Linear Algebra
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Lecture – 45
Results on Orthogonality

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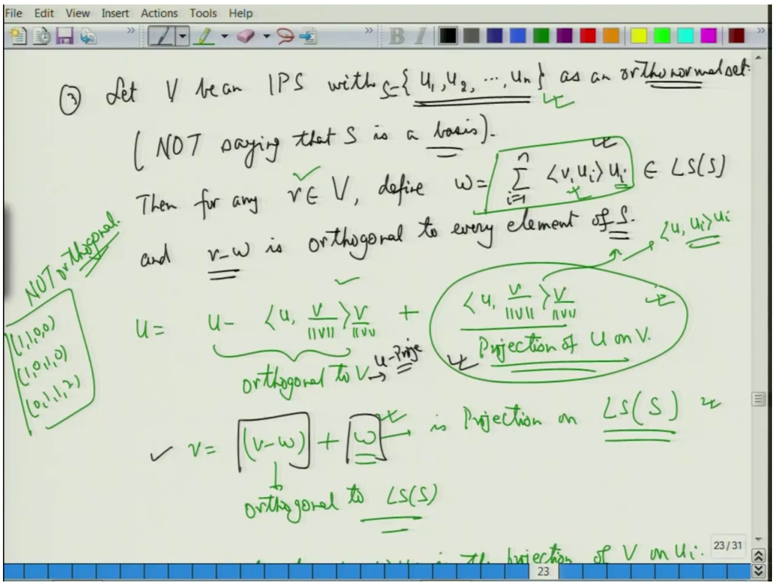

Alright so, we had stated this theorem yesterday if you remember. So, I will just open that page; we had stated it and gave an idea that why do we need such things. So, from the point of view of projection that is one dimension, looking at a vector which is u minus that so recall we had v as v minus w plus w which is similar to projection and the orthogonal projection u minus the projection, alright. So, we had looked at this yesterday and then based on that we wanted to generalize those ideas.

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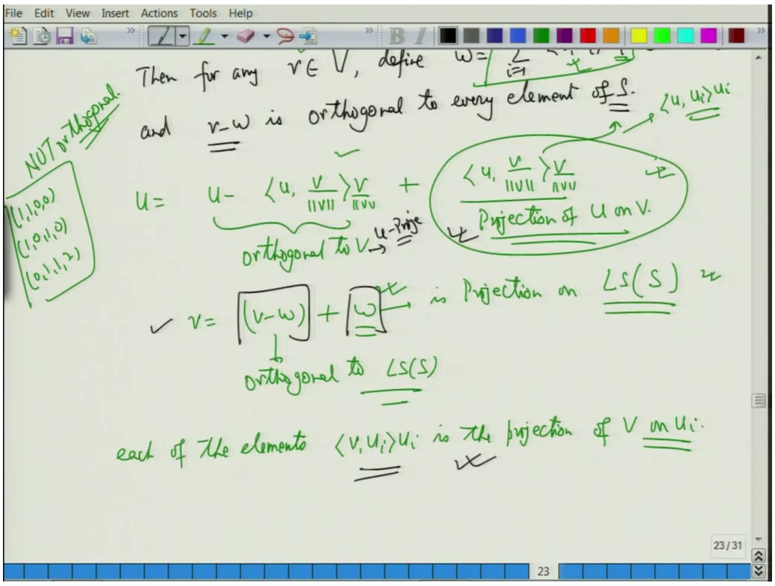

And that was the idea that we were trying to generalize that we are looking at the projection on u i, looking at linear combination and trying to proceed fine.

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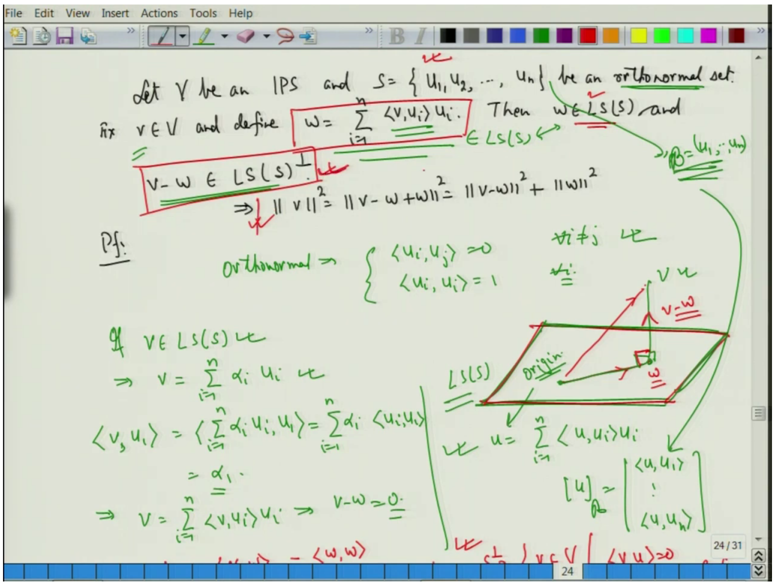

