Linear Algebra
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Lecture – 43
Projection on a Vector

Alright. So, in the last class I had defined what is cos theta?

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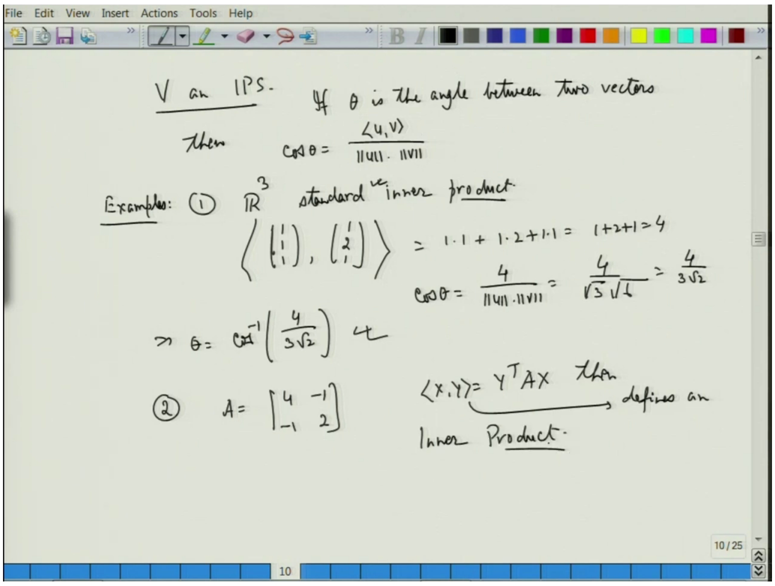

So, assume that so, I have a V, which inner product space V an inner product space. So, where, if theta is the angle between two vectors, then we defined cos theta is equal to u v divided by length of u into length of v, this is what we had done, alright.

So, there from giving this definition, I just went directly to the perpendicularity orthogonality fine, to get you some ideas about parallelogram law, Pythagoras theorem and so on. Now, I would like you to compute this and then proceed further in those direction. So, just some example here examples so, examples first example so, we showed that there are certain vectors in R 2 which are 0 under the standard inner product. So, let me again look at a R 2 itself with a standard inner product, fine.

So, under standard inner product, if I have look at the vector say while R 3 like put it R 3, then I want to find the vector this and 1 2 1 inner product of this is nothing, but just compute it, since I am looking at a standard inner product, it is 1 into 1 plus 1 into 2 plus 1 into 1, which is same as 1 plus 2 plus 1 which is, 4 fine.

So, the angle between these two vectors cos theta of the angle between them is nothing, but 4 divided by length of u into length of v, which is 4 upon length of u is root 3 and length of this is 1 plus 4 is 5 plus 1 is 6 what it is. So, this is equal to 4 upon 3 root 2, alright.

So, this implies theta is equal to cos inverse of 4 upon 3 root 2, fine. So, you can compute this yourself. Another example so, I think I had taken matrix A as in R 2 as 4 minus 1 minus 1 2 alright, with this we had that, if I define x y as Y transpose A X, then this defines an inner product, we have done that part, fine. So, under this let us compute the angle between sum vectors.

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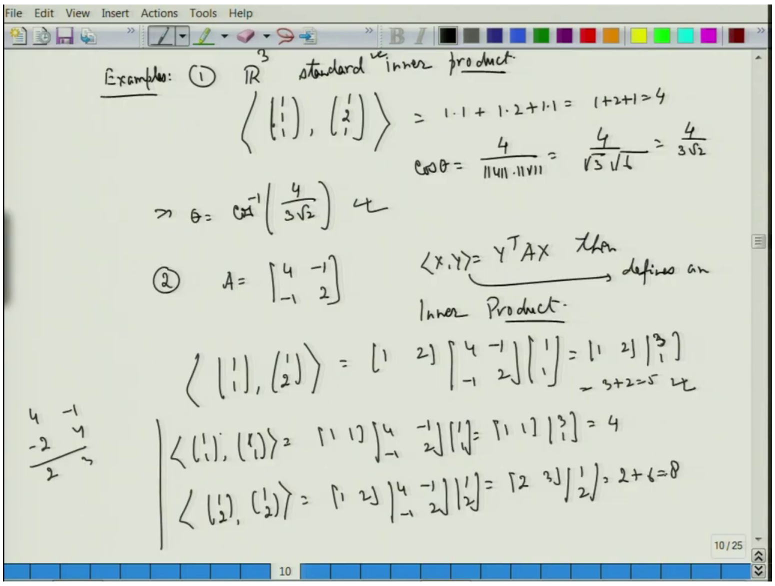

