Linear Algebra
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Lecture – 41
Inner Product Continued

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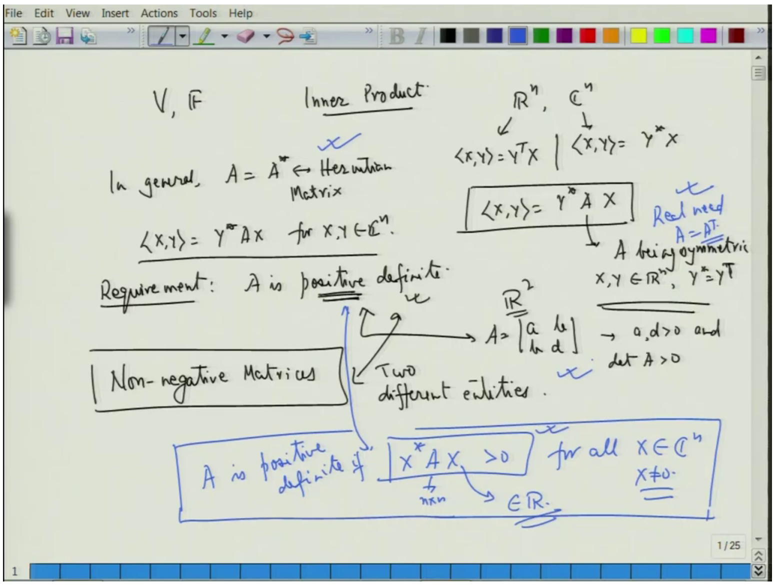

Alright. So, yesterday we had looked at what are called inner product. So, given vector space V, where the notion of inner product. So, vector space over F and there was a notion of inner product. And, we looked at examples where we looked at R n and C n, in C n we had X and Y as Y star x.

Here, we had X, Y; and Y transpose X the same thing fine. From there we went to X, Y is equal to Y star A X alright. But, at this only for A being symmetric and with the idea that X and Y are a R n itself X Y belongs to R n. So, Y star is basically Y transpose is that ok, that was the idea, but in general you can take A equal to A star, what is called a Hermitian matrix.

And, then define yourself X, Y is equal to Y star A X for X, Y belonging to R n belonging to C n, there is no problem. What do you require is requirement is; requirement A is positive definite this was the requirement this what we saw when we looked at R say so, when we looked at yesterday we looked at some examples like we took A as a b b d and then we said that a has to be positive, d has to be positive, and determinant of A has to be positive.

This corresponds to what is called positive definite fine. So, we looked at for R 2, this was done for R 2, you can do it for any R n or C n accordingly change it, and then look at things what are called positive definite. So, there is a difference between positive definite and there is a notion of what is called non negative matrices, these are two different things alright. This and this are two different entities.

So, positive definite does not mean that we are looking at integer positive signs. What you require is positive rate basically requires us to have that just the definition of inner product. That X star A X should be 0 for all X belonging to C n alright. A is n cross n, then I say that A is positive definite alright, definite if this is there for all X, X not equal to 0 alright and if X is 0, then any way it is 0 as such.

So, this is a requirement for you in general when I say something is positive, it means that I am indirectly saying that this belongs to R, this is a real number. Because complex numbers you do not have 1 plus i as positive number or negative number things like that. So, in complex numbers you cannot compare objects, but in real numbers you can compare.

So, when you say that X star A X is positive at the back of my mind I already have that X star A X is a real number fine. And, that is guaranteed because A is A star, but in many books you will not get the requirement that A equal to A star. Because, this having real number X star A X being a real number will imply that A has to be equal to A star fine.

But, when you are looking at real numbers, then you need that A has to be symmetric for so real, need A equal to A transpose, because X star A X real number does not imply that A has to be symmetric fine. So, for real numbers A is a real number positive definite means that A has to be symmetric we have to assume and then proceed fine. So, let us look at some more examples, not in this direction. In this direction I have done enough for you we will look at some other directions alright. So, let us do the next thing.