So, let us go back to this statement again and have a clarity of things. So, first thing we are saying here is that S is an orthonormal set. So, means orthonormal means orthonormal implies u i, u j; their dot product is 0 or the inner product is 0, for all i not equal to j. And length of each vector or the norm of each vector is 1 for all i, this is what it means fine.

Now, we are looking at w; w is linear combination of elements of S fine. This is what it says here then w belongs to this, because u 1 belongs u 2 belongs u n belongs; so there scalar multiple belong therefore, their sum belongs alright fine. What is more important is to understand this part that v minus w is orthogonal.

So, what we are trying to do is let us look at geometrically first that I have a plane here, this plane is generated by L S of S. So, every element here any u here is of the type summation u comma v i; v i, is that ok; so any combination here.

So, recall previous that if I taken this as my standard basis B as ordered basis as u 1 to u n; if I taken this as an ordered basis, then u in this ordered basis u in this ordered basis looks like u v 1 till u, I wrote u 1 here alright. So, u and u i here; u and u 1, so the last will be n here, fine.

So, let me correct it myself where did I write here v i wrote here, it should be u comma u i u i, we write anything else; no, so everywhere I wrote to u i yeah fine. So, what we are saying is that if I look at L S of S, it looks like this fine every element here is of this type. And I have vector v which may be in this or which may be outside; if v is inside, then it is a linear combination and therefore v minus w will be 0 alright.

So, if v belongs to L S of S itself alright, this will imply that v is linear combination of u i's; i is equal to 1 to n, this what will this will imply evacuate linear span. And from what we have understood or what I am I have written here in terms of B the ordered basis; if I look at the inner product of v with u 1 alright, v with u 1.

This gives me summation i is equal to 1 to n alpha i u i u 1 which by the definition of inner product that I can take out a scalars and so on alpha i times u i u 1 which will give me alpha 1 itself, alright because of the condition that i not equal to j it is 0. And this part because of orthonormality, I will get this part.

So, this tells me that alpha 1 is this and therefore, v is nothing but i is equal to 1 to n v comma u i u i itself. And therefore, fine if I look at v minus w, w is also of the same form itself will be equal to a 0 vector alright, fine. So, the idea is to look at things from that perspective that I have a subspace here L S of S and v is outside; if v is inside, it is already linear combination; if v is outside, now what I am doing is that from here I am dropping a perpendicular to this plane alright.

Once I have drop a perpendicular, this is the origin for me; because 0 0 is always in the linear span, the 0 vector is always in the linear span. So, given this v I am getting this vector this alright; and what is this vector, this vector is your vector w this what it says, w is this vector fine.

So, this is my w vector; what is v minus w? So, look at this; this is what v is and minus w means this, so from here I am going this way, so this is my vector v minus w alright. So, what it says is that v minus w is perpendicular to every element of this fine that is important. So, let us define what is perpendicular.

We have done it at the time of what is called fundamental spaces is associated with matrices, there we had by inner product as dot product. So, we want to define it for the general setup and then proceed alright. So, let me just define it here so definition, so given any set S as it is here.

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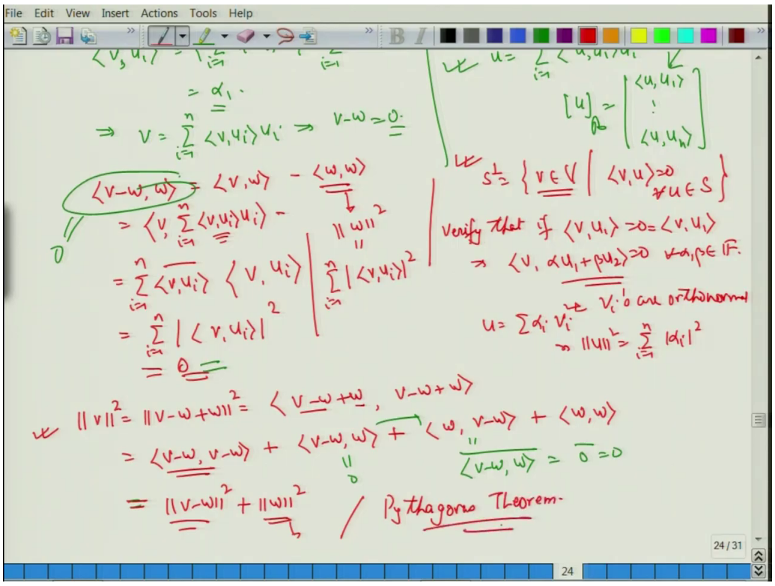