So, suppose, I want to look at the vector 1 1 so, I want to look at inner product of 1 1 with 1 2. Suppose I want to look at this inner product, then this is equal to 1 2, because I am looking at Y transpose X 1 2 that is 4 minus 1 minus 1 2 1 1, which is same as fine 1 2.

Now, just have to add these 2 1 and 1 is just add these 2 4 minus 1 is 3 and this is 1 here. So, this is same as 3 plus 2 which is 5 is that ok. So, this is a dot product between x and y. Now I have to look at x and x to get the length of x and then y y for the length of y. So, what is 1 1 comma 1 1? This is equal to 1 1 times 4 minus 1 minus 1 2 1 1, which is 1 1 times 3 1. I hope I am doing correctly, because you will find lots and lots of mistake in my calculations.

So, keep especially keep during the verification. So, 3 plus 1 is 4 with 1 2 1 2, if I look at it will be 1 2 4 minus 1 minus 1 2 1 2, which is same as whatever it is. So, let us see so, it is 4 minus. So, let me write it, because I do lots of mistakes so, 4 minus 1 and multiplied by 2 so, minus 2 and 2 2 4.

So, this gives me 2 and 3. So, this is 2 3 with 1 2 so, it is 2 plus 6 which is 8, please check alright, because I must have done a mistake alright. So, just let me look at 4 minus 2 is 2 fine and minus 2, sorry minus 1 plus 4 is 3 I hope it is correct fine.

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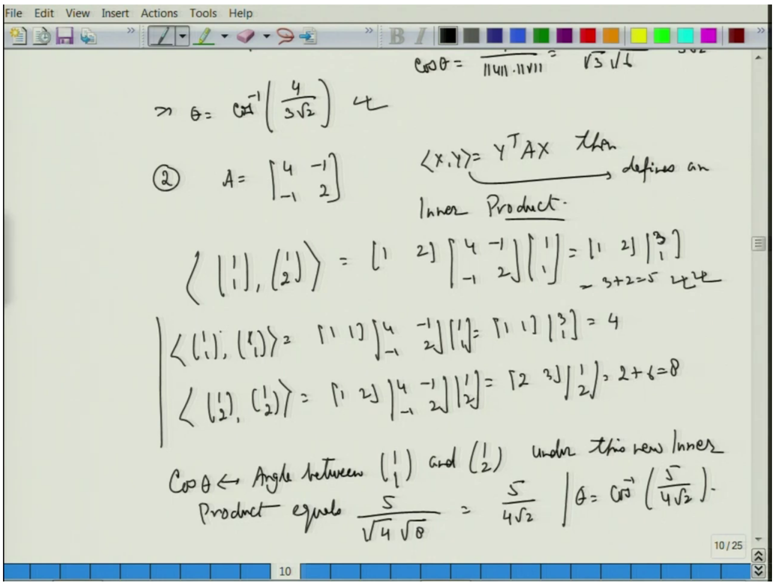

So, therefore, in this case cos theta cos of theta, which is the angle between 1 1 and 1 2 under this new inner product, product equals numerator will come from here 5 divided by a square root of 4 and a square root of 8. Which is same as 5 times 2 into 2, that is 4 root 2 is that ok.

And therefore, theta is cos inverse of 5 upon 4 root 2. So, please check it yourself, I hope I have done correctly. So, I would like you to learn them try to do them is that ok. Now, let us come to the next idea what is called projection.

