Linear Algebra

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Lecture – 42
Cauchy Schwartz Inequality

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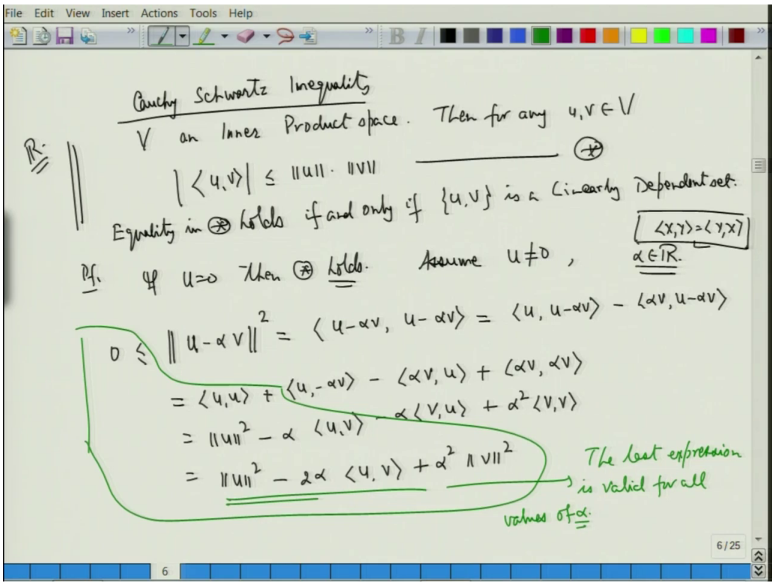

So, in the previous class I had stated this Cauchy Schwartz inequality and then shown that if u is 0. So, we had shown that if u is 0. So, as a proof we had shown that if u is 0, then a star holds, alright. So, to complete the proof let us assume that u is not the 0 vector. So, assume now so, assume u is not the 0 vector fine.

So, I can divide by length of u that is one way and we can try to prove it by some manipulation I will not get into that part fine. What we will do is that we will try to prove it only for the case when I have a scalar side real numbers alright. So, let us try to see in terms of real numbers where does we get this ideas alright, you can prove it for complex some manipulation is required, but for real everything is nice. So, let us try to understand that.

What we have is, look at this vector u minus alpha v, alright. Let us look at this we assume that u is non 0 I want to look at length of this square. Now, what we know is that this vector u minus alpha v is either a 0 vector or non 0 vector, but this scalar quantity square the length of something is always greater than equal to 0, alright. So, we already know that the square of this is greater than equal to 0, fine.

Now, let us write it back in terms of inner products. So, you can write it as u minus alpha v; u minus alpha v important we are assuming that alpha belongs to R, alright. We are looking at only real scalars therefore; I am able to do this manipulation otherwise something extra has to be done. So, I can write it as u comma u minus alpha v minus alpha v times, u minus alpha v, I am writing this expansion here taking time, but later on I will just go very fast.

I may do mistakes, but it will be your responsibility to make corrections in those expressions, alright. But I here I am doing it nicely. So, what we know is that since I am looking at over R. So, x, y is same as y, x, alright. I do not have bar coming into play, is that ok. So, since there is no bar I can write it as u, u minus u comma alpha v sorry plus this minus alpha times v, u; so, alpha times. So, let me again I think there may be confusion.

So, let me write it as alpha v, u minus minus plus alpha v comma alpha v; so, which is same as norm of u square plus is as I said inner product of x y same as inner product of y x fine. So, I can write this as minus alpha times u, v this is also minus alpha times u, v itself v u. So, let me write v, u plus alpha square v, v which is same as norm of u square minus 2 alpha inner product of u, v plus alpha square norm of v square alright fine, is that ok. So, I can write this.

Now, what I know is that, look at this expression this expressions if I look at this expression alright the last part says that this the last part the last expression; last expression is valid for all values of alpha, alright. So, what we are saying is that I have something in alpha which is always non negative.