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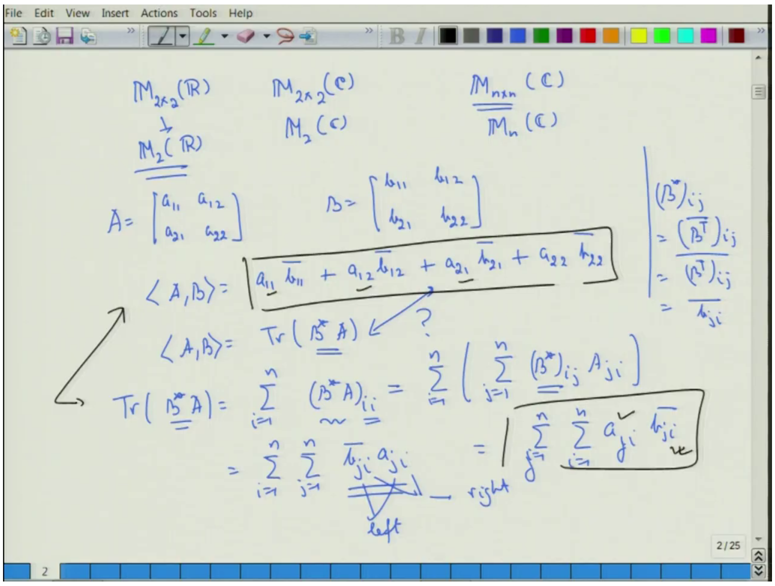

So, let us look at this for example, I want to look at set of all 2 cross 2 matrices, over real numbers or set of 2 cross 2 matrices over complex numbers, or in general set of n cross n matrices over complex numbers. So, recall that we mostly wrote it as M n itself. Only for M cross n we wrote M cross n, otherwise we just wrote it as M 2 itself alright. So, what we can do is that I have the matrix here as say a 1 1 a 1 2 a 2 1 a 2 2 this is the matrix that I am looking at.

A is this. I can have B as b 1 1 b 1 2 b 2 1 b 2 2, and I am thinking in terms of complex numbers. So, I can think of A, B as just multiply component wise. So, a 1 1 into b 1 1 bar plus a 1 2 into b 1 2 bar plus a 2 1 into b 2 1 bar plus a 2 2 b 2 2 bar, I can do this. There is no problem and this will define an inner product, because it does the standard inner product for you.

I would like you to see that, if I look at this A, B, which is equal to trace of B star A, then do I get back this that is the question alright. So, let us look at what is trace of B star A. So, trace of B star A is nothing but, what was the trace? Trace was some of the diagonal entries.

So, it is same as i going from 1 to n, B star A, i th entry, which is same as i going from 1 to n. Now, what is this entry? This is nothing but j equal to 1 to n because it is an n cross n matrix. So, everything will be that part.

And, then I have to write here B star of i j and A of fine j i here, this is what I like to do. Which is same as I equal to 1 to n, j equal to 1 to n, and what is B star of i j, B star of i j is. So, if I look at B star of i j is same as B transpose bar of i j, which is B transpose i j whole bar, which is same as B transpose i j is same as B j i fine.

So, therefore, it will be nothing, but j equal to 1 to n B j i bar into a j i fine. So, see here j i j i, i is here on the right hand side and j is on the left hand side left, or you can interchange the 2 things whatever way you want you do it you can also like it as because the summation is over both i and j. So, i is equal to 1 to n. So, you can write first j equal to 1 to n.

And, then i is equal to 1 to n which is a j i into B j i bar or whatever way you want to write to write it. So, we can see that these two expressions are the same fine. So, j i and B j i bar you are getting the same thing, j is 1, i is 1, a 1 1, if j is 2 alright and i is 1 you get this 1, if j is 1 i is 2 you get this and so on.

So, you can compare the 2 and say that they are the same thing fine. So, somehow the trace has come to our rescue in some sense and we can do all your calculation, because at least we have understood something about trace. So, some things can be said about it fine. So, this is another inner product fine.

Another inner product that I would like you to understand, which is very important is even though it is not in our syllabus, but it is used quite frequently in general fine what is called.