I can define S perp as all v belonging to V such that inner product of v with u is 0, for all u belonging to S, alright. So, it is collection of all the vectors which are perpendicular to elements of S alright, fine. So, here you phase you verify that verify, verify that if v is perpendicular to u 1 and so, v is so if v is perpendicular to u 1 and v is perpendicular to u 2, both this will imply that v is perpendicular to alpha 1 u 1 plus beta u 2, for all alpha beta belonging to whatever scalar you have, is that ok.

And therefore, what we are saying is that when some vector is perpendicular to the basis vector is perpendicular to everything else fine, so use this idea to let us prove that part. So, what we are trying to prove here let us look at it. We say here that this belongs to this that is by definition itself, because it is a linear combination. I have to prove that v minus w is perpendicular alright.

So, what we will show is that v minus w is perpendicular to all the elements of S, we are not going to prove it for every element of linear span; we will just prove it for every element u 1, u 2, u n. And therefore, the remark that I just gave here will imply that it will be therefore, everything.

So, let us look at what is the inner product of v v minus w i, w and what is this inner product. So, this by definition is v comma w minus w comma w. Now, what is v comma w? v comma w is v comma summation i is equal to 1 to n fine. I wrote it there, v comma u i u i, this is the definition alright and this is again same thing or the norm of v square you can say, it is just the norm whatever it is. So, I have just leave it for the time being we will see whether we need it or not.

So, this is same as i is equal to 1 to n v comma u i whole perp v comma u i fine, which is nothing but i going from 1 to n norm of or absolute value of v, u i whole square. And what is this, let us look at go back I think no, so anyway so let me do that also for you. So, this is nothing but norm of w square.

And what is norm of w square, so since I am looking at scalars here. So, if I am writing any vector u as summation alpha i v i alright and v i's are unit vector, unit u i's are or v i'is here, v i’s are orthonormal. And therefore, I get that norm of u square will be equal to summation i is equal to 1 to n mod of alpha i square.

And therefore, here w was already consisting of these elements, so it will be equal to norm of v minus u i whole square i equal to 1 to n. And therefore, they cancel out; so therefore this is 0 is that ok, so this is what you have to be careful.

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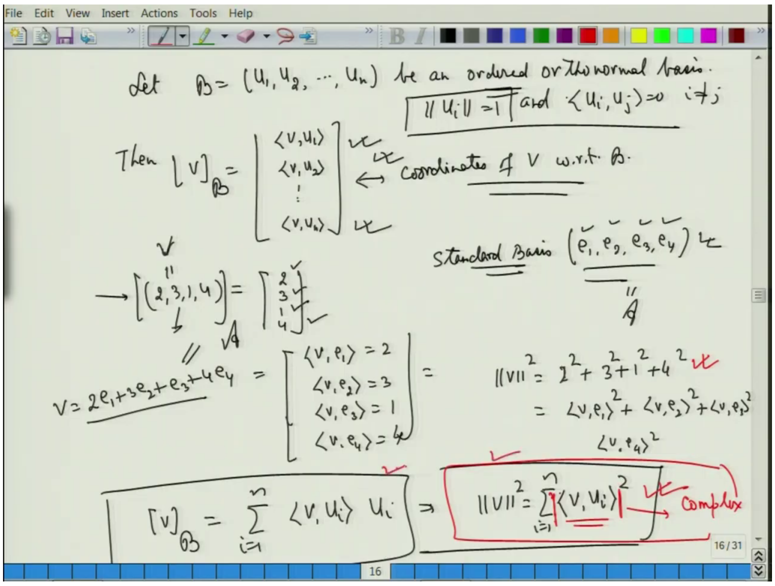

So, let me go to maybe the previous slide where I have done this I think, we are done this so look at here. We had shown that norm of v square is absolute value of v comma u i whole square absolute value of that alright. So, we had already proved it for any element, so I am just using that idea and nothing else fine.

So, we have shown that v minus w i w is perpendicular. So, we have given the proof in the sense that I have done this part, I have done this part, sorry, done this part. Now, if I want to prove this part norm of v square, then what happens is let us look at norm of v square. So, norm of v square is equal to norm of v minus w plus w whole square, which by definition is v minus w or how should I write? v minus w, let me expand it totally.