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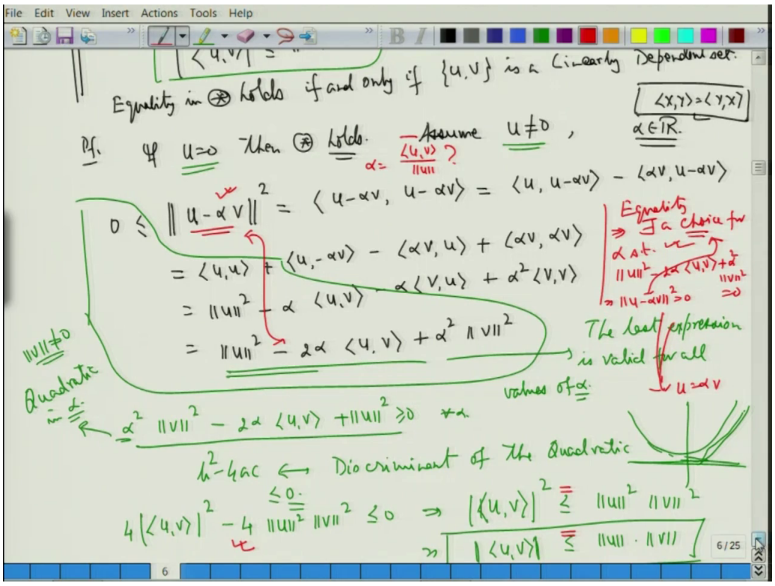

What does it mean? If I am saying that something is non-negative and alpha if I look at this expression we are looking at alpha square norm of v square minus 2 alpha u, v plus this is greater than equal to 0 for all alpha, it is quadratic in alpha, alright; it is quadratic in alpha fine.

So, if I assume that v is not 0. So, I should have taken v to be non-zero in place of u alright. So, that is a small mistake that I have done, but anyway. So, I could have started here with v to be 0 got the result, then assume v is non 0 and then proceeded here. So, since so assuming that norm of v is not 0.

So, norm of v is not equal to 0 will give me that this is expression is quadratic in alpha and it is non-negative for every alpha. Now, when I say that some quadratic is not 0 is positive or greater than equal to 0 for every alpha it means that the discriminant, alright the discriminant that is b square minus 4 a c; discriminant of the quadratic alright, should be greater than equal to 0 or less than equal to 0 what should it be? Fine.

So, since I am saying that this is non-negative it means that the curve is always above the x axis or it may touch the x axis at some point fine, but it is always above it means that I do not have anything below, it means that I do not have any real roots if there is a real root it is exactly one it is repeated twice fine. So, the discriminant has to be less than equal to 0. So, this has to be less than or equal to 0.

So, let us look at this part b square is 4 times u, v whole square; whole square is same as writing like this b square minus 4 times. So, I wrote it apposite. So, b square minus 4 c a I wrote here this has to be less than or equal to 0 this 4 cancels out. This implies u, v this square is less than equal to norm of u square into norm of v square. And therefore, I get that u, v is less than equal to this fine.

And, this is what we wanted the Cauchy Schwartz inequality is this what it is that the length of u, v alright the vector absolute value of this u, v is less than equal to this. So, let us recall let us see that part. So, this is what we have alright fine. So, it just follows for the real case it directly follows from the idea of quadratic that if the quadratic is always non negative then the discriminant has to be less than equal to 0 and we are done fine. What is more important is that when will this be equality fine.

So, there is a equality here, if there is a equality here it means that there has to be equality here now equality here means that the discriminant is 0; discriminant is 0 means what are the discriminant means 0 the vector has to be 0 in the sense that there is only one solution for you, alright.

So, it means that this has to be 0 fine, I am saying that this expression is same as this expression now we are saying that there is only one root and what is the root? There is a particular choice of alpha for which u equal to alpha v alright. So, equality implies equality here will imply that this part is 0. So, there is only one root for alpha if there is only one root for alpha it means that there is a choice of alpha for which this is 0 fine.

So, since there is a choice for alpha this is 0. So, there is a equality implies there exist a choice for alpha; a choice for alpha such that norm of u square minus 2 alpha this plus alpha squared times norm of v square is 0 for this particular choice of alpha whatever that alpha is. So, this will imply that norm of u minus alpha v square is 0, for the same choice of alpha alright for the same choice of alpha this is 0 and this implies that u is equal to alpha v because the norm of something is 0 means the vector is 0 is that fine.

So, this is a very important thing for us, you need to understand this with slight modification, you can prove it for the complex be careful, but it will not be the idea of discriminant because we do not have any complex number we will have to use some idea what is called we will have to write alpha as I do not know I have written here in terms of the other way round.

