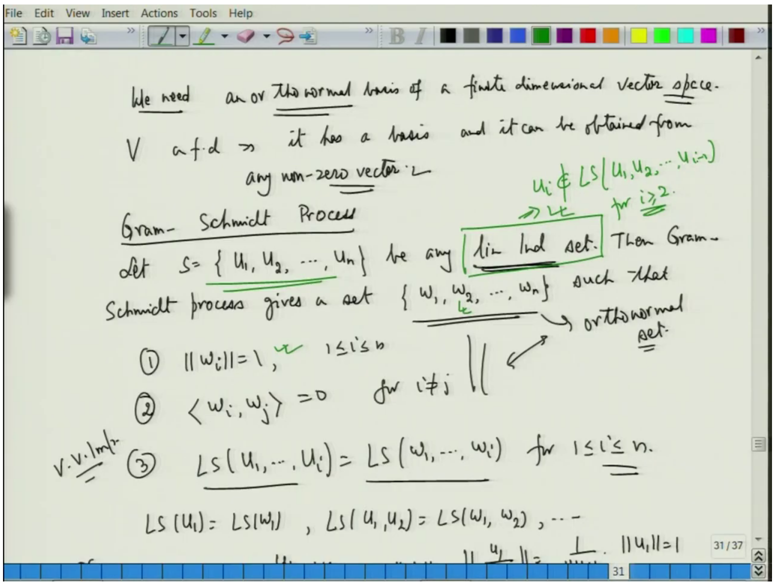

Alright, so, what we had seen was that, we need an ortho normal basis, an ortho normal basis of a finite dimensional vector space, fine. We need this fine, ortho normal basis. What we have know that, V a finite dimensional implies it has a basis and it can be obtained from any non-zero vector, alright. So, we would like to do the same thing again here, alright. So, first let us try to understand what is called the Gram-Schmidt orthogonalization process and then get this.

So, what is called Gram-Schmidt's process, alright? It is called orthogonalization process that I am going to make it orthogonal. So, let me add the statement here. So, let S is equal to u 1, u 2, u n be any linearly independent set. Then Gram-Schmidt's process gives a collection of vectors, gives a set w 1, w 2, w n such that 1; length of w i is 1, 1 less than equal to i less than equal to n. 2; w i and w j their inner product is 0 for i not equal to j.

And 3rd is very very important; very very important is that, linear span of u 1 to u i is same as linear span of w 1 till w I, for 1 less than equal to i less than equal to n, alright. So, what we are saying is that, I may starting with any linearly independent set, I am able to get an orthonormal set alright, this is an orthonormal set. How do I get orthonormal set? So, let look at this w 1, w 1 has the property that length of w i's is 1; inner product within w i and w j is 0, and therefore I get an orthonormal set.

What is extra that I am having in this process is that, the linear span is maintained. So, what we are saying is that, linear span of u 1 is nothing but linear span of w 1; linear span of u 1, u 2 is equal to linear span of w 1, w 2 and so on, at each stage the linear span is maintained, alright. And the idea of the proof is just using projections. So, let us try to prove this construction, alright. So, let us get into this, alright.

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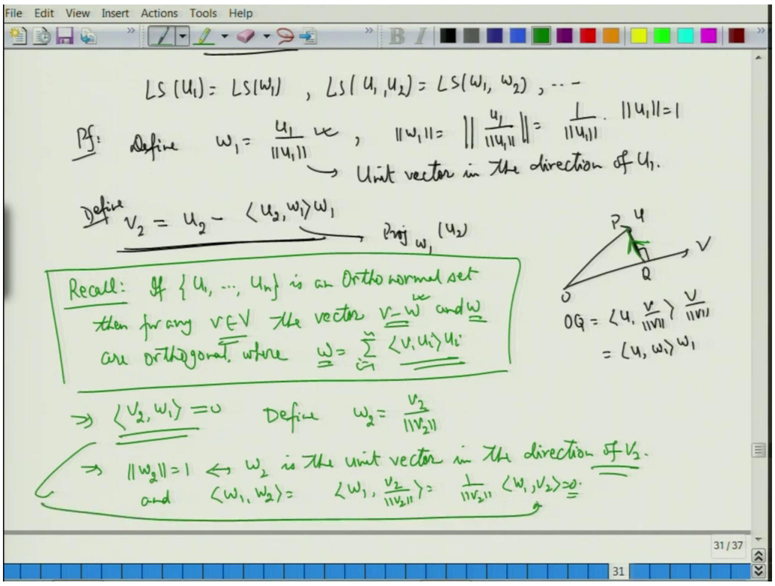