So, let me be careful otherwise you will do a mistake, plus w which is same as v minus w, v minus w plus v minus w, w plus w comma v minus w plus w, w alright; you just show that this is 0 alright, this is 0. Therefore, this part is 0 and this is nothing but this is equal to v minus w, w whole bar; so which will be equal to 0 bar, which will be 0 alright.

So, these two cross product terms are 0, cross product terms are 0, so what I am get a what I am left out is this thing which is nothing but norm of v minus w whole square plus norm of w square.

So, this is what we have the Pythagoras theorem, this is what we are saying in some sense that hypotenuse square is equal to the base square plus; so this is the base square and this is the perpendicular square alright. So, this is just say, rewriting what is called Pythagoras theorem. So, you have put it in a different framework that is all nothing else fine.

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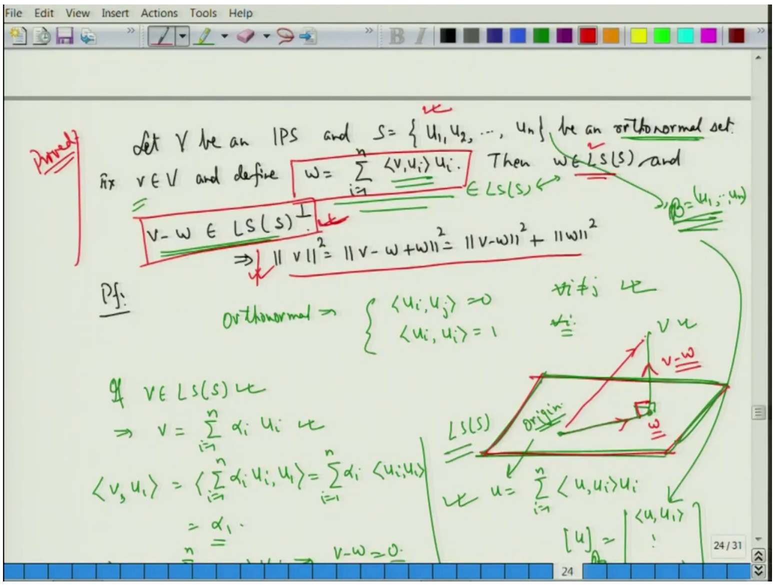

So, we have proved this theorem now so this theorem is proved. Let us try to understand its implication, it says lots and lots of implications fine, let us try to understand this idea again and again fine. First thing is that it talking about perpendicular, so when I am talking a perpendicular lot of nice things happens alright; so let us go one by one slowly and build up the ideas here alright, so let us look at the ideas.

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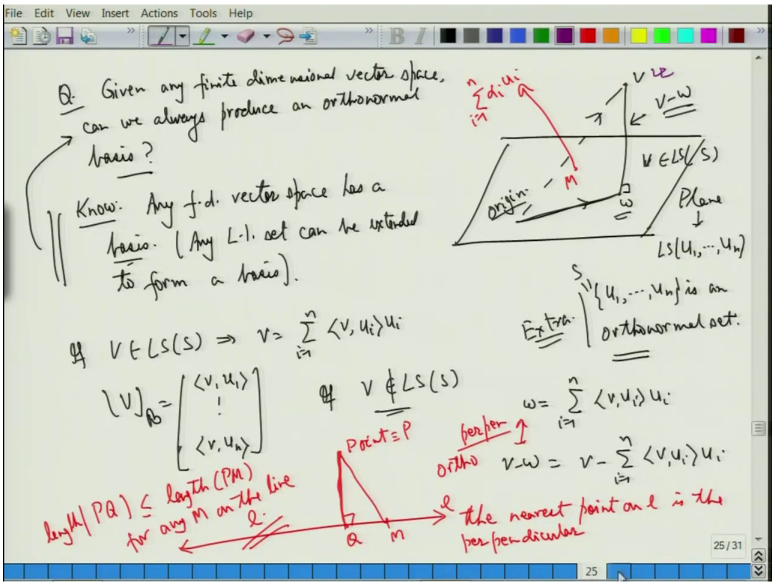

So, I have a plane here some plane p I have a plane, now this plane has is generated by certain spaces. So, it is generated by linear span of some vectors u 1 to u n alright, it is generated by this. It is given to me as a special case, as a special case it is given that the set u 1 to u n is an orthonormal set, so this is given to me extra. Till now, I am not been able to say that given any plane I can always get such a space, such a set that I am not able to do.