So, you may have to define alpha is equal to something u, v bar divided by norm of u and things like that something you have to do. So, just try it out or look at the notes and things like that fine. So, there is a small modification that you need to do.

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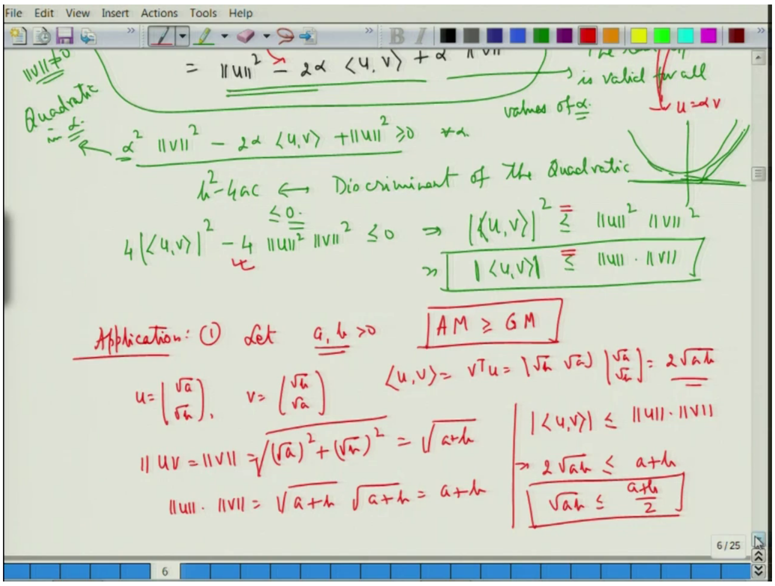

Let us look at the use of this use application, alright. So, let a and b be two real numbers which are positive fine they are positive numbers we can talk in terms of. So, we know what is called as arithmetic geometric progression. So, AP is greater than equal to GP fine. Can you use these ideas? So, let us try to prove this AP is greater than equal to GP fine.

So, I have assumed that a and b are positive fine since a and b are positive. So, I can define my vector x as u as square root of a, square root of b v as square root of b, square root of a. So, when I look at the inner product the standard inner product which is u, v inner product is nothing but v transpose u it will be square root of b square root of a times square root of a square root of b which will be equal to 2 times square root of a b. So, I get the expression corresponding to GP fine, alright.

So, I think it should be not AP GP, but AM, GM alright again. So, AM GM inequality fine. So, I got this and what about the length of u? Length of u is same as length of v which is same as if you look at a square root of a square plus square root of b whole square root which is same as square root of a b fine.

So, therefore, when I look at norm of u into norm of v I get a square root of a b sorry square root of a plus b and so it is square root of a plus b into the square root of a plus b which is a plus b alright. So, therefore, this when I say is less than equal to norm of u into norm of v.

What we get here is that 2 times the square root of a b is less than equal to a plus b which is what AM GM inequalities, AM GM inequality says this is less than equal to a plus b upon 2 alright. So, we have proved the AM GM inequality fine. So, this what you has to look at.

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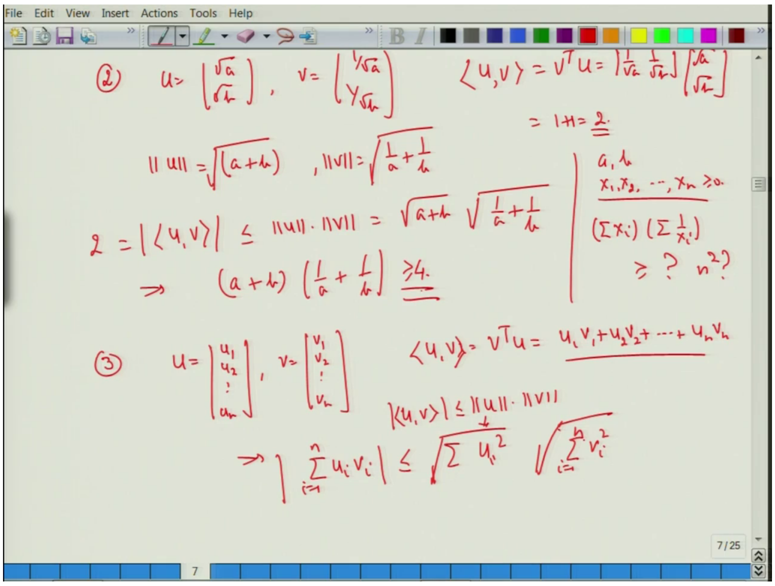

