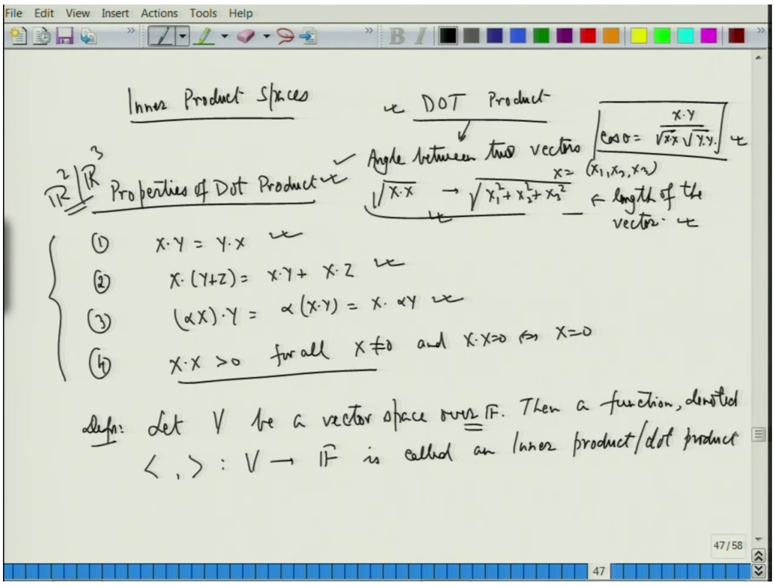
Linear Algebra
Prof. Arbind Kumar Lal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 40
Inner Product Space

(Refer Slide Time: 00:18)



Alright. So, now let us start the next chapter what is called Inner Product Spaces for us, so we are going to study what it called inner product spaces. So, in school, we had learned what is called the dot product and the vector product, it turns out that vector product cannot be generalized at the higher level because of certain restriction. It was possible only in three dimensions nowhere else, but dot product can be generalized for anything.

And the idea of dot product was let us try to understand, we could talk of angle between two vectors, angle between two vectors could be discussed. And, this also helped us to look at X dot X, and then the square root of that which was if I write X as X 1, X 2, X 3; then X dot X was X 1 square plus X 2 square plus X 3 square and a square root. So, we could talk of this which was called the length of the vector, alright fine.

So, we had two ideas. One was the dot product was gave us the angle as well as the length of a vector, and the angle between the vector was given by cos theta which was X dot Y divided by length of X into length of Y alright, this is what the idea was.

So, what we want to do is that we want to generalize these ideas, because these things are meaningful when I am looking at R 2, R 3. But if I want to discuss angles on the say surface of the earth or moon or anywhere else fine or any figure which is not as plain as R n that is flat, I will have issues with me.

Because if I look at suppose in on earth itself if I am starting from north pole alright, I go south, go east, again go north alright. So, what I am saying is that, I start from north pole, go south, east and north, it turns out that I will reach north pole itself, fine. But in Euclidean plane, if I go from any point to south, east, and again go north, I will get to a parallel place not to the same place alright, fine.

So, therefore, the geometry on earth and geometry on R n, they are quite different fine. So, we will have to define these things depending on the requirements, fine. So, let us try to do things at least for finite dimensional spaces, how should we go about doing those things fine.