So, question is given any finite dimensional vector space, can we always produce an orthonormal basis? Fine, what we know is that given any finite dimensional vector space we always have is know, what we know? Any finite dimensional vector space has a basis, as only we do not have a we have a basis, what we have is that any linearly independent set can be extended to form basis, we have any linearly independent set can be extend to form a basis alright.

So, there should be a way to go from here to here alright that I start with one element and keep building, and then I should be able to get the orthonormal basis this is what one question is that I should be able to get it fine. Another thing is that we are trying to emphasize here is that if there is a vector v inside the set itself alright, if I have a v which is inside L S itself, L S of S, L S of this whatever it is fine, L S of S itself then I do not have to do anything, everything is inside.

And v looks like, so if v belongs to L S of S, then v is nothing but summation v comma u i u i, i going from 1 to n or which in some sense we are saying that v in the ordered basis is I am just stressing it again and again, which is very important for me alright.

u 1 till v comma u n I have this; if not, if v does not belong to the linear span of S, then it is somewhere outside alright. If it is somewhere outside, then what I do? I drop a perpendicular from here fine, and I have some origin with me linear span always contains an origin, so I get a vector w fine. Now, what is this vector w? This w is summation i going from 1 to n v comma u i u i fine, I have not got the whole vector I have got this.

So, this vector v that I am looking at alright is this vector alright this vector is this vector plus this vector fine. Therefore, this vector is this vector plus this, so we are looking at this vector which is v minus w. And so v minus w we are looking at which is v minus i going from 1 to n v comma u i u i. So, what we see is that these two vectors are perpendicular, so these two are perpendicular or what we said orthogonal fine; so these two vectors are very nice.

Now, let us look at from the geometric point of view. In our school what we had learned to was that if I want to find alright, given anything any geometrical object. So, given any point here and a line fine, similarly, a plane here, that if I want to find the nearest point on this line; so, given this point p, point p; the nearest point; the nearest point on l is the perpendicular. In the sense that from here, I just have to draw a perpendicular and this will give me Q which is the; so this p point p I think I should have written here P fine.

So, this distance P Q is so length of this is less than equal to take any point M, then this line less than equal to length of P M for any M on the line l, alright. So, what we are saying is that the nearest point 2P on this line is nothing but the orthogonal alright. Similarly, here if I take any other point here, which is M for me; I will get something which will be summation i is equal to 1 to n, some alpha i u i alright, then this point will not be nearest to v, but this is the point which is nearest alright.

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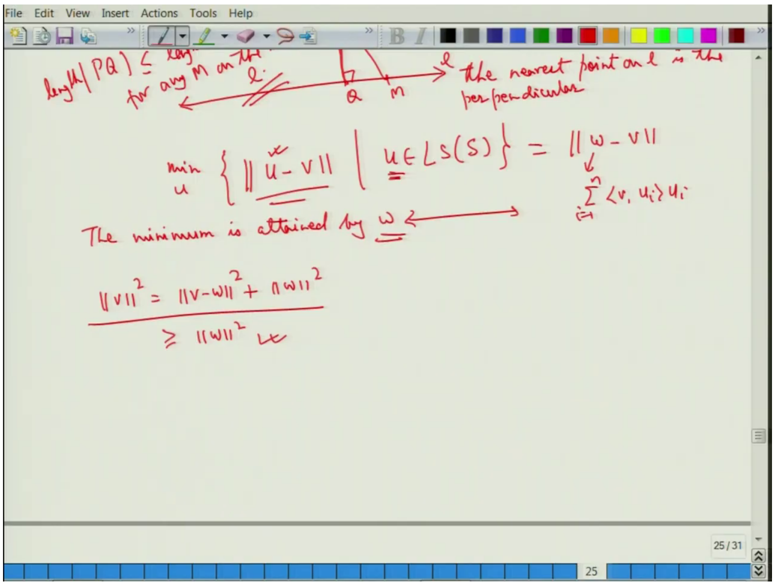

So, this is what you have to be careful. So, what we are saying is in another language is look at this minimum over. So, minimum overall so let me look at define it nicely. Summation; length of say, u minus v, length of this, such that u belongs to L S of S; look at this and minimum is overall u fine.

So, this is attained. So, this minimum which is a number this is attained for not u, so I am looking at any u here; it is attained for w which is so w which is nothing but i is equal to 1 to n v comma u i u i, the minimum is attained.

So, the minimum is attained by w alright, this is very important that the minimum is attained by w and why it is attained because of what we had proven here that or let me write here itself that norm of I am looking at v here, so norm of v square is equal to norm of v minus w whole square plus norm of w square.

So, therefore if I see this, we have already seen that this is equal to this. So, we are saying that this is greater than equal to this is greater than equal to norm of w square; we are saying this as well as this part because of this idea itself, because they are perpendicular.