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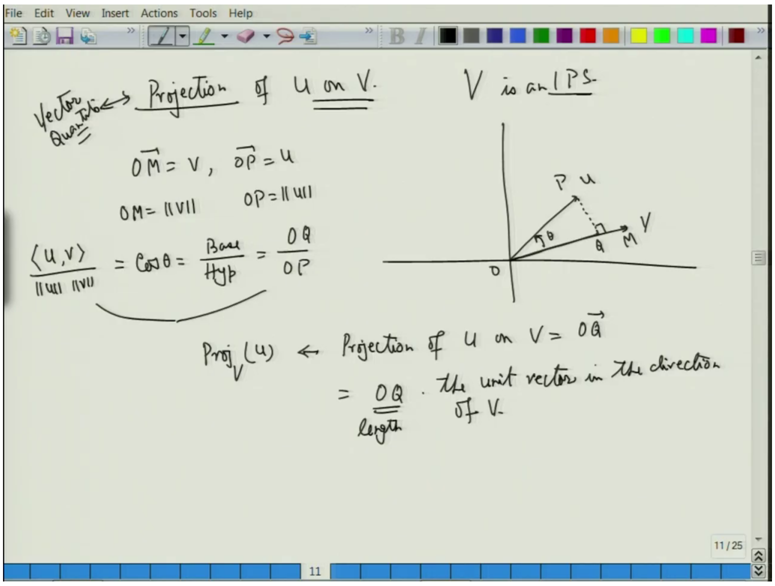

Projection of u on v alright so, I have a V inner product V and inner product space and I have got two vectors u and v, I want to look at the projection of u on v.

So, you have already done this in your school in physics, what is called u cos theta and u sine theta and so on. But, let us try to do it with vectors notations here alright. So, this is what I have fine? So, this is my v vector, this is my vector u and I want to look at projection here. So, this is 90 degree with me. So, let me write those point as 0 P Q M alright. So, the vector

OM is your vector v OP is your u those vectors the number OM is length of v OP length of so, OP the length OP is equal to length of v and so on alright.

If this is angle between both the vectors theta, then by definition cos theta so, cos theta is equal to what is cos theta? Base upon hypotenuse. So, base is so, base upon hypotenuse. What is base? Base is OQ and hypotenuse is OP, fine this is 1. This is also equal to u v divided by length of u into length of u this is also equal to this.

So, the idea is to relate these 2 ideas and get my answers fine here, it is a scalar quantity here it is a scalar quantity, but I want the projection so, I want projections projection. What is the projection? Projection is a vector quantity vector quantity, fine. So, what exactly we need is? So, projection of u on v, this is what it is projection of u on v. So, what I have to look at. So, by definition this will be equal to projection is nothing, but OQ vector.

So, it is a vector OQ which is same as length of OQ. So, this is the length of OQ into the unit vector in the direction of V alright. So, we have to know what is the unit vector in the direction of u?

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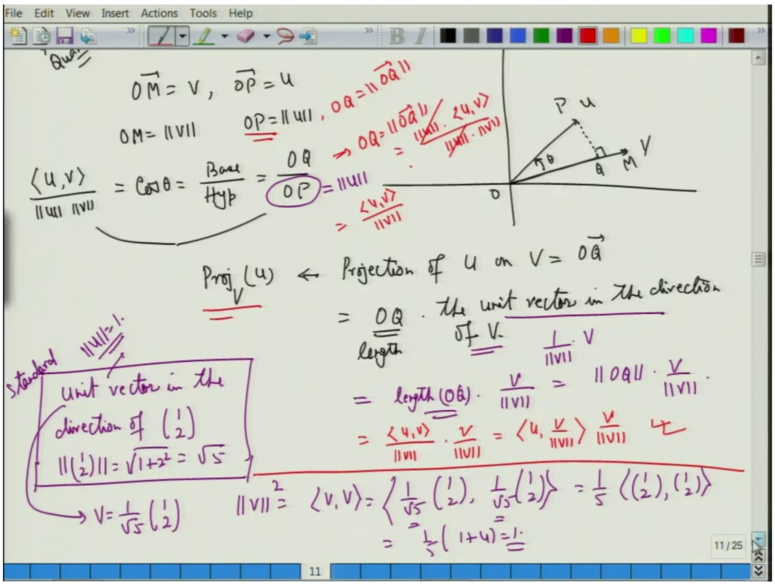