Another example that I would like you to see here is a slight change in this 2, take u as it is square root of a square root of b take v as 1 upon square root of a 1 upon square root of b, you take u and v as different fine there is a small difference. Therefore, if I look at u, v which is same as u, v here, so u, v. So, v transpose u will be equal to 1 upon root a 1 upon root b times root a root b this will be equal to this into this cancels out, you get 1 1. So, 1 plus 1 which is 2 fine.

But if I look at norm of u norm of u is alright, a plus b square root norm of v is equal to 1 upon a plus 1 upon b the square root. So, therefore, what I see from here is if I want to write here u, v which is 2. I hope I have done correctly this is equal to less than equal to norm of u into norm of v which is square root of a plus b into square root of 1 upon a plus 1 upon b and therefore, I get here is that a plus b into 1 upon a plus 1 upon b is greater than equal to 4, alright.

So, I get this inequality and this can be generalized for any n. So, I would like you to try that out that in place of looking at a and b just look at x 1 x 2 x n to be greater than equal to 0 fine. Then, you will have here x 1 plus x 2. So, you will get here summation x i here we will get summation 1 upon x i here and this should be greater than equal to what fine. So, just look at the idea here since you are looking at this it will be 1 plus 1 plus 1 n times. So, will it be n square? That is the question alright. So, try that out yourself fine.

Third one that we are going to look at is suppose you have got u as u 1 u 2 u n, v as v 1 v 2 v n. Again let us look at the standard inner product we are looking at over R n itself. So, u, v is equal to v transpose u which will be nothing but u 1 v 1 plus u 2 v 2 plus u n v n and therefore, this part is less than equal to this into this implies that.

So, implies that summation I going from 1 to n u i v i this mod is less than equal to summation u i square root because of this u and again square root of summation i going from 1 to n v i square, alright fine. So, I can do such a thing and this is what we required when we looked at C 2 or R 2 R 2 or R 3 that.

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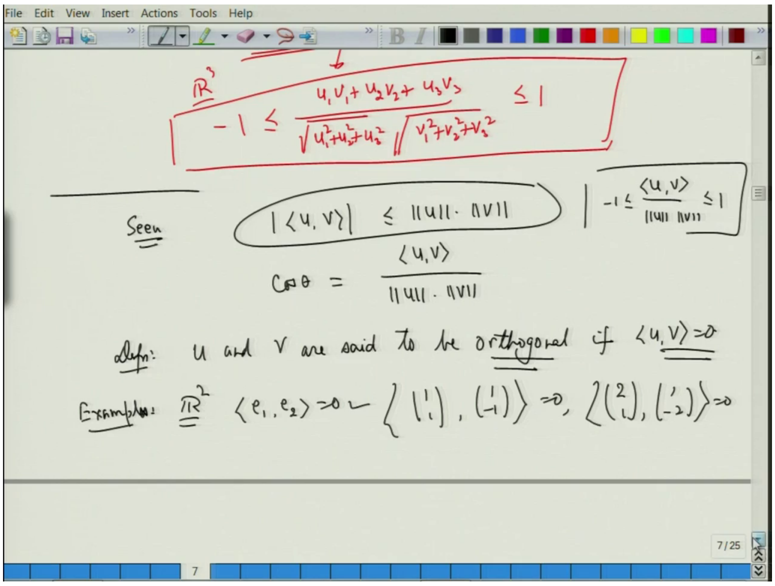

So, look at this part in case of R 3 it just corresponds to looking at u 1 v 1 plus u 2 v 2 plus u 3 v 3; plus u 3 v 3 upon look at these two u 1 square plus u 2 square plus u 3 square the square root into a square root of v 1 square plus v 2 square plus v 3 square is less than equal to 1 and this absolute value tells me that this is also this, fine. So, this is the one that we had in our school fine we assumed it and defined the cos theta fine.