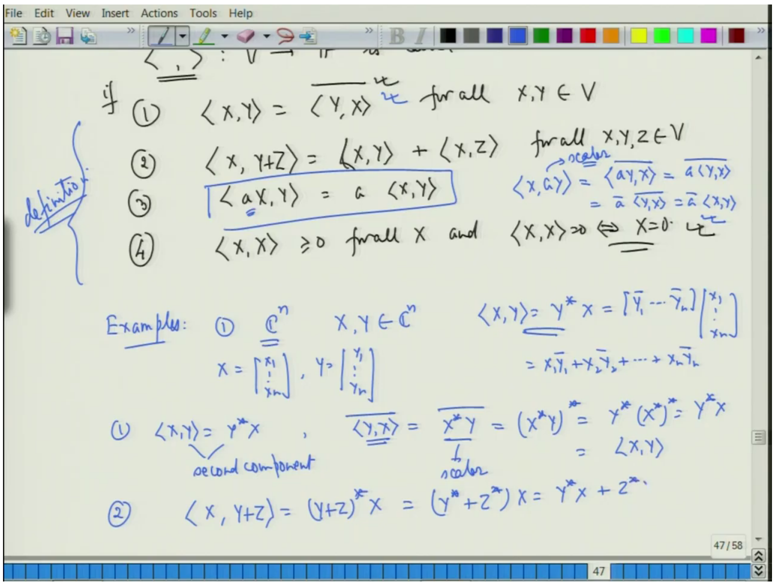
So, let us try to look at what are the properties of dot product; properties of dot product. So, recall what we have done, for vector spaces also what we had started words was we started the property on R 2 and R 3, and took them as our basic requirement that was the basic requirements, and then we define things alright. Similarly, whatever the properties dot products are going to satisfy that will be our minimum requirements.

So, what are the things, recall; X dot Y was same as Y dot X, then we had X dot Y plus Z was X dot Y plus X dot Z; 3, alpha X dot Y was same as alpha times X dot Y, which was also equal to X dot alpha Y, fine. And the fourth was related to the distance sorry, not the distance this length that X dot X is positive for all X not equal to 0 and X dot X is 0, so in some sense what it saying is this itself alright, fine.

So, this was the minimum property of dot product that we followed in our school that is the dot product is commutative. It distributes over additivity, then there is a notion of a scalar multiplication for vector, so it behaves nicely with a scalars; and then the length is always positive whenever the vector is non-zero fine.

So, we take it as our basic requirements for our inner product. So, definition of an inner product definition let V be a vector space over F, alright. There we looked at things over R 2 and R 3, fine; here also we are looking at over F, so we will be looking at F n in some sense, so V for us. So, then a function; so there is a different way of writing is denoted this, from V to F is called an inner product or dot product if 1.

(Refer Slide Time: 05:43)



So, this is the way we write do not write F any more, we write like this. So, what we are saying is that X comma Y is same as Y comma X for all X, Y belonging to V alright. So, the first property was commutativity of X dot Y same as Y dot X, we assumed it as it is fine; I will make some changes to it after some time. 2, X comma Y plus Z will be equal to X comma Y plus X comma Z for all X, Y and Z belonging to V.

3, a X comma Y we will write it as a times X, Y alright. I am not putting it on the right the last part, here also I have not I have done something what not exactly the one that I require fine, so I will do that. 4th which is again, which is important is very important is this part of the length. So, what we are saying is that X comma X is greater than equal to 0 for all X and X comma X is 0, if and only if X is 0 alright.

So, the same thing that I wrote here, we are writing the same thing here alright. Now, the only thing is that if I am over real numbers everything is nice this definition is fine, but our syllabus also requires us to understand a little bit of complex theory. And therefore, for complex numbers I need to talk of complex conjugate and therefore, the first one gets replaced by a bar here alright.

So, we look at this bar that is important for us as far as complex conjugate is concerned. And, therefore because of this complex conjugate what happens is that if I want to look at X comma a Y fine, it will turn out to be just look at this part here. So, this will turn out to be a Y comma X whole bar this is what the bar definition was which is same as, now if I go with this part this definition.

The first component I can take out a from the first component, I am not talking about if we look at this definition, there is no talk of taking out a from the second component fine. So, now I can take out a from the first component, but here it is with a bar. So, it is a times Y, X whole bar which is same as a bar Y, X which is same as a bar of X, Y fine.