So, again understand when I have a geometrical object, I have an angle, I have perpendicularity, then the smallest distance is nothing but the one which is a perpendicular and this is what we have got that this is perpendicular, would like to use this idea to build up things. So, let us now look at some implications of this as far as our fundamental spaces are concerned. So, application to fundamental spaces.

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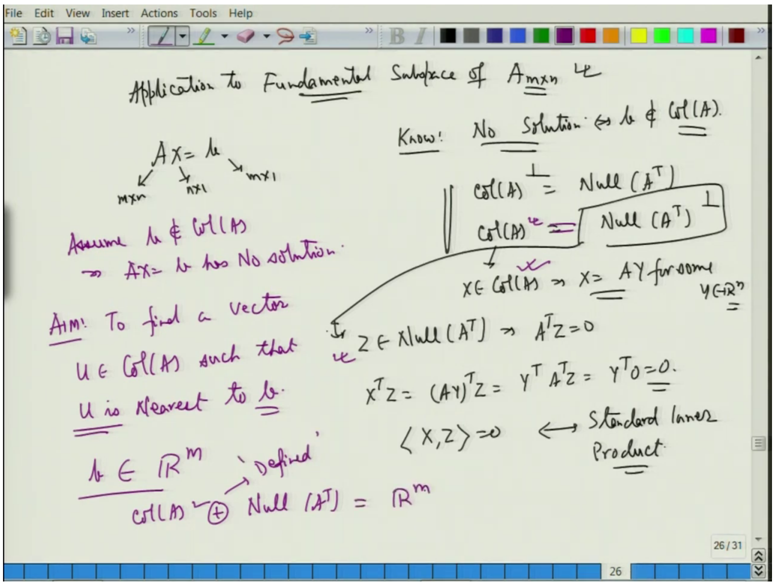

Application to fundamental subspace of A, which were m cross n, fine. So, I want to solve the system A X is equal to b; A is m cross n, X is n cross 1, b is m cross 1 fine. We have already seen when so already know the requirement for no solution alright. We already know the requirement for no solution and what it says is that no solution if and only if b does not belong to column space of a, alright.

So, this system has no solution if and only if b does not belong to column space of A, this is what we had seen. What we had also seen was when we are trying to study fundamental spaces that look at column space of A, the orthogonal of this is nothing but null space of A transpose, and why did we do that, understand.

So, look at this part or which is same thing as saying that look at column space of A is equal to null space of A transpose whole perp, whatever you want to understand, whatever you want to do fine. So, we have proved this part first, recall here that take any X belonging to column space of A, this will imply that X is equal to A times Y for some Y belonging to R n fine.

If I take anything here, Z belonging to null space of A transpose fine, this will imply that anything Z belonging to null space of A transpose not the perp, this will imply that A transpose Z is 0, fine.

So, now let us look at this, X and Z what happens to X and Z from here. So, I want to look at what is X transpose Z. X transpose Z is A Y transpose Z which is Y transpose A transpose Z, which is Y transpose 0 which is 0, alright. So, what we are saying is that X and Z, they are perpendicular with a standard inner product. Since, I am looking at R n for A of m cross n you are looking at only a standard inner product.

We have proving results for general setup, but when we come to applications it is only the fundamental spaces R n or C n that we are looking at. So, what we have shown here is that any element of column space, any element of column space and any element of the null space they are perpendicular, and therefore these two are true, is that ok.

Now, we are looking at the case when I do not have a solution; because if I have a solution, I know how to solve it alright. So, my case is looking at as I said in the previous slide also, we want to look at the case when v is above, v does not lie in the linear span alright. So, again we are doing the same thing, so we are assuming, assume b does not belong to the column space of A implies A X is equal to b has no solution.

So, aim: to find an find a vector find a vector u belonging to column space of A such that u is nearest to b. I cannot get b using columns, but so I want to find the nearest alright. So, what we do? We write b as so we know that b is in R m whatever it is, so b belongs to R m.

We also know that column space of A is perpendicular to this, so column space of A is perpendicular to this. So, I can use column space of A and null space of A transpose, I can look at this, I have not yet defined what it is, define to get whole of R m.