So, we have been able to do things so, in general so if I let us go back to the general setup. So, I had enough examples here you would like to look at there are some examples in assignments also. So, you need to understand and try them out fine. Now, so, application of this as another thing is that seen that length of absolute value of this is less than equal to this fine.

Therefore, we can define cos theta as this divided by this, now somebody can ask that why are you defining it as cos theta, why not as sine theta and things like that, why not cos 2 theta. I would like you to understand them go to properties of triangles and see that we need to define it as cos theta there is no other choice. This part tells me that minus 1 is less than equal to u, v divided by this that is all it says, but why it has to be cos theta that one has to understand fine.

So, we define this cos theta as this, you can compute angles and so on, for us what is more important is we say. So, definition, u and v are said to be orthogonal if inner product of u with v is 0, is that ok. Now, whether it is a real inner product space or it over complex whatever it is we say that they are orthogonal if the inner product between them is 0. Is that ok? That is important for us.

So, example in R 2 the vector e 1 is orthogonal to e 2 with the standard inner product fine; similarly the vector 1 1 is orthogonal to 1 minus 1 the standard inner product is 0, again you can also have 2 comma 1 and 1 comma minus 2 their inner product is 0 standard inner product fine.

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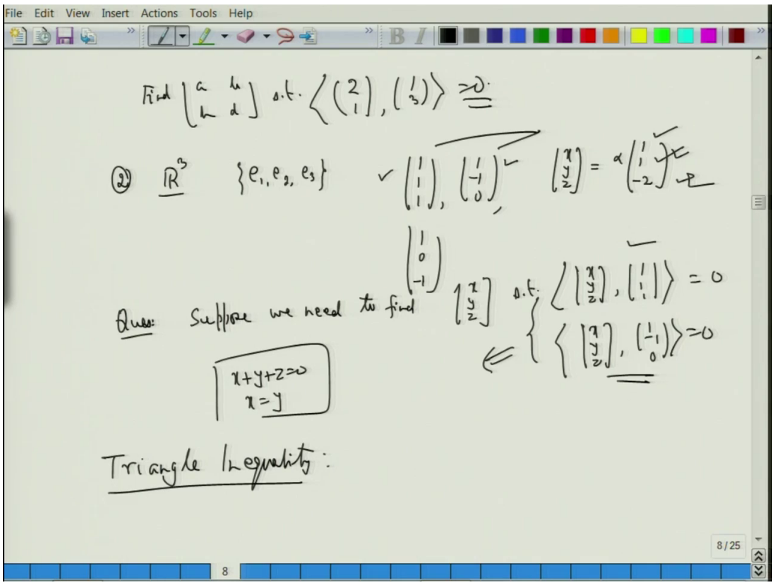

But we had asked you. So, recall that we had asked you a question in which what we wanted was find a b b d such that the inner product of 2 1 with 1 3 this was 0, alright fine. So, depending on requirement you can try them out, but as I said we will already looking at a standard inner product. So, do not worry you should know how to find it out, but in the exam if I saying something it will be with respect to the standard inner product fine. So, be careful. So, try them out, alright.

For example, now look at example another example in R 3 then the set e 1 e 2 e 3 if I look at the set then one is perpendicular to the other e 1 is perpendicular to e 2 e 2 with respect to e 3 e 3 with respect to e 1 and so on. Not only that if I look at this vector 1 1 1 this is perpendicular to 1 minus 1 0, this is also perpendicular to 1 0 minus 1 fine.

Something which is perpendicular to both of them, x y z, I want x y z to be perpendicular to this as well as this I would like you to see that this is nothing, but alpha times 1 1 minus 2 alright.

How do you find it out? We will just look at it one by one. So, question suppose we need to find x y z or in terms of column find x y z such that x y z comma 1 1 1 this is 0 and x y z times 1 minus 1 0 is also 0 both of them are 0. So, this part tells me the first example the first condition tells me that x plus y plus z is 0, the second condition tells me that x is equal to y, alright. So, now, I can solve it to see that you can see that x is equal to y here and was x is equal to y z is equal to minus 2 x. So, this is what I get here fine.

And there is infinite number of solutions things like that. We will come to that afterwards again. So, I would like you to keep track of that the next thing that I would like to understand make you understand what is called the triangle inequality. So, what are the triangle inequality? If you remember from your school days triangle inequality says that that if some things are sides of a triangle, alright.