So, when I am taking out a from the scalar from second component I get a bar into play is that ok, so this is a bar that is playing the role. So, you have to be careful; you can if you want to take out a scalar from the first component, it comes as it is; but if you are taking out from the second component, the bar comes into play. So, this is the way we define that is the definition alright, definition.

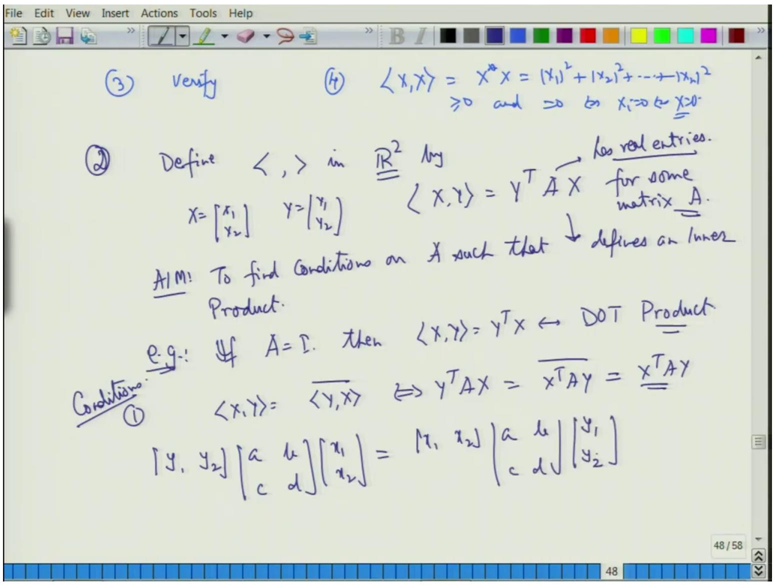
And let me give some examples first, and then go into it example. First example our dot product itself, so we are define so in C n alright, I take X and Y belonging to C n. I define X comma Y as Y star X; so X here is some X 1 to X n, Y is Y 1 to Y n, they are complex numbers alright.

So, therefore this will give me Y 1 bar till Y n bar times X 1 to X n which will be nothing but X 1 Y 1 bar plus X 2 Y 2 bar plus X n Y n bar. So, you can check that the first condition will be satisfied; second will also be satisfied, because just look at the second part.

So, first is satisfied first by definition itself, because X, Y is Y star X that is the way we have defined here fine. And if I want to look at Y, X whole bar this by definition will be equal to, so this will be equal to bar of Y, X is nothing but X star Y, because star is coming from the second component alright, the second component fine.

So, which is same as taking a star, because it is a scalar quantity the scalar; so it is same as looking at X star Y whole star which is same as Y star X star star which is Y star X, which is same as by definition is X, Y. So, we can check that first condition is satisfied. Second condition X comma Y plus Z will be equal to Y plus Z whole star X, which is same as Y star plus Z star X which is same as Y star X plus Z star X fine, by matrix multiplication fine.

(Refer Slide Time: 11:32)



3 verify it yourself, 3 verify. Fourth-one let us look at X comma X. So, X comma X will be X star X, which will be equal to mod of X 1 square plus mod of X 2 square, so mod of X n square which you know is always greater than equal to 0 and equal to 0 if and only if X i is 0, which is same as saying that X is the 0 vector, alright. So, you can check that everything is nice you are able to get things as per as this definition is concerned, alright.