So, let us compute that unit vector in the direction of suppose I have got this vector in this. So, what is the unit vector in the direction of this? Unit vector means the length of the vector. So, unit vector means the length of the vector should be 1 alright. Now, under the standard inner product; under the standard inner product alright, length of this vector is 1 plus 2 square root, which is the square root of 5. So, if you want a unit vector in this direction, I need to look at 1 upon root 5 into 1 2 fine.

Because, if I write this vector as V, then length of v square is equal to v, V which will be equal to 1 upon root 5 1 2 comma 1 upon root 5 1 2, this will come out as scalar quantities. So, 1 upon 5 and then it will be inner product of 1 2 with 1 2 will be equal to upon 5 times 1 plus 4, which is 1 alright.

So, when I want to look at the unit vector in any direction, I need to look at the vector divided by the length of the vector. So, any unit vector in the direction of V will be equal to v into 1 upon length of v is that ok. So, it is what you have to careful. So, therefore, projection of u on v will be equal to length of OQ into V upon norm of v length of OQ, in our notation is this length into V upon norm of u is that, ok.

So, we need to compute this part. So, we need to compute OP is already known to us What is OP? OP is a scalar quantity. So, it is nothing, but length of u itself, I want length of OQ. So, let us go back to OQ here fine so, from here what I get here is that this implies OQ, which is same as length of the OQ vector, because it is what we had taken we wrote OP, we did not write OQ alright. So, since I am writing OP here like this. So, OQ here will be length of the vector OQ, fine.

So, length of OQ will be equal to OP times this. So, this implies this is equal to OQ is equal to just this OP will get in the numerator, norm of u into u v divided by length of u into length of v, this part and this part cancels out you get it as u v divided by norm of v is that, ok. So, this by definition projection is nothing, but u v divided by norm of v, this is the projection the length from here, and then V upon length of v is that ok.

So, this is same as u comma again be careful here, you are putting a scalar quantity here. If I mean real number then, this is fine for us is that see if a real number, then this number is the length. I can put it inside this is, what I get is that ok. So, this is what you have to remember that the projection is this vector. So, let me write it down for you. So, that you remember it.

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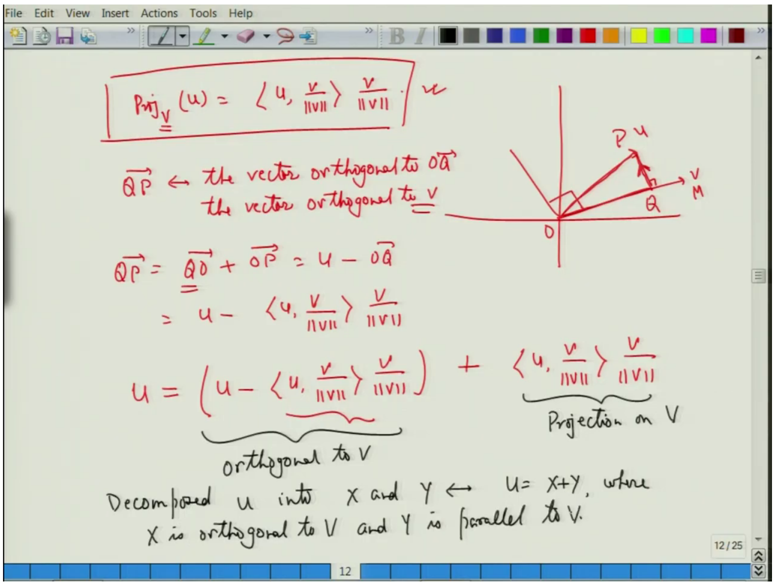

So, somewhere so, projection of u on v is just write u as it is, since we are talking projection on v it is a unit vector in v that I have to look at. So, v upon length of v divided by v upon the length of v is that ok. So, this is what you have to be careful you have to understand this part this is what the projection is, fine.