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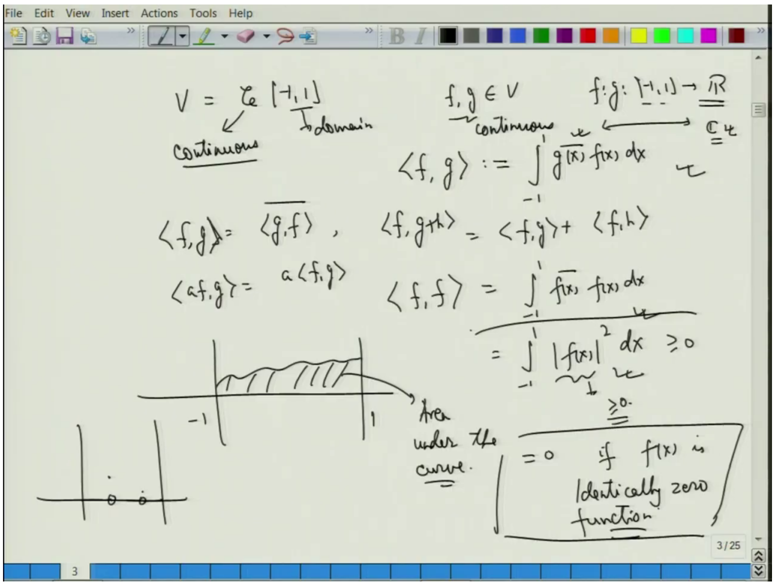

So, look at all vector space V, which is C of say minus 1 to 1. So, minus 1 to 1 is the domain and C for continuous, see if you recall I had said that set of all continuous functions is a vector space. Since, it is a vector space I can talk of inner product, it is not always possible that you can do it.

Sometimes you can do, sometimes you may not do it. We will look at examples where we do not have inner products fine, or I can define an inner product for finite dimensional vector space, but then you may not be happy with it in some sense. So, let us not worry about it. So, what we will assume that we have inner product as far as we are concerned and things like that.

So, let me look at this. So, here I have got 2 vectors f and g, which are continuous functions. So, f g continuous, I can define inner product of f and g as. So, f and g are functions from minus 1 1 to R fine. If, you (Refer Time: 10:14) over complex numbers still you can do it, we have to change it according to the definition.

So, this is equal to the domain is minus 1 to 1. So, the integration is from minus 1 to 1 that is one thing. And, then we are looking going to look at g x bar f x, d x alright. So, I am looking at complex numbers so, the notion of bar has come, if I do not want complex numbers, if I want real number that bar will go off. So, I am just looking at this.

So, we can check here that f of g h will be equal to g f this f of g plus h will be equal to f g plus f h, we also check that a f of g will be equal to a of f g fine. Now, the only thing that I need to check is what happens to f comma f. So, f comma f is nothing but, minus 1 to 1, f x bar f x d x, which is same as minus 1 to 1, this is squared d x. Now, this is greater than equal to 0 alright fine.

So, in integration theory we must have learnt somewhere that, when you are looking at functions which are always non negative, that was the starting point of integration. So, I am looking at minus 1 to 1 and the function is always non negative. So, I am looking at a function like this and in that case I get the area, area under the curve.

And, this area say this is non negative, this will always be greater than equal to 0 and will be equal to 0 and equals 0, if f of x is identically 0 function is that ok. Now in general when you do not have a function which is continuous, if I have a function, which is continuous at every point except a finite number of points integration makes sense, but I will not be able to conclude this part that from here, this will happen, that I cannot conclude.

Because, I need that, so, what I am trying to say is that if I function like this alright, which is 0 here, then 1 point is here, again it is 0 here again another point is here and so on. So, these are the points of discontinuity suppose I am looking at these points, then still the integration will be 0 because look at your integration, Riemann integration you can do that fine, math 1 1, but I cannot conclude this part. So, for this to be concluded I need that f and g has to be continuous. So, therefore, we wanted two continuous part alright.

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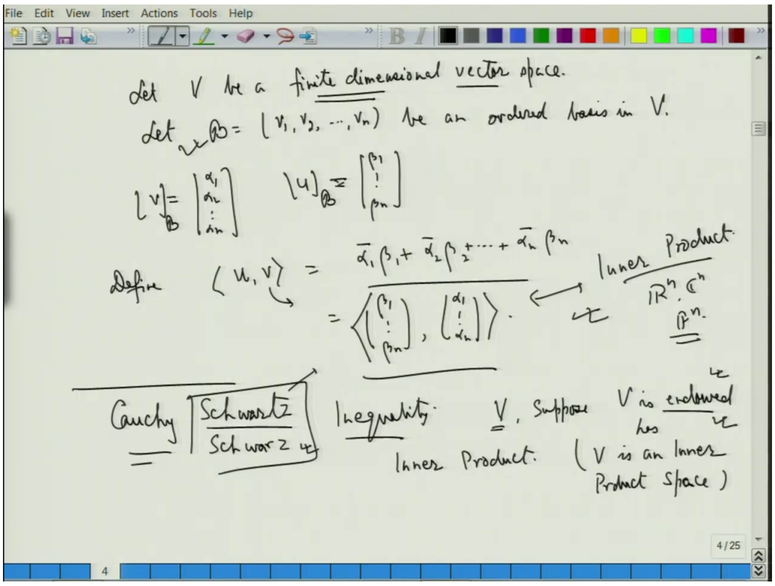