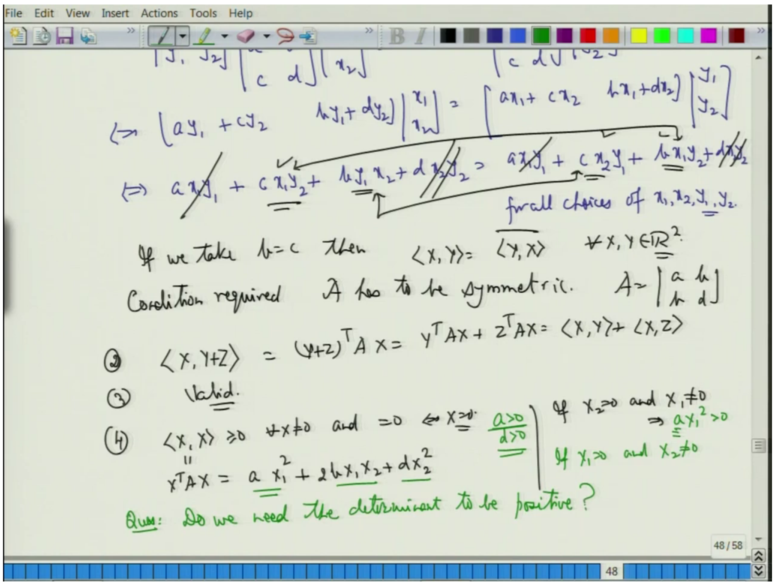
So, if this definition is there, everything is nice fine. So, let us go to the next definition, next part, another example, example 2. So, define this in R 2 by so in R 2, I have got vector say X, Y. So, X is X 1 X 2, Y as Y 1 Y 2; I am defining my this as Y transpose A X for some matrix A, alright. I want to check, so aim to find conditions on A such that this defines an inner product, fine.

So, for example if I look at example for example, if A is identity, then X comma Y is Y transpose X which is nothing but the dot product, alright. So, what we are trying to do is that we are trying to generalize the idea of dot product fine. So, let us take another example to see that whether it can work or not; so what are the conditions we need? The first condition we need is alright conditions.

Conditions that X, Y should be equal to Y, X bar fine which is same thing as saying that X, Y is Y transpose A X and I am taking A to be again real, so A is has real entries. I can take them as complex entries, but since I am looking at R 2 I am looking at real entries only alright.

So, Y transpose A X, so it will be Y X is X transpose A X, X transpose A Y bar of this; everything is real, so there is no complex conjugate which is same as X transpose A Y. So, what we are seeing here is that we have got y 1 here, y 2. Let me write A as a b c d fine, times x 1 x 2, I want this to be equal to look at this part x 1 x 2 a b c d y 1 y 2, fine.

(Refer Slide Time: 15:02)



So, this tells me that just multiply it out; a times multiply this into this a y 1 plus c y 2 b y 1 plus d y 2 times x 1 x 2 is equal to a x 1 plus c x 2 b x 1 plus d x 2. Please verify, because I do lot of mistakes alright, I am very good at it, so you have to spend time and verify things, is that ok. So, this is same as a times x 1 y 1 plus c times x 2 y 2 plus b times, c times this into this, this; no, so there is a mistake here as usual again.

So, I am multiplying x here so c times x 1 y 2 plus b times y 1 x 2 plus d times x 2 y 2. Here it is equal to a times x 1 y 1 plus c times x 2 y 1 plus b times x 1 y 2 plus d times x 2 y 2. Now, this has to be true for all choices of x 1, x 2, y 1, y 2. See this has to be true for all choices fine, you can see here that this and this there is no problem, similarly this with this is no problem. The problem is here, here it is c times x 1 y 2 whereas, it is here c times x 2 y 1 and it is b times x 1 y 2 and b times y 1 x 2.

So, just match it, c times x 1 y 2; c is here where it is x 1 y 2 is coming with b and this is coming with this part alright. So, if we take so if we take b is equal to c, then we get that X, Y is indeed equal to Y, X alright, for all X, Y belonging to R 2; is that ok. So, therefore we need that the matrix A, the first condition required such that matrix A has to be symmetric fine; condition required A has to be symmetric. So, let us write A as a b, b d for us fine and proceed further now.