So, again let me look at the diagram in that I have this is my v this was my u this was the projection so, O P Q m, fine. So, I am getting this as the vector, which is the projection fine. Now what is this vector, if I want to find out? This vector is same as this vector fine.

Now, what is this vector? This is minus of this plus this. So, if I want to look at vector QP, which is the vector orthogonal, to OQ or the vector orthogonal to P orthogonal to v alright. Then QP by definition is equal to QO plus OP OP is same as u for me, and this is same as minus of OQ, which is same as u minus u times v divided by length of v into v upon length of v is that ok.

So, therefore, what exactly we have done is so, let us look at this nicely. So, I have got the vector u as I am writing u as u minus u comma v upon length of v v upon length of v, this is one vector plus just this part u comma v upon length of v v upon length of v is that ok.

So, I got two vectors here what are the two vectors? This is projection on V, this is orthogonal. So, what we have done is I have decomposed, decomposed u into X and Y what do I mean by that? I am writing u as x plus y, where X is orthogonal to V and Y is parallel to V is that ok. So, I have done the decomposition. This is what you have to be careful about you have to understand, what has been our process.

So, my process has been very simple that I wanted a projection onto V, and I have given vector u I have given a vector v, I want the projection of u on v, fine looking at projection on u and v means at this I have to so, from here I am to look at a, which is parallel to this like this and something, which is perpendicular to this. So, this was your so, this is the vector that is u is this was the angle theta, you are looking at u cos theta and u sine theta, fine.

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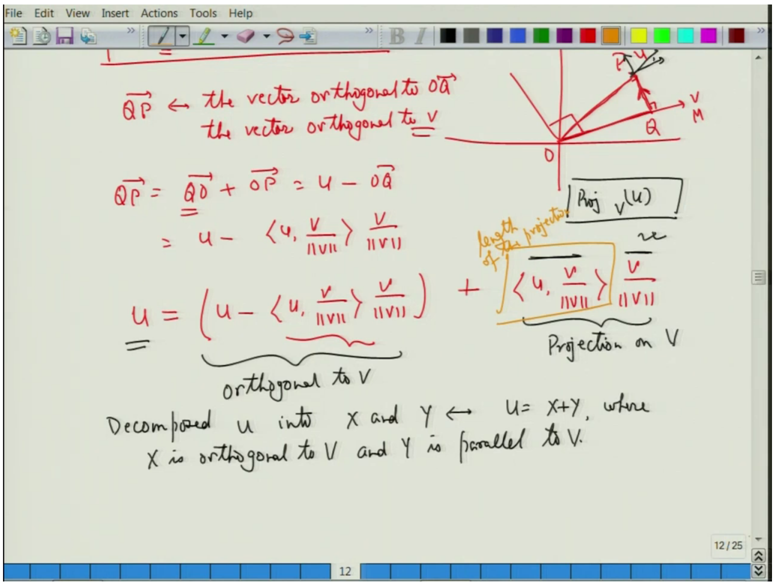

So, u cos theta is your this part and this is the u sine theta part. So, in the language of vectors I do not talk in terms of there, what we are saying is that I have decomposed u into two things, one which is perpendicular to V and the one which is parallel to V is that that is what you have to understand and idea is very simple.

Since we are looking at u with respect to v here projection, I am stressing it on again and again its very important idea that you have to look at inner product of u with the unit, vector v this is what we have done u with inner product v here and then multiplied by v.

So, this gives me the scalar quantity the left length in some sense and this is the direction of that. So, this is the length of the projection. So, this part is length of the projection. Why it is the length of the projection?

Because this is a unit vector so, when I took the length of the overall this will give me only one and this will give me the length of the vector OQ, fine. So, I have decomposed u into 1 orthogonal and the projection, fine. Let us look at some example to try to understand it example.