The last example as far as we are concerned is that let V be a finite dimensional vector space. So, it is finite dimensional. Since, it is finite dimensional I can talk of basis of this. So, let v 1, v 2 v n be an ordered basis in V, was this an ordered basis so any element v can be written as so, v with respect to B is some alpha 1, alpha 2, alpha n.

And, any u with respect to B will be equal to some beta 1 to beta n fine, and then I can define. So, now, we can define our inner product of u and v as. So, v is in the second component. So, it will be alpha 1 bar beta 1, plus alpha 2 bar beta 2, alpha n bar; alpha n bar beta n alright or the usual that you do which is inner product of so, v is on the right so, right there. So, beta 1 to beta n comma alpha 1 to alpha n is that.

So, this is what we have. So, again you can see that given an ordered basis they are able to talk in terms of inner product. And, this inner product is nothing but, inner product in R n, or C n, or f n wherever you want to talk of you can talk of that is that ok. So, that is important for us that any finite dimensional vector space can be endowed with an inner product alright. I can always do it fine.

And, but we will look at examples where I have something, but I do not have something that is come that will come afterwards. So, now, in next time what to now we what to look at is what is called Cauchy Schwartz inequality. So, many books they will write like this, many book will use just Schwartz like this.

So, whatever you want to follow you follow, I follow this Cauchy Schwartz inequality alright. So, you follow whatever you like. I also can follow this or this depending on choice, mood fine, because I always forget things. So, this is very important inequality, what it is says is alright.

That take any vector space V and suppose, V is endowed or V has endowed means has inner product alright. So, whenever I have vector space V which is endowed with the inner product or V has an inner product, we say that V is an inner product space. So, inner product space means, V is endowed with an inner product, there is some inner product in V that is all it says fine.

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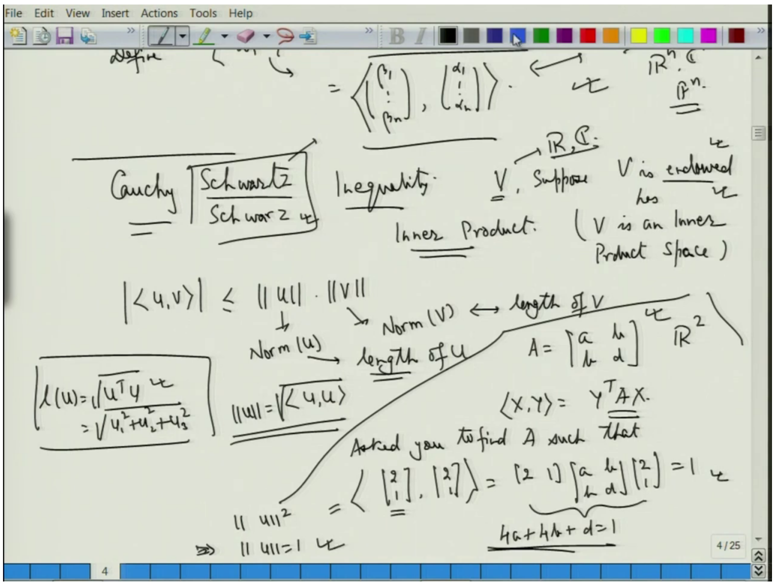

So, if I have such a thing v is a inner product then it says that look at any u and v in v, look at the absolute value of that fine. So, let us assume that you that V is over R or over C fine. So, that, I can talk about absolute value here. So, inner product of u and v is some scalar quantity, I am looking at the absolute value of this it says that this is less than equal to norm of u into norm of v.

Now, what is norm? I have to say what is norm of u or this is norm of v, they are nothing, but the lengths, length of v this is length of u. So, you already know what do you mean by length as far as your dot product is concerned.