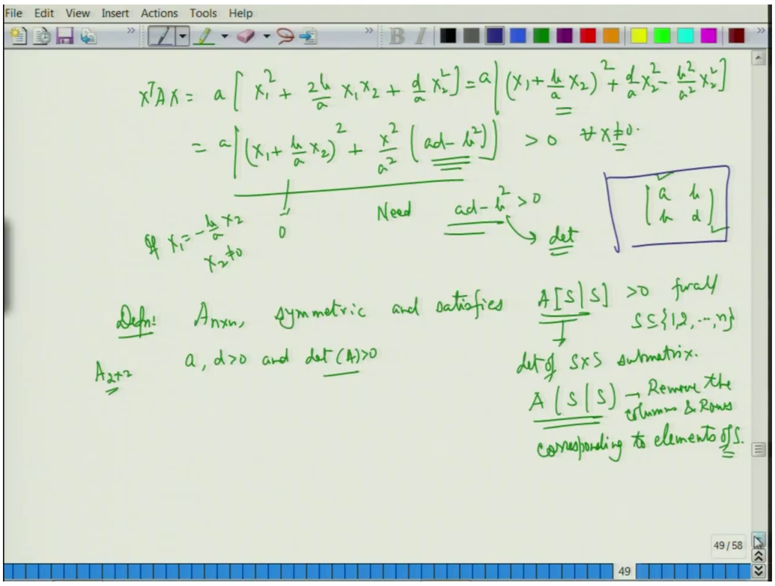
So, once this is done then you can check that second condition is valid as well as third, because X times Y plus Z will be equal to Y plus Z transpose A X, which by matrix multiplication will be Y transpose A X plus Z transpose A X which is X, Y plus X, Z. And third part will also be valid alright, so check it out valid no problem fine.

Now, the problem is with the fourth part, we want that that X comma X should be greater than equal to 0 for all X not equal to 0, and equality if and only if X is 0 alright, this is what we want; so let us compute that part. So, this gives me X transpose A X which I would like you to see that it is nothing but A times X 1 square plus 2 b X 1 X 2 plus d times X 2 square, alright.

So, therefore if X 2 is 0 and X 1 is not 0 will imply that X 2 is 0 means; this part will be 0, this part will be 0, you will get only left out with a X 1 square. You want this to be greater than 0 for every X which is non-zero, and therefore a has to be positive alright. So, this gives me a condition that a has to be positive, fine.

Similarly, if I take X 1 to be 0 and X 2 to be non-zero, it will imply that d has to be positive alright, so the diagonal entries are positive. What about the determinant, can I say that determinant is also positive alright. So, question do we need the determinant to be positive? Alright, so let us try that out fine. So, I already know that a is positive d is positive, so let us try to do what is called completing the square.

(Refer Slide Time: 20:09)



So, X transpose A X by for us is nothing but a times so a times X 1 square plus 2 b upon a X 1 X 2 plus d upon a X 2 square, which is same as a is outside X 1 plus b upon a X 2 whole square; just let us check whether it is or not, X 1 square plus 2 b a X 1 X 2 yeah, fine plus d upon a X 2 square minus a square coming from here which is b square by a square X 2 square.

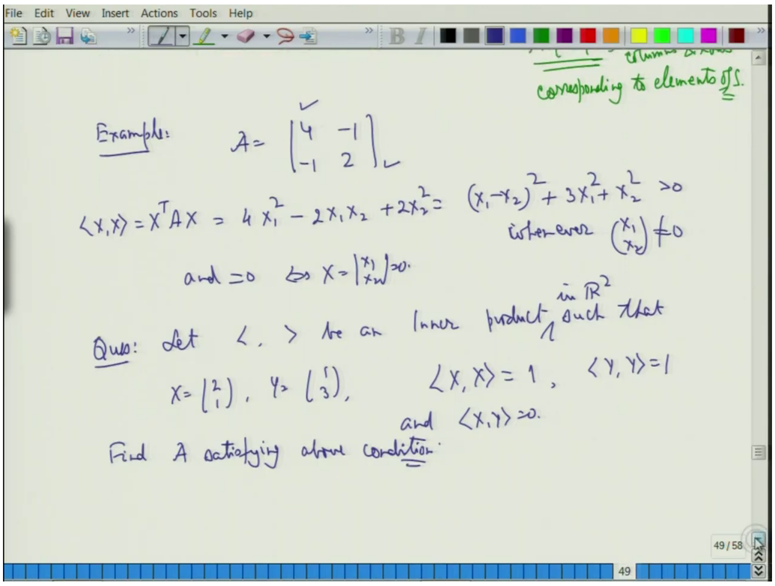
So, this is a times X 1 plus b upon a X 2 whole square plus X 2 square common, so X 2 square by a square common; I get here a d minus b square a d minus d square alright, fine. So, now what I want is that this has to be greater than 0 for all X non-zero alright, fine.