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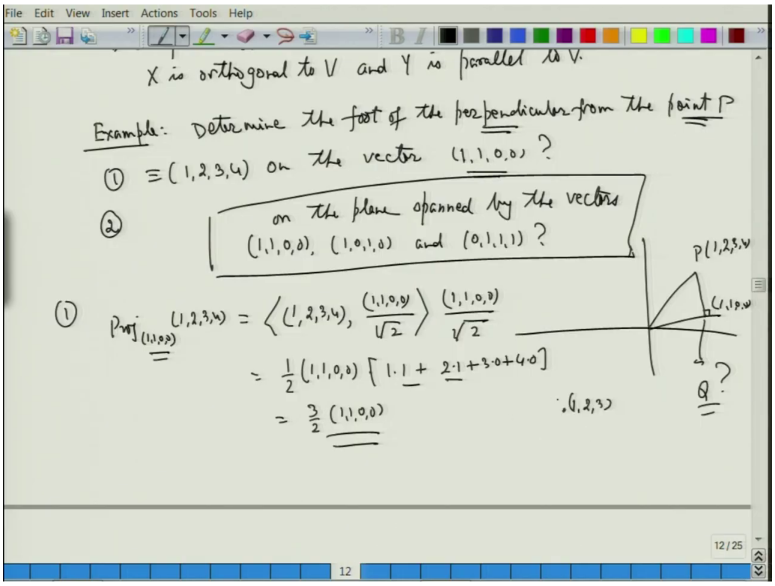

Alright. So, my question is determine the foot of the perpendicular from the point Q, which is 1 2 3 4 on the vector 1 1 0 0, suppose I am looking at this, that is my first question. Second is the same thing, but on the plane, on the plane spanned by the vectors 1 1 0 0, then 1 0 1 0 and 0 1 1 1.

So, these are the two questions that I would like to answer alright, and then end this class with some ideas alright. So, let us write that out. So, I am looking at determine the foot of the perpendicular from the point Q, which is this on the vector this fine. So, when I am saying the foot of the perpendicular, again what I am doing is that I am looking at the coordinates of the point Q.

So, sorry I should have written P here in my notes it is Q, but here I am looking at P here. So, this is my vector which is 1 1 0 0 suppose this is my vector 1 1 0 0 in some plane this is my vector P which is 1 2 3 4 I have to draw a perpendicular.

So, this is a point that I am looking at what is this point alright. So, in our notation this is supposed to be the point Q, fine. So, I want to find out this. So, what I am supposed to look at fine just look at the projection. So, what is the projection? So, projection of so, 1 projection of 1 2 3 4 on the vector 1 1 0 0, so this by definition is 1 2 3 4, then I have to look at this vector, which is 1 1 0 0 divided by the length of this sorry alright.

So, I am not here using the column notation, because it requires a lot of space. So, I am just writing like this, but at the back of my mind I have everything with respect to column representation alright. So, this is same as root 2 root 2 gets multiplied it is half, 1 1 0 0 and the inner product of these 2 is nothing, but 1 plus 1 1 into 1 plus 2 into 1 plus 3 into 0 plus 4 into 0 which is same as 2 plus 1 is 3. So, 3 upon 2 times 1 1 0 0 is that, ok.

So, this is the foot of the perpendicular that is why we are asking for so, I have got this part fine. Now I want this on the plane spanned by this. So, what do I mean by this on the plane, fine. So, there are two ways of going about it one way is that.

So, if I want to look at so, let us try to understand geometrically first fine. So, if I look at say the point say 1, 2, 3. So, suppose I have the point 1, 2, 3 then I can write so, 1, 2, 3 in some sense if I want to get it in so, projection of so let me do it for R 2 first alright.