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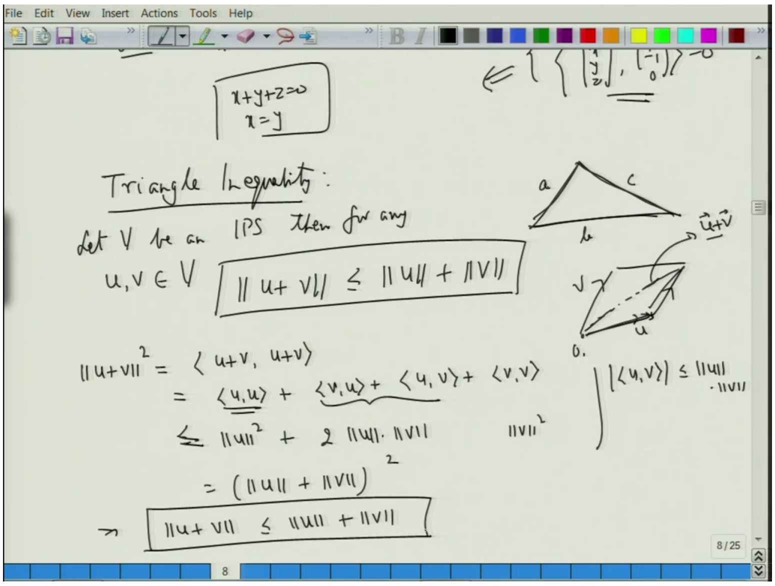

So, you have some sides here if this length is a, this length is b, this length is c then the length of a plus length of b is greater than equal to length of c alright. So, we need to talk of length of a length of b length of c things like that fine. So, in terms of our vector space it turns out that if let V be an inner product let V be an inner product space then for any u, v belonging to capital V length of u plus v is less than equal to length of u plus length of v this is what the triangle inequality is fine.

So, in some sense what you can see is that starting from 0. So, this is 0 vector that I have, I go to u, I go to v. So, this is the vector which is u plus v fine, then we are saying that length of this vector is less than equal to length of this vector plus the length of this vector that is what we are saying here, is that ok.

Or the length of the (Refer Time : 22:10) the length of the diagonal is less than equal to the length of the two sides that is all we are saying here fine. So, let us try to prove this part which is very important. So, when you are saying this. So, length of u plus v I will look at a square of that, I do not want to look at just this. So, this by definition is equal to u plus v comma u plus v this is equal to by definition is u, u plus v, u alright, u with v, v with u this is what I have done plus u with v plus v with v.

Now, this is equal to alright, in absolute value; absolute value so modulus of this we will look at modulus of this if I look at this is less than length of u. So, Cauchy Schwartz inequality says that u, v is less than equal to length of u into length of v.

So, let us use that part this is equal to length of u square this is equal to length of v square and this part is less than equal to this so plus. So, this is less than equal to 2 times the length of u into length of v and therefore, I get that this is u plus v whole square and therefore, I get that length of u plus v is length equal to length of u plus length of v, is that ok, fine you get this part.

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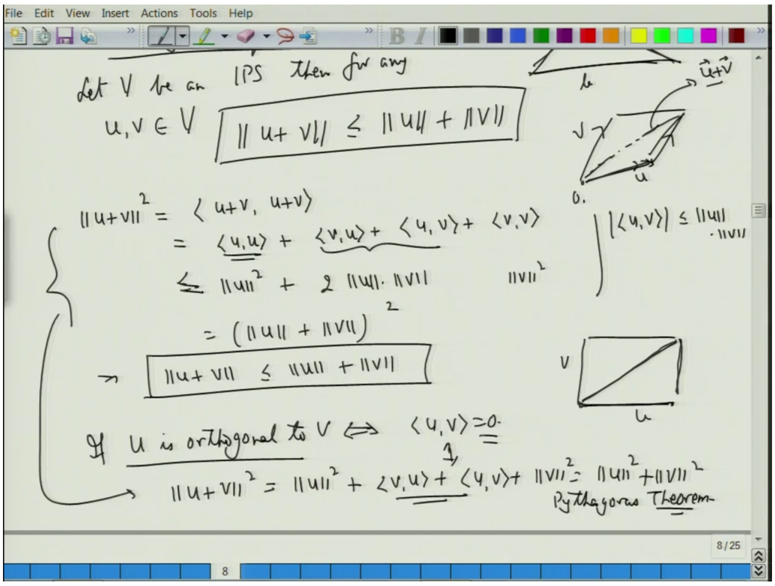

You also have what is called so, if u is orthogonal to v. So, orthogonal to v means look at the definition it says that u comma v is 0 fine. In that case this will turn out to be equal to norm of u plus v whole square will be equal to norm of u square plus v, u plus u, v plus norm of v square is nothing but norm of u square plus norm v square alright because this part is 0 from here fine.