So, proof. How do I, what is this algorithm about, alright? So, define w 1 as u 1 upon length of u 1, alright. So, the first is. So, here I am giving the first element has u 1, second as element as u 2 and u n and therefore, I am building w 1 as u 1 upon this. So, you can see that length of w 1 is nothing but length of the whole thing u 1 upon length of u 1 which is nothing but 1 upon length of u 1 into length of u 1 which is 1, alright. So, this basically the unit vector in the direction of u 1, alright.

So, once we have got that this u 1 vector. So, recall for us that, what we are done was I have this vector v; I had u with me, there was this projection vector Q, P, O. Then the projection OQ this vector was nothing, but; it was u inner product with v upon length of v into v upon length of v, alright. So, here for us it will be u comma w 1 and w 1; because w 1 is a unit vector for us, alright.

So, now, so I am going to look at this part. So, what we are going to look at, define v 2; I have got w 1, now I will defining v 2 that is very important; v 2 is, define v 2, define v 2 has u 2 minus u 2 comma w 1 w 1, this what we are defining, fine.

So, what we are doing is, look at u 2; from u 2 remove the composition, the component, the projection. So, this is the projection of u 2 on w 1; we are removing that part. Since we are removing this part, we are left out with this vector which is perpendicular, alright. So, you can see here or you can use the previous theorem, where we had shown that if I keep it removing the components where.

So, recall this theorem, recall, which is very important theorem that I being saying it again and again that; if u 1 to u n is an orthonormal set, then for any v belonging to V, the vectors v minus w and w are orthogonal, where w was looking at alright, v comma u i u i. This is what I had proven one of the previous classes that things are perpendicular, fine.

So, this I am trying to recall for you, very very important that if I keep removing any component the whatever is left out is again perpendicular. So, this part tells me that, v 2 is perpendicular to w 1 alright; because look at here it says that, this is perpendicular to this. So, whatever I am obtaining, the linear span, it is perpendicular to that part.

So, v 2 is perpendicular to this I already know; you can verify that also, but this theorem tells me that, this implies that this is 0, fine. That v 2 is perpendicular to w 1; I do not know whether the length of v 2 is 1 or not, idea was to get w 2. So, the idea of the theorem was, I want to get w 2 such that the length of w 2 is 1, fine. Length of v 2 may not be 1; so I have to get w 2 to get things. So, I define, so this implies this because of this theorem.

So, define w 2 is equal to v 2 divided by length of v 2. Once I have defined this; this implies that length of w 2 is 1. So, what is w 2? w 2 is the unit vector in the direction of v 2, which is this direction; in this particular case this is this direction that I am looking at fine, direction of v 2. And inner product of w 1, w 2 is same as inner product of w 1 with v 2 upon norm of v 2 which is same as 1 upon norm of v 2 upon w 1, v 2 which is already 0 fine, because of this part, fine.

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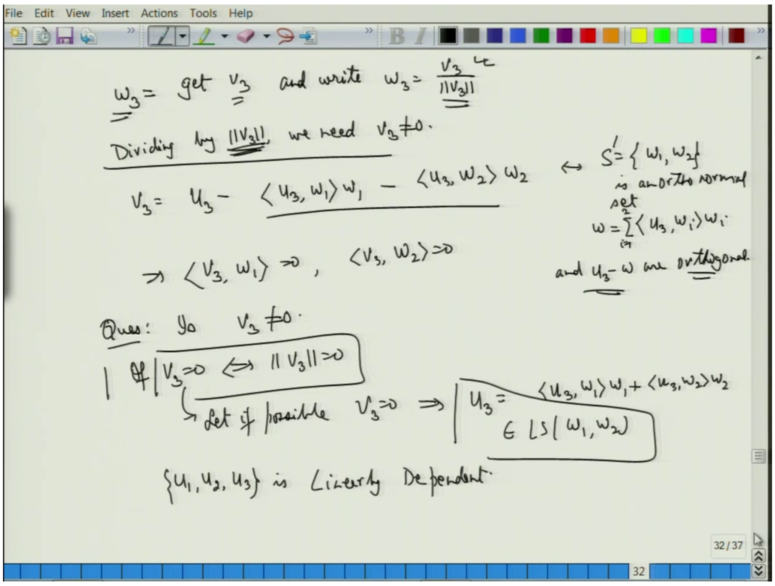