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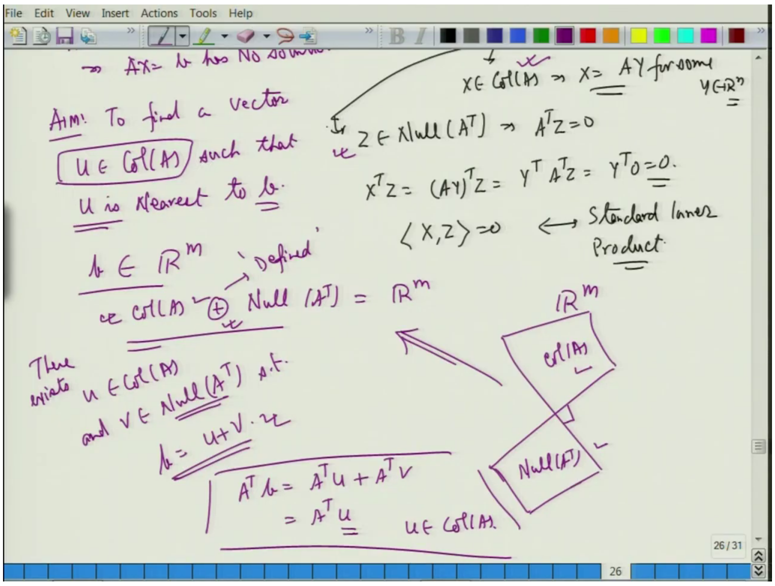

So, if you remember your fundamental spaces, I had done that part this part I did not write this direct sum here this notion. But what I had done was that I wrote R m as this is what I had done here fine. We wrote here column space of A and with they were this null space of A transpose which was perpendicular alright.

So, this is what I am putting here R m, I wrote like this, and which is this language is same as this they are perpendicular for me alright. This has other meaning, but it is since they are perpendicular. So, this will imply this part alright fine.

So, what we are saying is that any vector b can be written as sum of this and sum of this. So, we can say that there exist u belonging to column space of A, and I am writing V belonging to null space of A transpose such that our b is equal to u plus V. Is that ok?

Now, I want to understand this using everything fine. So, this is what is important for me. So, let us look at. So, I know that b is this. Let us multiply by A transpose what we know is that A transpose of V is 0. So, we multiply by A transpose of b, I get a transpose of u plus A transpose of V which is same as A transpose of U. And what is U? u belongs to column space of A.

So, in some sense, we can see here that I have been able to get something which is near. So, A transpose u is inside column space of A alright. And this is what we wanted; I wanted something which is inside fine. So, I have got something here. So, let us try to rewrite this in terms of what I am trying to say.

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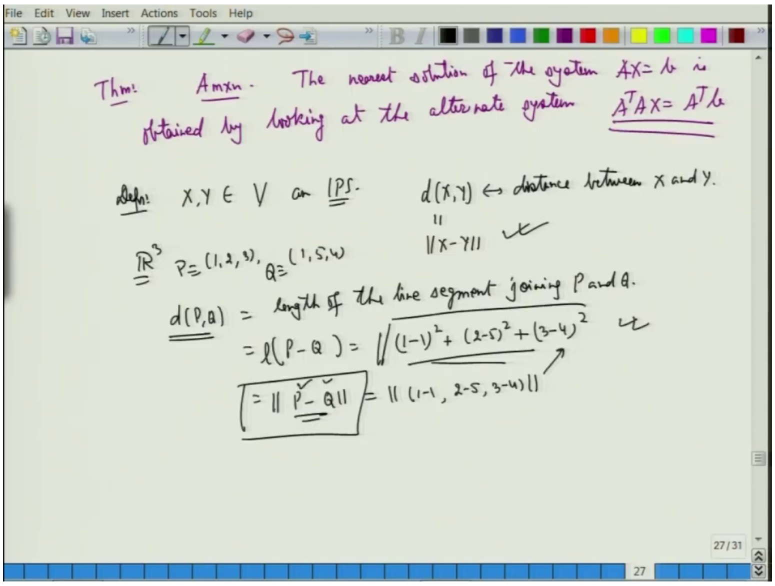

So, theorem: A is m cross n fine. The system the nearest solution of the system A X is equal to b is or any system this is obtained by looking at the alternate system A transpose A X is equal to A transpose b. So, let me try to rephrase this in some other language. So, to do that, I will start with this following definition alright.

So, given two vectors X and Y belonging to any vector space V an inner product space, I can talk of distance between X and Y. So, this is the distance between X and Y to be equal to length of X minus Y, alright. So, this is what we have been doing that if I am taking it R n or say R 3 I have vector 1, 2, 3 and I have vector say 1, 5, 4, P and Q here fine.

Then the distance between P and Q was nothing but the length of line segment, length of the line segment joining P and Q. So, which, was same as looking at so we looked at this minus this.