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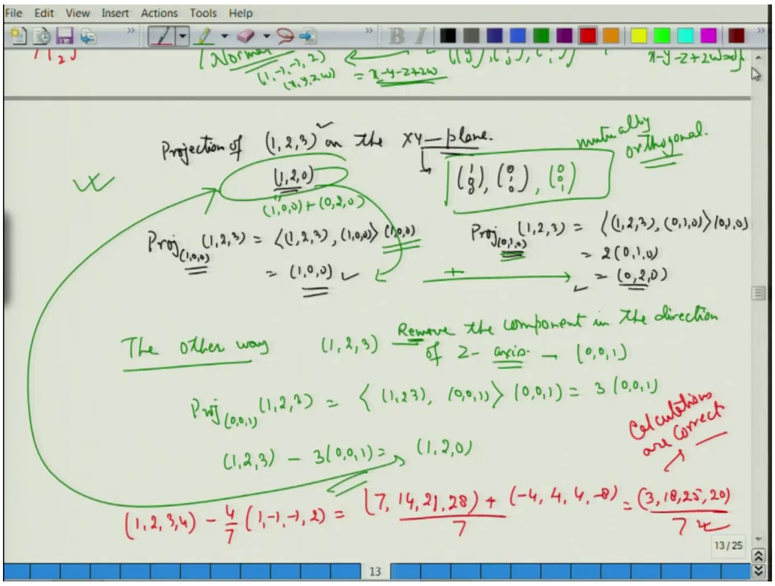

So, projection of 1, 2, 3 on the xy plane alright. Now, looking at the projection of this on the xy plane we know it is nothing, but 1 2 0 fine and from where do I get 1 2 0, we know this part how do I get it. So, the xy plane is generated by the vector 1 0 0 and 0 0 1 0 fine. So, let us look at the projection of this. So, we want to compute projection of 1, 2, 3 on 1, 0, 0 and I want to compute projection of 1, 2, 3 on the 0, 1, 0 these 2.

So, this by definition is 1, 2, 3 into 1, 0, 0 the length of this vector is already 1 nothing to do. So, this is same as 1, 0, 0 you can check here, if I look at this this is 1, 2, 3 into 0, 1, 0 again the length of this vector is 1 so, nothing to do 0, 1, 0 which is 1 into two times 0 1 0, which is 0, 2, 0 alright, fine.

So, we will look at 1, 2, 0 is basically some of this and this fine. So, this is equal to this and this together that is all fine 1, 0, 0 plus 0, 2, 0. This is one thing the other way is to think of that you look at so, the other way the other way will be that from 1, 2, 3, remove the component in the direction of z axis is that ok.

So, what is the direction component is direction of z axis? The foot of the perpendicular on z. So, what is z axis? z axis is 0, 0, 1. So, I have to look at projection of 1, 2, 3 on the vector 0, 0, 1 so, this is nothing, but 1, 2, 3; 0, 0, 1 again it is a unit vector. So, it remains it is so, it is same as 3 times 0, 0, 1.

So, from these 1, 2, 3 if you look at this I have to remove. So, 1, 2, 3 minus 3 times 0, 0, 1 which is nothing, but 1, 2, 0 alright so, I have got this 1, 2, 0 once more fine. So, there are two ways of doing it either I look at the component in each of these directions or I remove the component, which is perpendicular to that, fine. So, these are two things very very important I would like you to understand here, that these vectors 1, 0, 0; 0, 1, 0 and 0, 0, 1, they are perpendicular vectors.

So, these three are mutually orthogonal. So, that is very important we need to look at mutually orthogonal vector, fine. So, what I would like you to do is that in the previous example, these three vectors they are not mutually orthogonal, fine you can see that this and this are even not perpendicular, because there is one which is common here.

Similarly this and this is not because there is third component common and here the second component common. So, we will have to find out something which is orthogonal to all of them.