So, there in the dot product the length of u vector u was u transpose u square root, which was same as u 1 square plus u 2 square plus u 3 square square root fine. Now, this we are changing it to u comma u this is what we did. We just replaced the inner product u transpose u with this notation that is all we did fine. So, therefore, when I talk of norm of u for me norm of u will be square root of this is that ok.

So, you can compute them for example, yesterday we had 1 example in R 2 where I had A as a b c d alright no. So, it was d itself we showed that has to be symmetric a b, b d in R 2 and we defined our X Y inner product as Y transpose A X. Recall that I had asked you, to find A such that this vector 2 1, 2 1 this inner product, which was nothing, but 2 1 here a b, b d 2 1 to be equal to 1.

I asked you that find a b c d a b d such that this depends and I had got your expression for this which was I think 4 a plus 4 B plus d this was 1 of the condition that was required. So, therefore, under this inner product that I am defining here alright, that I am defining here this vector 2 2 2 1 has length 1 alright.

So, we are saying that this is nothing but norm of u square. And, this is norm u square is 1 a norm of u is a positive number norm of u is a length. So, length is always a positive quantity and is 0 only when it is 0. So, you can see that norm of u will also be 1 fine.

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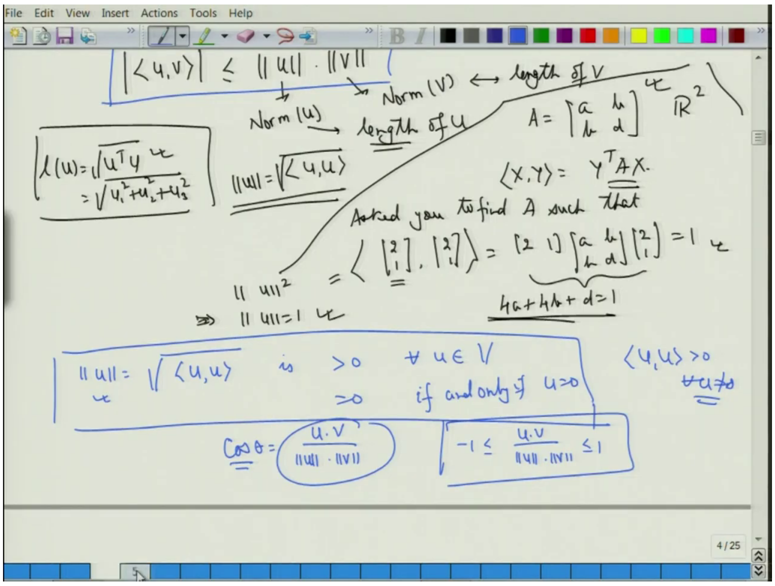

So, let me write it. The norm that we are defining norm of u is equal to square root of u u is for greater than 0 for all u belonging to wherever vector space we are looking at, equal to 0, if and only if u is 0. And, this follows because of the condition that we had, that u u is positive for all u not 0 alright.

So, that was the restriction for our inner product and because of that we are getting that the length of u is always a positive quantity fine. So, now, we would like to prove this part alright. Why do we need to prove it? That you can see that, in our school we had learnt what is called cos theta. So, we had learned what is cos theta. Cos theta was u dot v divided by length of u into length of v.

So, that when defining something to be cos theta, I will have to say that this number has to lie between minus 1 and u, because cos theta is a number lying between minus 1 and 1. So, I will have to prove that such a thing is happens, which is same thing as the Cauchy Schwartz inequality. So, it is not true only for the dot product, it is true for any inner product, let us try to prove that now alright.

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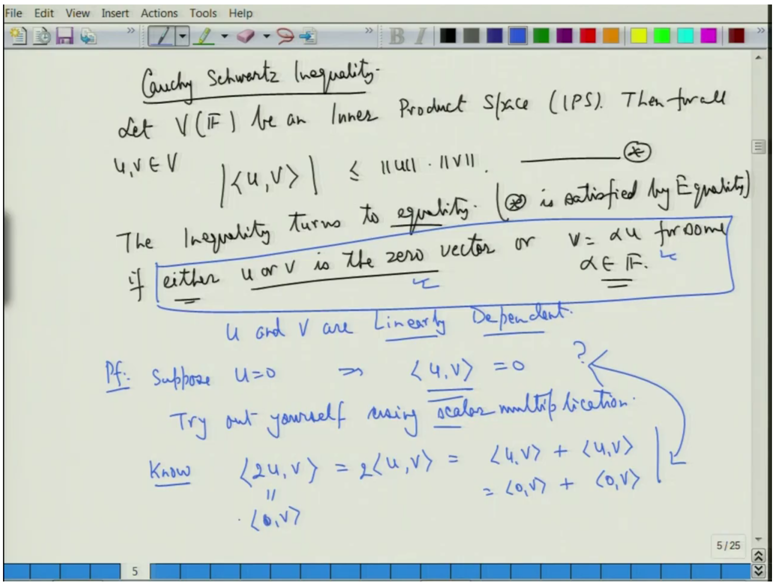