So, we looked at P minus Q looked at the length which was nothing but; so just keep recalling things one after the other it was 1 minus 1 whole square plus 2 minus 5 square plus 3 minus 4 square; squared this is what we had alright. So, this is nothing but, look at this part, this is nothing but length of P minus Q for us because this is the way we define P minus Q.

So, first length of P minus Q means just remove the components here. So, take their minus. You are supposed to look at P minus Q is 1 minus 1. Then second component is 2 minus 5, third component is 3 minus 4, we looking at this. And whose length is nothing but this alright. So, this is what we have learnt in school and this is what we have. So, defining distance between P and Q as length of P minus Q, so, this you can do it for the general setup also. So, whenever I have an inner product, I can talk of this.

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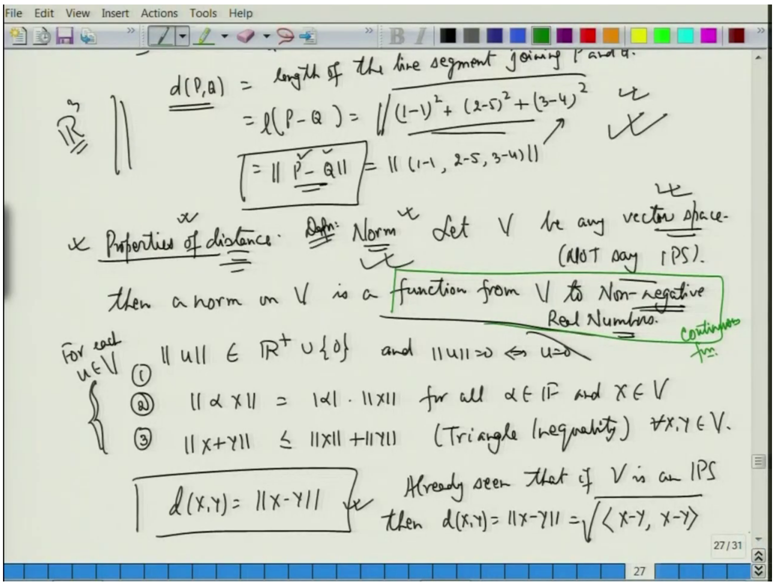

Now, what are the properties of a distance that you should have? So, properties, properties of distance that you would like to have distance. So, in place of distance, what you use the word, alright, norm. So, let me look at norm. So, what is a norm? So, definition of norm, so, given any inner product let V be any vector space, note, I am not saying it inner product space not saying IPS. I am just saying it is a vector space alright.

Then a norm on V is a function from V to non-negative real numbers for us alright. So, it is a map from V to R plus union 0 alright. So, it is R plus union zero non-negative means 0 is included, alright fine. So, the map is so it should have certain property. So, generally we write it as this. So, for each u belonging to V this belongs to R plus union 0 alright.

And what you need is that norm of u is 0 if and only if u is 0, we need this part alright for us. So, let me just have the right things, otherwise there are many problems. So, this is the first thing for us alright. So, 1, 2 norm of alpha X should be equal to mod of alpha into norm of X, for all alpha belonging to a scalar whatever it is, and X belonging to V.

3, what we need is the triangle inequality that norm of X plus Y should be less than equal to norm of X plus norm of Y, alright the triangle inequality, for all X Y belonging to V, this is what we need for a norm alright. Now, using this norm, we have this definition of distance which is distance between X and Y is nothing but norm of X minus Y. Is that ok?

So, I will come to this norm afterwards, you can see that they are lot of norms, you can also we have already seen already seen that if V is an inner product space, then distance between X, Y which is norm of X minus Y is same as X minus Y, X minus Y inner product, and then a square root of that alright.

So, this we have already seen this is the example here also that I have in R 2 or I looked at R 3 in R 3 also I have the same thing. But in general there are other norms also and we look at them at the end of the lecture on inner product spaces. So, I would like you to see that such things are there. So, let us go back and try to understand that what we are talking about. So, we are talking about there should be a distance, distance is basically coming from the norm that we are doing here.

Once we have norm, norm is nothing but looking at the distance in some sense or these properties that I have written here. I would like you to see that this is a continuous function alright. See, so this function that I am defining here alright, the function from V to non-negative real numbers this is a continuous function alright.

Since, this a continuous function in any part in any interval, it will attain its maximum and minimum closed interval. And therefore, when I am trying to look at distance of anything from a convex part which is closed things will be nice. So, for us linear span we are looking at which is a plane. So, it is a very nice object.

I am looking at, so I have a plane I have something I am dropping a perpendicular. So, I will get something which is the minimum distance and that minimum distance will be a unique point because of Pythagoras theorem alright. So, I am doing that part.

Thank you.