So, we have already got w 1 and w 2; how do I get w 3? So, w 3 again I define w 3 to be is equal to. So, the idea is, if I want to define w 3; I have to get v 3, get v 3 also, get v 3 and write w 3 as v 3 upon length of v 3, fine. Since I am doing this I am dividing; so dividing by length of v 3, we need v 3 to be non-zero, fine. So, let us try to do this part also. So, define first row. If I want to get w 3, I have to get first v 3 and I have to show that whatever v 3 I get that should be non-zero, alright.

So, let us be defined v 3 as, v 3 minus, sorry u 3 minus; from u 3 remove the component corresponding to w 1 alright and remove the component corresponding to w 2, alright. So, what I have done basically; here is I have got the set S prime as another set S prime which is w 1, w 2, this is an orthonormal set, S prime w 1, w 2 is an ortho normal set. I am defining w as fine, i is equal to 1 to 2; u 3 comma w i w i. So, what we know is that, w and u 3 minus w are perpendicular, are orthogonal, fine.

Also, so what we from, this part I get that. So, this implies that, v 3 with w 1 is 0; v 3 with w 2 is 0. Question is, is v 3 non-zero, alright? So, if v 3 is 0 that will imply that. So, or which is same thing as saying that, v 3 is same as this is 0; because we wanted to look at, we are dividing by norm of v 3, not by v 3 as such.

Since we are dividing by norm of v 3, norm of v 3 should not be 0. And if norm of v 3 is not 0 means, v 3 should not be 0; because this is the condition that I have that v 3 is 0, if and only if the norm or the length of the vector is 0, alright.

So, I will assume that. So, suppose. So, let if possible, let if possible v 3 is 0; this implies that u 3 is equal to u 3, w 1 w 1 plus u 3, w 2 w 2. And this imply that u 3 belongs to linear span of w 1 and w 2, fine. So, if you remember your results about linear independence and dependence it says that, as soon as the u 3 is in the linear span; implies that the set u 1, u 2, u 3 is linearly dependent, alright.

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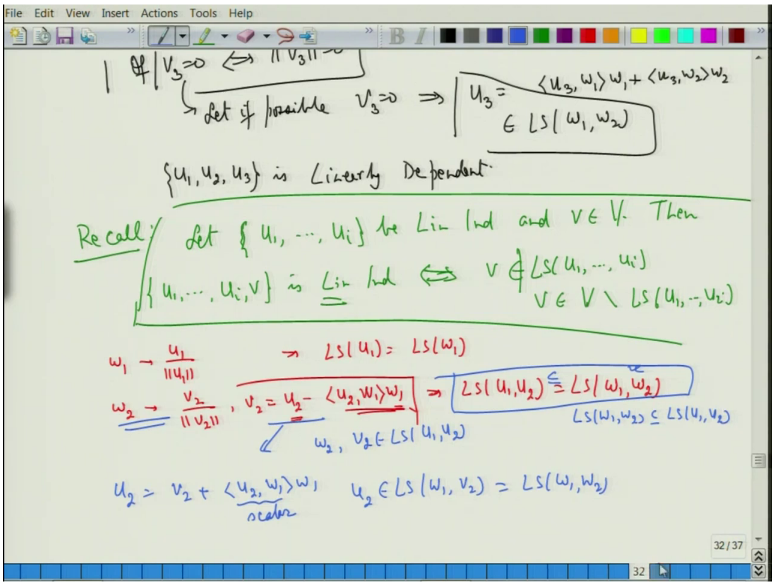

So, recall. So, again recall this theorem; to recall let u 1 to u i be linearly independent and v belong to V, then u 1 to u i comma v is linearly independent if and only if, linearly independent if and only if v does not belong to L S of S or which we say that v belongs to v minus L S of u 1 to u i.

So, not S here; so u 1 to u, this is what our theorem was in linear independence dependence, this is what we are using at each stage. So, let us go back; the statement here is that this is a linearly independent set, our set S is linearly independent.

So, at no stage; so linear independence implies that, u i does not belong to linear span of u 1, u 2 till u i minus 1 for i greater than equal to 2, alright fine. So, this is what is important that, there are some theorems which we may not have thought that so important; but they are playing a role that, we are taking that u i belongs to the linear span will imply that this becomes linearly dependent set. Recall also that the first time it happens alright; I have a problem, fine.