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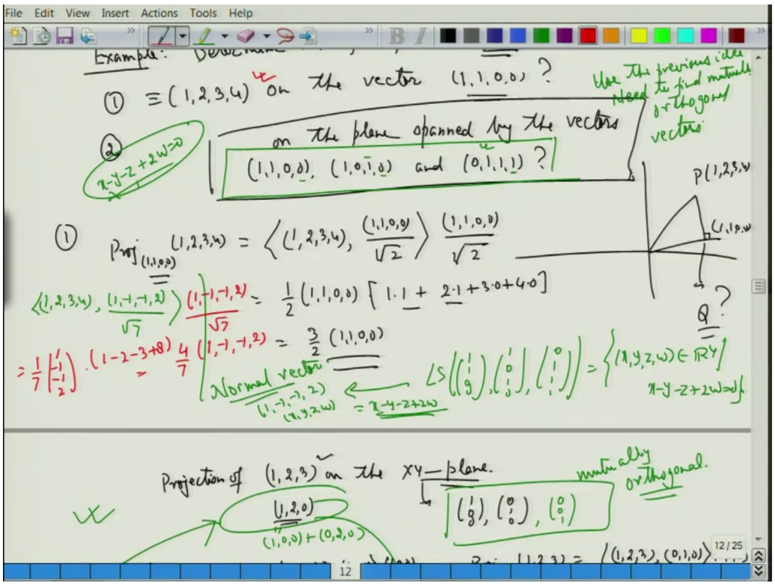

So, if I want to use that idea so, one way use the previous idea use the previous idea, need to find mutually orthogonal, mutually orthogonal vectors alright. And then do the work, which of the first part here, then writing it out or go to something which is perpendicular to it, fine. So, let us look at this vector these three vectors are linearly independent, that you can check yourself there is nothing special just verify it for yourself that they are linearly independent why they are linearly independent.

So, be careful look at this third component third the fourth component fourth component is non-zero here 0 here 0 here. So, therefore, this vector is perpendicular to these two. Now, look at this vector third component this is 1 here 0 here. So, therefore, this is independent of this. So, you can check that these are linearly independent vectors and a plane which contains all of them is this vector x minus y minus z and plus 2 w is 0.

You can check that if I put this vector 1 1 x and y x is 1 y is 1 z and w are 0. So, this will be 0 1 0 1 so, x is 1 z is 1 the rest are 0 and here it is minus y minus z plus w 0. So, this is the one that I am looking at fine. So, this is the plane which contains all the 3 fine so, the linear span. So, what I am saying is that look at the linear span of the vectors 1 1 0 0 1 0 1 0 0 1 1 1 is nothing but, all x y z w belonging to R 4 such that x minus y minus z plus 2 w is 0, fine.

Normal of this vector. What is the normal vector? Normal vector means the one which is perpendicular to all of them. So, normal vector here is nothing, but 1 minus 1 minus 1 2 just look at this with a dot product x y z w is this vector. So, this with x y z w is nothing, but x minus y minus z plus 2 w alright, fine.

So, this is a normal vector. So, what I can do? I can look at inner product of 1, 2, 3, 4 with this vector that I am looking at 1 minus 1 minus 1 2 divided by length of this. So, length of this should be 4 plus 5 plus 1 5 6 7 the square root of 7 times this vector again.

So, let me write use another ink 1 minus 1 minus 1 2 divided by root 7. So, this will be 1 upon 7 times, this vector which is 1 minus 1 minus 1 2 into this, which is 1 minus 2 minus 3 4 plus 2 is 8, which is 8 plus 1 is 9 minus 5 9 minus 5 is 4 upon 7 times 1 minus 1 minus 1 2. So, therefore, the actual vector that I am going to look at is this vector 1 2 3 4 minus this.

So, the answer for the question is 4 upon 7. So, 1, 2, 3, 4 minus 4 upon 7 times 1 minus 1 minus 1 2, which is same as 7, 14, 21, 28 minus 4 so, I always do mistakes so, plus 4 plus minus 4, then it is 4 4 minus 8 divided by 7.

So, this is equal to 7 minus 4 is 3 upon 7 3 14 plus 4 is 14 plus 4 is 18 21 plus 4 is 25 and 28 minus 8 is 20 this upon 7. So, this is the projection that I would like you to try that out fine and this matches with my answers so, my calculations are correct or might so, calculations are correct.

So, let us finish what I am trying to say. So, in place of finding something which is perpendicular to it, I went and found out something which is orthogonal, I could do it because there was only one vector, which was perpendicular fine.

So, the in the next class what we are going to do is that we are going to find ways and means to understand that given a set of vectors, how do I make them perpendicular is that that is our aim in the next class

Thank you.