So, what we are saying is that if u and v are perpendicular. So, I have u here. So, starting with 0 I have got a u here, I have got a v here fine, make this parallelogram for you. This is the hypotenuse the length of this is square is same as length of this square plus length of this square alright this is what it says.

So, this is what the Pythagoras theorem is. So, when something is perpendicular everything is nice, we will look at these applications in the next class, fine. I had also like you to as a last thing I would like you to understand this that in here is a certain notion what is called a Parallelogram law. So, what does the parallelogram law; let us try to understand it.

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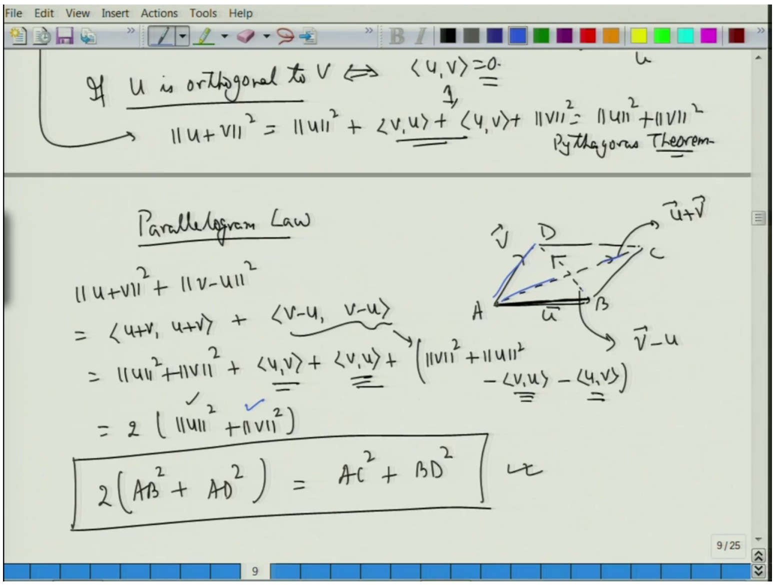

So, it says that I have the vector u here, vector v here just join it here this is the vector u plus v and this is the vector. So, I am going from this suppose I am going from this direction fine it means that I am going u and v here. So, this vector for me is alright going like this, v minus u I am looking at this part.

So, let us look at the lengths here. So, I want to compute what is norm of u plus v whole square plus norm of v minus u whole I want to compute this fine. So, this by definition is equal to look at this I have already computed it for you. So, it is u plus v comma u plus v this is nothing, but v minus u v minus u.

So, this gives me u square plus v square plus u, v plus v, u alright. This is what it gives, what about this one? This gives us norm of v square plus norm of u square again minus u minus u gives you plus u. Now, it is v and v, u with the minus sign minus v, u minus u, v alright this is what I get.

So, if you just look at this and this cancels out u, v with u, v cancels out v, u with v, u cancels out alright. So, therefore, what I get is 2 times length of u square plus length of v square. This is what the parallelogram law says, parallelogram law says that. So, the parallelogram law says that the sum of squares the length of the sum of squares the sum of the squares of the. So, I think language will be problem.

So, let me it as A B C D, then it said that look at AB square alright, fine. This length AD square at twice of this, this is same as AC square plus BD square. So, we use to prove it using Pythagoras theorem same thing is happening here also. This is very important theorem Parallelogram law; even though the proof is very simple, the idea is very simple, but it is very very important I would like you to keep track of that we will be using this idea to say something is possible something is not possible is that ok.

So, keep track of this part that AB square plus AD square or look at the sides alright, square of this which is nothing, but u square. Similarly, square of this which is v square twice of that is equal to the sum of the squares of the diagonals is that ok. So, that is all for now.

Thank you.