So, if I take X 2 to be or X 1 to be minus of this, this part will be 0 fine; and therefore, I need that this has to be positive alright. So, if I take X 1 to be equal to minus b upon a X 2 this part is 0, so I will need that need and X 2 not equal to 0 will imply that a d minus b square should be positive fine. And what is a d minus b square our matrix were a b b d, this is nothing but determinant, fine.

So, we need that a has to be positive, d has to be positive and the determinant has to be positive fine, such matrices has special name they are called; so a matrix A n cross n symmetric and satisfies A of S S 0. So, recall that what was that this was the determinant of something alright positive or for the 2 by 2 case we just need that for so A is 2 cross 2, we need a has to be positive and determinant of a is positive. In general, I need this to be 0 for all S subset of 1 to n alright; these are the determinant of S cross S sub matrix alright.

We also add A of S S, so here it was in the chapter 1 determinant this is what we had, this means remove the columns and rows; rows from rows corresponding to elements of S, corresponding to elements of S, is that ok. So, what we see here is that we have got any matrix which has this form 2 cross 2; where a is positive, d is positive, a d minus b a square is positive, then it is indeed a indeed it will give me a inner product, alright.

(Refer Slide Time: 23:42)

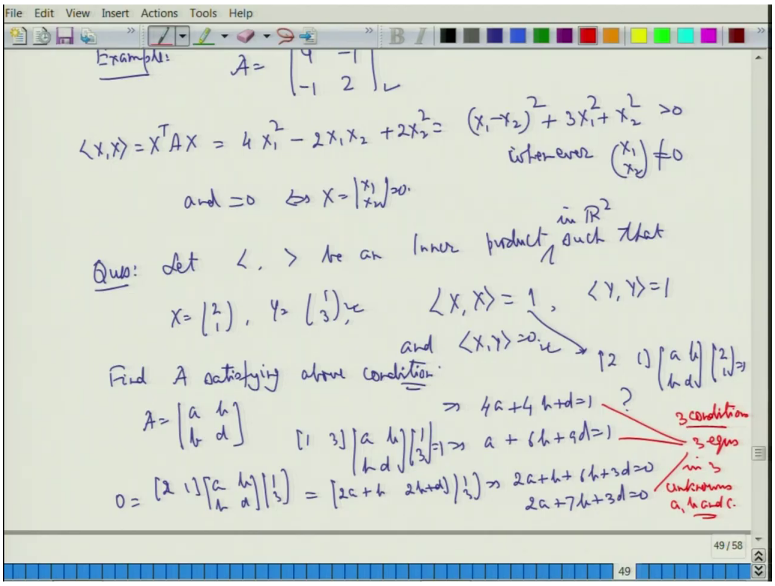


So, example proper example, I take A as 4 minus 1 minus 1 4, suppose I take or say 2, I take this as my matrix it is symmetric. So, you can check that all the conditions will be valid here, and because 4 is positive, 2 is positive, this is this.