So, let us prove it now; so, the Cauchy Schwartz inequality. So, let V over F. So, F is either R or complex number for us for the time being be an inner product space IPS. Then for all u comma v belonging to v inner product space, in inner product space v length of or norm of u v is less than equal to norm of u into norm of v fine. This happens, the equality, the inequality turns to equality.

So, we are saying that star if I look at star is satisfied by equality. If, either u or v is the zero vector or either this is zero vector or v is equal to alpha u for some alpha belonging to F alright.

So, either they are so, in some sense we are saying that so, if we look at say either u or v is 0 vector or v equal to l u I can combine them together and say that u and v are linearly dependent alright. So, recall the definition of linearly dependent. If there is a 0 vector then that set is linearly dependent and here we are already saying that is alpha times this.

And, that will imply that if I take alpha to be 0, then v is 0 and things like that for otherwise it is dependents let us try to prove it so proof. So, suppose so, let us look at this case suppose u is a 0 vector alright. If, u is a 0 vector this will imply that we are looking at u, v inner product, I did not prove that. So, I need to prove that.

So, this imply that this is 0. Now, why is so, that we need to understand, that why should the inner product of u and v should be 0 when u is a 0 vector alright. So, that is an exercise try that out yourself, fine; so, try that out try out yourself, using the idea that 2 times u u should be equal to u plus u and so on.

So, try it out yourself using a scalar multiplication alright or just look at this part. So, what we know 2 times u v alright will be equal to 2 times u v, which is same as alright.

So, what is should I write, same as u v plus u v, but u is 0. So, what we are saying that this is same as 0 comma v, this is also 0 comma v plus 0 comma v. So, together will imply this part. One of them will cancel out and therefore, we will get 0 comma v has to be 0 fine, that is one thing fine.

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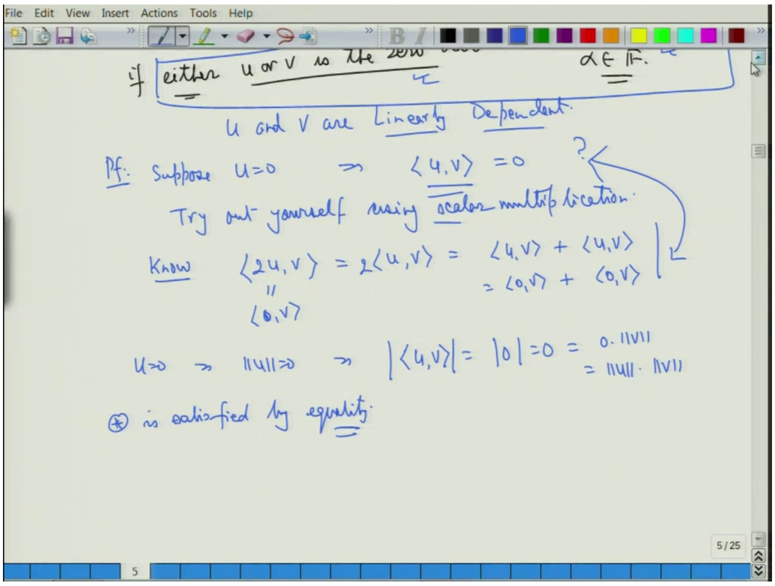

So, if u is 0 you are done there is no problem so, you can see that suppose u is 0 this implies. And, we also know that u is equal to 0 implies that length of u is 0 implies, this is equal to this which is same as 0 is equal to 0 times norm of v, which is norm of u into norm of v alright. So, the star is satisfied by equality alright.

So, you can see that at least here we are able to do it equality. Now for the general setup when we want to prove I think I will just stop here; so, that is all for now.

Thank you.