Because if I am saying that this set is linearly dependent; it means that there exist a k, such that u k is linear combination of the previous ones that also we had proved. So, what we are having here is that, S is the linearly independent set; and therefore u i is not the linear combination of the previous ones, and therefore at each stage everything is going smoothly for us, alright fine.

So, now, let us go back again, so that you have clarity with you. So, what we have done at this stage is, I have got w 1 which was u 1 upon length of u 1. So, this implies that linear span of u 1 is same as linear span of w 1; then we define w 2 as v 2 divided by length of v 2. And what was v 2? v 2 was equal to u 2 minus u 2 w 1 w 1 alright, this is how we have defined.

So, can I say from here that this implies that, linear span of u 1, u 2 is same as linear span of w 1, w 2, can I say this? So, understand here; if I want to look at linear span of u 1, u 2, then w 1 is already linear span of u 1. So, this is in u 1, I am looking at u 2. So, this part implies that, this part implies that, v 2 belongs to linear span of u 1, u 2. And w 2 is nothing, but v 2 upon this; so, this imply that, w 2 also belongs to.

So, w 2 also belongs to this part, fine. So, both this belongs to this therefore, this is a subset of this. So, we have shown that L S of w 1, w 2 is a subset of L S of u 1, u 2, fine. Now, the other way around; since we have already shown that v 2 is nonzero. So, I can write u 2. So, from here I can write u 2 as v 2 minus u 2, w 1 w 1, this is just a scalar quantity.

So, I am writing u 2 belongs to linear span of w 1 and v 2; but v 2 and w 2 is the same vectors as the unit vector difference. So, this is same as linear span of w 1, w 2. So, you have shown that this also a subset and therefore, this is satisfied, fine.

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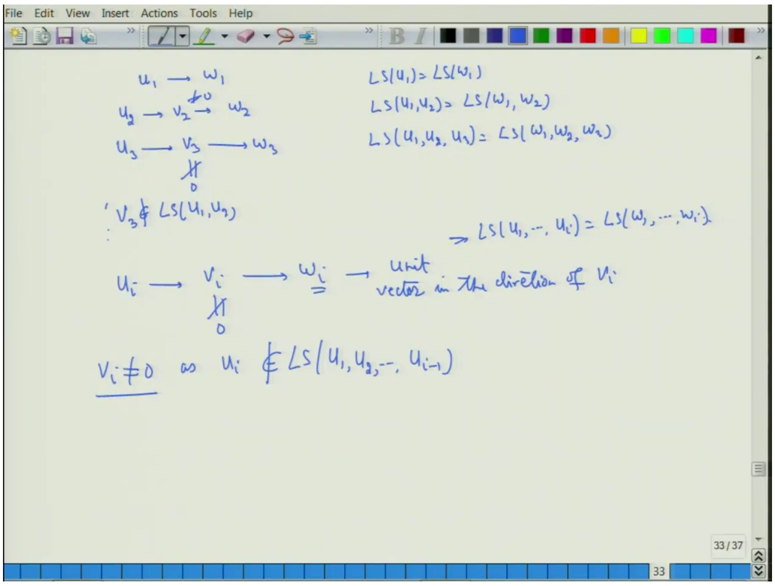

So, what we are done here is, we have shown that; I am going from u 1 to w 1; linear span of u 1 is same as linear span of w 1, u 2 gives rise to v 2, which gives rise to w 2. I get LS of u 1, u 2 is equal to LS of w 1, w 2; at the next stage I have u 3, I go to v 3. This is not 0. So, this was not 0; because of linear span of. So, v 3 does not belong to linear span of u 1 and u 2; I got v 3, from v 3 I get to w 3 and again LS of u 1, u 2, u 3 will be equal to LS of w 1, w 2, w 3, is that ok?

So, this way you can build up the whole thing; at any stage if I look at u i. So, u i will give rise to v i. Now, why is v i not 0? v i is not 0 as u i does not belong to the linear span of u 1, u 2, u i minus 1, alright. So, this is not 0; so, I can go from here to w i.

And w i, what is w i? Unit vector in the direction of v i, alright. So, in the process that, the way we are building up ideas this will imply that; LS of u 1 to u i will also be equal to LS of w 1 to w i, alright fine. So, let me end it here; will we will look at applications of this in the next class.

Thank you.