So, if I want to look at X transpose A X which is for me going to look like X comma X will be equal to 4 times X 1 square minus 2 times X 1 X 2, fine; 2 times X 1 X 2 plus 2 X 2 square, which is same as X 1 minus X 2 whole square plus 3 X 1 square plus X 2 square, alright which is always greater than 0; if whenever X 1 X 2 is not equal to 0, and equal to 0 if and only if X is equal to X 1 X 2 is 0, alright.

So, we can check that everything is nice here. So, we can ask questions here, so the question that are going to come here which can be asked is question that let this be an inner product, inner product in R 2 such that 2 comma 1 is inner product of. So, inner product of 2 let me write X as, 2 1; Y as 1 3, alright such that length of X, I have not defined what is length of X, but anyway. So, such that X comma X is a 1; Y comma Y is 1; and X comma Y is 0 alright. Find A satisfying above condition, alright.

(Refer Slide Time: 26:06)

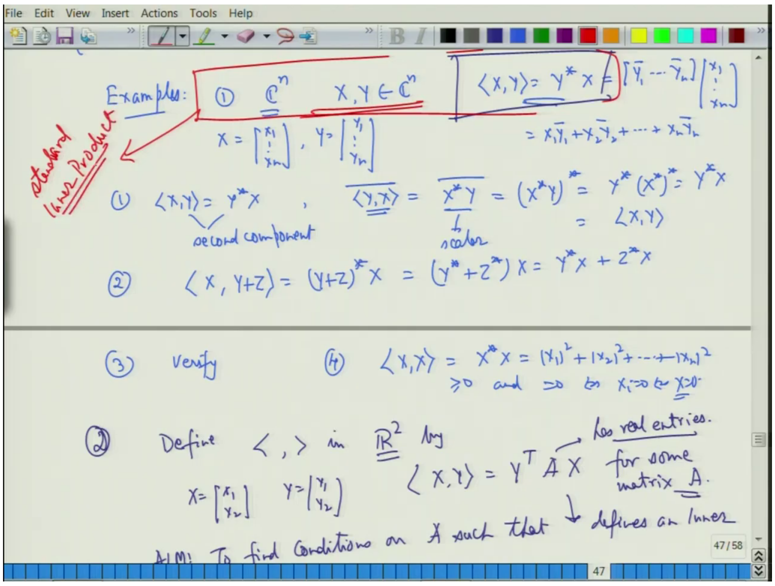


So, what we will have to do is that in such a case, we will have to take A as a b c d fine. Now, X, X is 1 implies that 2 1 times a b b d times 2 1, should be 1; so this should be b itself, because it has to be symmetric. So, this will imply that 4 a plus 2 b plus d is 1 just check it. 2 into 2 is 4 times a plus (Refer Time: 26:37) 4, I think 4 b not this, plus 4 b plus d is 0; please check all these calculations, alright.

1 3 I am saying here, so again 1 3 a b b d is equal to 1 will imply that times 1 3 is equal to 1 will imply a plus 6 b plus 9 d is equal to 1. And this part which is 0 will give me, 2 1 a b b d times 1 3, 0 is equal to this will give me just multiply. So, 2 a plus b 2 b plus d 1 3, will imply that so 2 a plus b plus (Refer Time: 27:39) 6 b plus 3 d is 0 or which is same thing as saying that 2 a plus 7 b plus 3 d is 0, alright.

So, see here I have got 1 condition, 2 condition, 3 conditions; 3 conditions or 3 equations in 3 unknowns a, b and c; solve it, to get your answer alright. So, I would like you to try these outs, because they are important for us, as far as the calculation is concerned. In the next example, next class we will look at some more examples here and try to relate different ideas, but most of the times we will be looking at only a standard inner product alright. A standard inner product means, the dot product itself is that ok; dot product or we will be looking at with that.

(Refer Slide Time: 28:37)



So, this part; part we will be looking at complex which will be this part, is that ok. So, mostly we will be looking at this, what is the standard inner product; but we will look at some more examples, so for clarity fine; that is all for now.

Thank you.